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On the uniqueness of minimal extrapolations

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1. Introduction

Let E be a non-empty closed proper subset of $-\infty < t < \infty$ and let f_0 be a complex-valued function on E such that the class D of all functions F of bounded variation which satisfy

$$f_0(t) = \int_{-\infty}^{\infty} e^{itx} \, dF(x),$$

when $t \in E$, is non-empty. In case there exists a function $F_e \in D$, such that

$$\text{Var } F_e = \inf_{F \in D} \text{Var } F,$$

we call the corresponding function

$$f_e(t) = \int_{-\infty}^{\infty} e^{itx} \, dF_e(x), \quad -\infty < t < \infty,$$

a minimal extrapolation of f_0.

The concept of minimal extrapolation was introduced by Beurling [1]. Esseen [2], p. 13, pointed out that Lemme 1 in Beurling [1] implies that a minimal extrapolation always exists if E is the closure of its interior. Actually, Beurling's methods can be used to prove more general results in this direction. We shall, however, in this paper instead turn to the following problem, which does not seem to have been treated before in the literature: If a minimal extrapolation exists, is it then unique?

In § 2 and § 3 we collect some preliminary results, and in § 4 we prove the uniqueness if E is a half line. The remaining cases are studied in § 5. It turns out that we necessarily have to lay extra conditions on f_0 in order to secure uniqueness. The particular case when E is the complement of a finite interval is discussed further in § 6, and it is shown that we have uniqueness if f_0 has the property that one (and hence every) corresponding function F has an absolutely continuous part with a continuous derivative.

2. Minimal extrapolations with absolutely continuous F_e

We denote by L_E^1 the class of functions $G \in L^1$ for which

$$\int_{-\infty}^{\infty} e^{itx} G(x)\,dx = 0$$

for every $t \in E$, and by L_E^∞ the class of functions $\varphi \in L^\infty$ for which

$$\int_{-\infty}^{\infty} \varphi(x) G(x)\,dx = 0$$

for every $G \in L_E^1$. Thus L_E^∞ is the weak closure of the linear set which is spanned by the functions e^{itx}, $t \in E$.

The following lemma is not entirely new, since parts of it can be traced back to Sz. Nagy [4] and seminars by Beurling.

Lemma. Suppose that

$$f_1(t) = \int_{-\infty}^{\infty} e^{itx}\,dF_1(x),$$

where $F_1 \in D$ is absolutely continuous.

Necessary and sufficient for f_1 to be a minimal extrapolation is that there exists a function $\varphi \in L_E^\infty$ with

$$|\varphi(x)| \leqslant 1$$

almost everywhere and such that

$$F_1'(x)\,\varphi(x) = |F_1'(x)|$$

almost everywhere.

Proof of the necessity. If F_1 is absolutely continuous and corresponds to a minimal extrapolation we must have

$$m = \operatorname{Var} F_1 = \int_{-\infty}^{\infty} |F_1'(x)|\,dx \leqslant \int_{-\infty}^{\infty} |F_1'(x) - G(x)|\,dx,$$

for every $G \in L_E^1$. Hence m is the distance in L^1 between the element F_1' and the closed linear set L_E^1, and by the Hahn–Banach theorem there exists a bounded linear functional F^* with norm $\leqslant 1$ such that

$$F^*(G) = 0, \quad \text{if } G \in L^1(E),$$

while

$$F^*(F_1') = m.$$

The space of bounded linear functionals on L^1 can be identified with L^∞. Hence there exists a function $\varphi \in L^\infty$ such that

$$|\varphi| \leqslant 1 \tag{1}$$

almost everywhere, such that

$$\int_{-\infty}^{\infty} \varphi(x)\, G(x)\, dx = 0,$$

if $G \in L_E^1$, i.e. which belongs to L_E^∞, and finally such that

$$\int_{-\infty}^{\infty} \varphi(x)\, F_1'(x)\, dx = m = \int_{-\infty}^{\infty} |F_1'(x)|\, dx,$$

which by (1) implies that

$$\varphi(x)\, F_1'(x) = |F_1'(x)|$$

almost everywhere.

Proof of the sufficiency. We skall use the auxiliary function

$$K(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-y^2}\, dy.$$

Let F_1 and φ satisfy the conditions of the lemma. If F is an arbitrary function in the class D, and if $*$ denotes the ordinary convolution operation in the class of functions of bounded variation, then it is easy to see that the functions

$$F(x) * K\left(\frac{x}{\sigma}\right) \quad \text{and} \quad F_1(x) * K\left(\frac{x}{\sigma}\right)$$

are absolutely continuous for every $\sigma > 0$ and that the derivative of their difference belongs to the class L_E^1. Hence

$$\int_{-\infty}^{\infty} \varphi(x)\, d\left[F(x) * K\left(\frac{x}{\sigma}\right) - F_1(x) * K\left(\frac{x}{\sigma}\right)\right] = 0,$$

which may be written

$$\int_{-\infty}^{\infty} \varphi(x)\, d\left[F(x) * K\left(\frac{x}{\sigma}\right)\right] = \int_{-\infty}^{\infty} \varphi(x)\, d\left[F_1(x) * K\left(\frac{x}{\sigma}\right)\right]. \tag{2}$$

The function K is monotonically increasing with variation 1, and hence, by a wellknown convergence theorem for Lebesgue integrals,

$$\lim_{\sigma \to +0} \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} H(x-y)\, dK\left(\frac{y}{\sigma}\right) - H(x) \right| dx = 0$$

for any function $H \in L^1$. Using this we see that the right hand side of (2) converges to

$$\int_{-\infty}^{\infty} \varphi(x) F_1'(x)\,dx = \int_{-\infty}^{\infty} |F_1'(x)|\,dx = \text{Var } F_1,$$

as $\sigma \to +0$. On the other hand the left hand side has absolute value

$$\leqslant \text{Var}\left\{F(x) * K\left(\frac{x}{\sigma}\right)\right\} \leqslant \text{Var } F(x) \cdot \text{Var } K = \text{Var } F.$$

For that reason

$$\text{Var } F \geqslant \text{Var } F_1$$

for every $F \in D$, which shows that F_1 corresponds to a minimal extrapolation.

3. On the uniqueness problem when E contains a half line

The difference between any two functions in D has a Fourier-Stieltjes transform which vanishes on E. If E contains a half line, i.e. an interval $(-\infty, a)$ or an interval (b, ∞), this implies by a theorem by F. and M. Riesz [3] that the difference is absolutely continuous. Hence the functions in D differ only by their absolutely continuous parts, and as for questions of existence and uniqueness of minimal extrapolations there is no restriction in assuming that f_0 has the property that every function in D is absolutely continuous. We shall assume this in the rest of this paragraph, and we exchange for simplicity's sake the derivatives $F^1, F_1', F_2', \ldots$ of functions in D to $H, H_1, H_2, \ldots$ which then are functions in a certain subclass D' of L^1.

Suppose that f_1 and f_2 are two different minimal extrapolations of f_0, corresponding to H_1 and H_2, respectively. The function

$$f_3(t) = \tfrac{1}{2}(f_1(t) + f_2(t)) = \int_{-\infty}^{\infty} e^{itx}\, \tfrac{1}{2}(H_1(x) + H_2(x))\,dx$$

is another extrapolation of f_0, and since

$$\int_{-\infty}^{\infty} \tfrac{1}{2}|H_1(x) + H_2(x)|\,dx \leqslant \tfrac{1}{2}\int_{-\infty}^{\infty} |H_1(x)|\,dx + \tfrac{1}{2}\int_{-\infty}^{\infty} |H_2(x)|\,dx, \tag{3}$$

it is a minimal extrapolation. We must furthermore have equality in (3), and this is the case only if

$$\frac{|H_1(x)|}{H_1(x)} = \frac{|H_2(x)|}{H_2(x)}, \tag{4}$$

for almost every x, for which the two members are defined. Using the function φ which in the sense of the lemma corresponds to the minimal extrapolation f_3, (3) and (4) imply that

$$\varphi(x)\,(H_1(x)+H_2(x)) = |H_1(x)+H_2(x)|,$$
$$\varphi(x)\,H_1(x) = |H_1(x)|,$$
and
$$\varphi(x)\,H_2(x) = |H_2(x)|,$$

almost everywhere. But

$$\int_{-\infty}^{\infty} e^{itx}\,(H_1(x)-H_2(x))\,dx = 0,$$

if $t \in E$, and since we assume that E contains a half line, well-known properties of Fourier transforms in L^1 show that $H_1 - H_2$ coincides almost everywhere with the boundary values of a function, analytic in a half plane. Since the function is not identically zero we may conclude that it is $\neq 0$ almost everywhere (cf. [3]). This implies that for almost every x at least one of the functions H_1 and H_2 is $\neq 0$. We may conclude that

$$H_1 + H_2 \neq 0$$

and

$$|\varphi(x)| = 1$$

almost everywhere.

Apparently we have moreover

$$\varphi(x)\,(H_1(x)-H_2(x)) = \psi(x)\,|H_1(x)-H_2(x)|, \tag{5}$$

where ψ is real-valued and

$$|\psi(x)| = 1$$

almost everywhere.

4. The uniqueness when E is a half line

Theorem 1. The minimal extrapolation is unique when E is a half line.

Proof. It is apparently enough to give the proof if E is the half line $0 \leqslant t < \infty$ and, by § 3, if the functions in the class D are absolutely continuous.

We give an indirect proof, and we thus assume that there exist two different minimal extrapolations

$$f_\nu(t) = \int_{-\infty}^{\infty} e^{itx}\,H_\nu(x)\,dx, \quad \nu = 1, 2.$$

For any $\lambda \geqslant 0$ the function

$$e^{i\lambda x}\,(H_1(x)-H_2(x))$$

has a Fourier transform which vanishes if $t \geqslant 0$, i.e. if $t \in E$. The function φ which in the sense of the lemma corresponds to the minimal extrapolation

$$\tfrac{1}{2}\,(f_1 + f_2),$$

(cf. § 3) belongs to the class L_E^∞ and hence

$$\int_{-\infty}^{\infty} \varphi(x)\, e^{i\lambda x} (H_1(x) - H_2(x))\, dx = 0, \tag{6}$$

if $\lambda \geqslant 0$. But (5) shows that

$$\varphi(H_1 - H_2)$$

is real-valued almost everywhere. By conjugation of (6) we therefore obtain

$$\int_{-\infty}^{\infty} \varphi(x)\, e^{-i\lambda x} (H_1(x) - H_2(x))\, dx = 0,$$

if $\lambda \geqslant 0$, which shows that (6) holds for every real-valued λ. Hence by the uniqueness theorem for Fourier transforms in L^1

$$\varphi(x)\,(H_1(x) - H_2(x)) = 0$$

almost everywhere, and since $|\varphi| = 1$ almost everywhere (§ 3) we can conclude that

$$H_1(x) = H_2(x)$$

almost everywhere. But the minimal extrapolations were assumed to be different, and hence this contradiction proves the uniqueness.

5. The case when E is not a half line

Theorem 2. If E is not a half line, then there exist functions f_0 which have non-unique minimal extrapolations.

Proof. We have to consider two separate cases.

Case 1: The complement of E is connected. Since the complement is open and is not a half line, it has to be a finite interval. It is apparently no restriction to assume that the complement of E is the interval $-1 < t < 1$.

Case 2. The complement of E is disconnected. It is no restriction to assume that $0 \in E$ while the two intervals $(-a-b,\ -a+b)$ and $(a-b,\ a+b)$ are in the complement. (a and b are some positive numbers, $b < a$.)

The proof will be quite similar in the two cases. We start from a real-valued function H, which $\in L^1$, is $\neq 0$ almost everywhere and finally has the property that

$$\frac{|H(x)|}{H(x)} \in L_E^\infty.$$

By the lemma in § 2 the function

$$f(t) = \int_{-\infty}^{\infty} e^{itx} H(x)\, dx$$

is a minimal extrapolation of the restriction of f_0 to E. Then let G be a real-valued function $\in L_E^1$, which is $\not\equiv 0$ and satisfies

$$|G(x)| < |H(x)|. \tag{7}$$

Obviously
$$\frac{|H(x)|}{H(x)} = \frac{|H(x)+G(x)|}{H(x)+G(x)}$$

and
$$\int_{-\infty}^{\infty} e^{itx}(H(x)+G(x)) = f(t), \quad \text{if } t \in E.$$

By the lemma also this function is a minimal extrapolation, and by the uniqueness theorem for Fourier transforms in L^1, the two minimal extrapolations are different. Hence the only problem is to show that we can find functions H and G with the mentioned properties.

Case 1. We form the function

$$\varphi(x) = \begin{cases} 1 & \text{if } \cos x \geqslant 0, \\ -1 & \text{if } \cos x < 0. \end{cases}$$

It is easy to show that if $G_0 \in L^1 \cap L^2$, then

$$\int_{-\infty}^{\infty} \varphi(x) G_0(x)\, dx = \sum_{-\infty}^{-1} + \sum_{1}^{\infty} \frac{2}{\pi n} \sin n \frac{\pi}{2} \int_{-\infty}^{\infty} e^{inx} G_0(x)\, dx.$$

Hence, if $G_0 \in L_E^1 \cap L^2$

$$\int_{-\infty}^{\infty} \varphi(x) G_0(x)\, dx = 0, \tag{8}$$

and since obviously $L_E^1 \cap L^2$ is dense in L_E^1, (8) holds for any $G_0 \in L_E^1$. For that reason $\varphi \in L_E^\infty$, and we can choose

$$H(x) = \frac{\varphi(x)}{1+x^2}.$$

It can be proved by a simple contour integration that any entire function of exponential type 1, which belongs to L^1 on the real axis, has the property that the Fourier transform of its restriction to the real axis vanishes outside $(-1, 1)$, i.e. in our terminology belongs to L_E^1. Hence we can choose

$$G(x) = \lambda \cdot \frac{1-\cos x}{x^2}$$

if the positive constant λ is assumed to be so small that (7) holds.

Case 2. Since $0 \in E$,

$$\int_{-\infty}^{\infty} G_0(x)\, dx = 0$$

for every $G_0 \in L_E^1$, i.e. every constant function belongs to L_E^∞. For that reason we can choose

$$H(x) = \frac{1}{1+x^2}$$

The Fourier transform of

$$\frac{1-\cos b x}{x^2} \cdot \cos a x$$

vanishes outside $(-a-b, \ -a+b)$ and $(a-b, \ a+b)$. Hence we can choose

$$G(x) = \lambda \frac{1-\cos b x}{x^2} \cdot \cos a x$$

if the positive constant λ is assumed to be so small that (7) holds.

6. A sufficient condition for uniqueness in a special case

It follows from Theorem 2 that in the cases which are covered by that theorem the class of functions f_0 with unique minimal extrapolations is a proper subclass of the class of all possible f_0 under consideration. The extent and properties of this subclass seem to depend very heavily on the algebraic properties of the set E. If, for instance, E is a subset of the set $\{a n + b\}_{-\infty}^{\infty}$, for some $a \neq 0$ and b, and if f_e is a minimal extrapolation, then

$$f_e(t) \cos \frac{2\pi (t-b)}{a}$$

is also a minimal extrapolation. Hence $f_0 \equiv 0$ is the only function with a unique minimal extrapolation. On the other hand, if E is not of that kind it is easy to see that also the functions

$$f_0(t) = \text{constant}$$

have unique minimal extrapolations.

We shall turn to the special case when E is the complement of a finite interval, and we shall prove the following theorem which gives a sufficient condition for f_0 to have a unique minimal extrapolation.

Theorem 3. Suppose that E is the complement of a finite interval, and that f_0 has the representation

$$f_0(t) = \int_{-\infty}^{\infty} e^{itx} \, d F_0(x),$$

where $F_0(x)$ has an absolutely continuous part with a continuous derivative. Then it has a unique minimal extrapolation.

Proof. By § 3 it is enough to give the proof, when $F_0(x)$ is purely absolutely continuous. We can furthermore assume that the complement is the interval $-1 < t < 1$.

The difference between any two functions in the class D' (§ 3) has a Fourier transform which vanishes outside $-1 < t < 1$. By the inversion theorem this has the consequence, that this difference coincides almost everywhere with the values on the real axis of an entire function of exponential type 1. Since by assumption one of the functions in D' is continuous, we may assume that every function in the class is continuous.

Let us therefore assume that we have two different minimal extrapolations

$$f_\nu(t) = \int_{-\infty}^{\infty} e^{itx} H_\nu(x)\, dx, \quad \nu = 1, 2,$$

where H_1 and H_2 belong to L^1 and are continuous. By the above arguments the function

$$G(x) = H_1(x) - H_2(x)$$

is the restriction to the real axis of an entire function of exponential type 1. The function φ, which in the sense of the lemma corresponds to the minimal extrapolation

$$\tfrac{1}{2}(f_1 + f_2)$$

(cf. § 3) can be assumed to be continuous except at certain points, where both H_1 and H_2 vanish. Hence it is continuous and satisfies

$$|\varphi(x)| = 1$$

except at certain of the zeros of G. It then follows from (5) that the real-valued function ψ can be assumed to satisfy

$$|\psi(x)| = 1$$

everywhere, and it changes its sign, i.e. $\psi(x+0) \neq \psi(x-0)$, only at certain of the zeros of the entire function G.

Let us first discuss the case when there are only a finite number of changes in the sign of ψ. Let the changes occur when

$$x = \lambda_1, \ldots, \lambda_n.$$

We form

$$K(x) = G(x) \frac{1}{(x-\lambda_1) \ldots (x-\lambda_n)}.$$

K is the restriction to the real axis of an entire function of exponential type 1, and it belongs to L^1. As mentioned before (§ 5) this implies that it belongs to L_E^1, and since $\varphi \in L_E^\infty$, we have

$$\int_{-\infty}^{\infty} \varphi(x)\, G(x) \frac{1}{(x-\lambda_1) \ldots (x-\lambda_n)}\, dx = 0,$$

hence by (5)

$$\int_{-\infty}^{\infty} |G(x)| \frac{\psi(x)}{(x-\lambda_1) \ldots (x-\lambda_n)}\, dx = 0.$$

But the integrand has a constant sign, and it is not identically vanishing. This contradiction shows the uniqueness if the number of changes in the sign is finite.

If there is an infinite number of changes in the sign of ψ, there exists a point x_0, where there is no change, and such that the number of changes for $x < x_0$ and for $x > x_0$ is either even or infinite. We denote these points by λ_i in such a way that

$$\cdots < \lambda_{-n} < \cdots < \lambda_{-1} < x_0 < \lambda_1 < \cdots < \lambda_n < \cdots.$$

We can then choose the finite open interval (a, b) so large that there is an even number of changes for $a < x < x_0$ and for $x_0 < x < b$, say $2m$ and $2p$, respectively, and such that

$$\int_{\lambda_{-1}}^{\lambda_1} |G(x)|\, dx > \int_{-\infty}^{a} + \int_{b}^{\infty} |G(x)|\, dx. \tag{9}$$

We form the functions

$$K_+(x) = \frac{(x-\lambda_2) \ldots (x-\lambda_{2p})}{(x-\lambda_1) \ldots (x-\lambda_{2p-1})}$$

and

$$K_-(x) = \frac{(x-\lambda_{-2}) \ldots (x-\lambda_{-2m})}{(x-\lambda_{-1}) \ldots (x-\lambda_{-2m+1})}.$$

The functions are positive if $x \in (\lambda_{-1}, \lambda_1)$ and if $x \notin (a, b)$, and it is easy to see that

$$\inf_{x \in (\lambda_{-1}, \lambda_1)} K(x) > \sup_{x \notin (a, b)} K(x), \tag{10}$$

is fulfilled with K exchanged for either of the functions K_+ and K_-. Hence the function

$$K = K_+ \cdot K_-$$

also satisfies (10).

The function $G \cdot K$ belongs to L^1 and is the restriction to the real axis of an entire function of exponential type 1. Hence it must satisfy

$$\int_{-\infty}^{\infty} \varphi(x)\, G(x)\, K(x)\, dx = 0,$$

i.e.
$$\int_{a}^{b} \varphi(x)\, G(x)\, K(x)\, dx = -\left(\int_{-a}^{a} + \int_{b}^{\infty} \varphi(x)\, G(x)\, K(x)\, dx \right). \tag{11}$$

But (5) shows that the left hand member of (11) has the value

$$\int_{a}^{b} |G(x)|\, \psi(x)\, K(x)\, dx,$$

and since $\psi \cdot K$ has a constant sign in the whole interval (a, b), we obtain from (11)

$$\int_a^b |G(x)|\cdot|K(x)|\,dx \leqslant \int_{-\infty}^{a} + \int_b^{\infty} |G(x)|\,|K(x)|\,dx.$$

Hence we have, using (10)

$$\int_{\lambda_{-1}}^{\lambda_1} |G(x)|\,dx \leqslant \int_{-\infty}^{a} + \int_b^{\infty} |G(x)|\,dx,$$

which is a contradiction to (9). This proves the uniqueness in the general case.

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