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On a class of normed rings[1]

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Introduction

A. BEURLING, in [1], introduced the class of normed rings L'_σ with the norm $\|f\| = \int_{-\infty}^{\infty} |f(x)|\,\sigma(x)\,dx$, $\sigma(x)$ being a weight function such that $\sigma(x+y) \le \le \sigma(x)\,\sigma(y)$. This paper has its origin in Prof. Beurling's suggestion to use the methods and results of [1] in studying a more general class of spaces. The writer wishes to take this opportunity to express his sincere thanks to Prof. Beurling for his stimulating advice and kind encouragement

We are concerned with certain spaces of functions defined on the real line. Let L be a Banach space of functions summable on $(-\infty, \infty)$, and let $\bar{L}$ consist of all φ with $\int_{-\infty}^{\infty} |f(x)|\cdot|\varphi(x)|\,dx < \infty$ if $f \in L$. We shall use the following notation: if $f,\ g \in L$, $f * g\,(x) = \int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy$, while if $f \in \bar{L}$, $\varphi \in L$, $f * \varphi\,(x) = = \int_{-\infty}^{\infty} f(y-x)\,\varphi(y)\,dy$. We shall consider the following conditions on L:

(1) L contains the characteristic functions of all finite intervals.

(2) If $f \in L$ and $g \in L$ and $|f(x)| = |g(x)|$ a. e., then $\|f\| = \|g\|$.

(3) If for a measurable function f we have $\int_{-\infty}^{\infty} |f(x)|\cdot|\psi(x)|\,dx < \infty$, for all ψ in $\bar{L}$, then $f \in L$.

(4) Every bounded linear functional α on L is of the form $\alpha(f) = \int_{-\infty}^{\infty} f(x)\,\varphi(x)\,dx$, where $\varphi \in \bar{L}$, for all f in L; conversely, every φ in $\bar{L}$ defines a bounded functional in this way.

(5) The translation operator $T_\tau : T_\tau f(x) = f(x-\tau)$ is bounded on L for each real τ.

(6) L is a normed ring under convolution, i. e., if $f,\ g \in L$, then $f \star g \in L$ and $\|f \star g\| \le k\,\|f\|\cdot\|g\|$, where k is a constant.

[1] Most of this paper forms part of the author's dissertation (Harvard, 1951).

If we assign to each φ in $\bar{L}$ a norm equal to the norm of the corresponding functional on L, then $\bar{L}$ becomes a Banach space isomorphic to the conjugate space of L. It is easily seen that every space L satisfying (1) through (4) is separable. If in addition L satisfies (5) we may note the following: $\log \| T_\tau \|$ is a subadditive function finite for all real τ. By known properties of subadditive functions we have that $\| T_\tau \|$ is bounded on every finite interval of values of τ. (Cf. Hille, [7], Chap. VI.) Since the constant 1 is in $\bar{L}$, $\| T_\tau \| \geq 1$ all τ. If τ_n converges to τ, $T_{\tau_n} f$ converges to $T_\tau f$ for every f in L in the norm of L. Finally, as $|\tau|$ tends to ∞, $\| T_\tau \| = 0\,(e^{A|\tau|})$ for some finite A.

If at last L satisfies (6), i. e. L is normed ring, we can say the following: Every complex-valued homomorphism of L has the form: f maps into $\hat{f}(s) = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$ where s is some complex number determined by the given homomorphism. We shall consider in this paper only such spaces L for which $\| T_\tau \| = 0\,(e^{\varepsilon|\tau|})$ for each positive ε. Hence e^{isx} is in $\bar{L}$ if and only if s is real and hence the space of regular maximal ideals of L coincides with the real line.

We shall denote by J the class of normed rings L satisfying (1) through (6). In Section 1 we shall consider for the spaces L in J the following questions which are of interest in the general theory of normal rings:

(A) Given a point p on the real line and an open set around p. Does L contain a function f with $\hat{f}(p) = \int_{-\infty}^{\infty} f(x)\, e^{ipx}\, dx \neq 0$ while $\hat{f}(\lambda)$ is identically 0 outside the open set?

(B) Is the set of f in L with $\hat{f}(\lambda)$ identically 0 outside a compact set dense in L?

(C) Does every closed ideal I in L which is not all of L have a zero, i. e. can we find p with $\hat{f}(p) = 0$ for all f in I?

In section 2 we shall consider spaces L satisfying (1) through (4) and discuss a problem concerning trigonometric approximation in $\bar{L}$. In Section 3 we shall use the results of Section 2 to answer the following question for a subclass of J:

(D) How may one characterize the ideals which have precisely one zero?

Examples of spaces L satisfying (1) through (4) may be obtained as follows: Let $r \geq 1$, $\frac{1}{r} + \frac{1}{s} = 1$, and let $p(x)$ be positive and summable and $\sigma(x) = (p(x))^{-\frac{s}{r}}$. If L_σ^r denotes the space with norm $\| f \|^r = \int_{-\infty}^{\infty} | f(x) |^r \sigma(x)\, dx$ and L_p^s the space with $\| \varphi \|^s = \int_{-\infty}^{\infty} | \varphi(x) |^s p(x)\, dx$, then L_σ^r is a space L satisfying (1) through (4) and L_p^s is its conjugate space $\bar{L}$. Furthermore, if $\sup_{-\infty < x < \infty} \frac{p(x - \tau)}{p(x)} < \infty$ for all real τ, then L_σ^r also satisfies (5). Regarding (6) we have the following Lemmas:

Lemma 1: *Let L be a space which satisfies (1) through (5). For such φ in $\bar{L}$ let $\Psi(\tau)=\|\varphi(x+\tau)\|$, $-\infty<\tau<\infty$. If, for each φ in $\bar{L}$, $\Psi\in\bar{L}$ and $\|\Psi\|\le k\|\varphi\|$ where k is a constant, then (6) holds for L.*

Proof: We first note that if $\varphi\in\bar{L}$ then for every real τ and f in L,

$$\int_{-\infty}^{\infty}|\varphi(x+\tau)|\,|f(x)|\,dx=\int_{-\infty}^{\infty}|\varphi(x)|\,|f(x-\tau)|\,dx<\infty$$

since $f(x-\tau)\in L$ by (5). Thus $\varphi(x+\tau)\in\bar{L}$. If now $f, g\in L$, then f and g are summable and so $f*g$ is summable. Take any φ in $\bar{L}$. Then

$$\int_{-\infty}^{\infty}|\varphi(x)|\,dx\left|\int_{-\infty}^{\infty}f(x-y)\,g(y)\,dy\right|\le\int_{-\infty}^{\infty}|g(y)|\,dy\int_{-\infty}^{\infty}|f(x)|\,|\varphi(x+y)|\,dx$$

$$\le\int_{-\infty}^{\infty}|g(y)|\cdot\|f\|\cdot\Psi(y)\,dy\le\|g\|\cdot\|\Psi\|\cdot\|f\|.$$

Therefore $f*g(x)\cdot\varphi(x)$ is summable for every φ in $\bar{L}$ whence by (3) $f*g\in L$. From (4) we get that

$$\|f*g\|=\sup_{\|\varphi\|=1}\left|\int_{-\infty}^{\infty}f*g(x)\,\varphi(x)\,dx\right|$$

and so, since $\|\Psi\|\le k\|\varphi\|$, we have $\|f*g\|\le k\|f\|\cdot\|g\|$.

Lemma 2: *Let L_p^s have the same meaning as above and $s<\infty$. If $\varphi\in L_p^{\bar{s}}$ let $\Psi(\tau)=\|\varphi(x+\tau)\|$. Then, for every φ, $\|\Psi\|\le k\|\varphi\|$ where k is a konstant, if and only if $\int_{-\infty}^{\infty}p(x-\tau)\,p(\tau)\,d\tau\le k^s\,p(x)$.*

Proof:

$$\|\varphi(x+\tau)\|^s=\int_{-\infty}^{\infty}|\varphi(x+\tau)|^s\,p(x)\,dx=\int_{-\infty}^{\infty}|\varphi(x)|^s\,p(x-\tau)\,d\tau$$

$$\|\Psi\|^s=\int_{-\infty}^{\infty}\|\varphi(x+\tau)\|^s\,p(\tau)\,d\tau=\int_{-\infty}^{\infty}|\varphi(x)|^s\,dx\int_{-\infty}^{\infty}p(x-\tau)\,p(\tau)\,d\tau.$$

Clearly $\|\Psi\|\le k\|\varphi\|$ if and only if $\int_{-\infty}^{\infty}p(x-\tau)\,p(\tau)\,d\tau\le k^s\,p(x)$.

We note that the condition $\int_{-\infty}^{\infty}p(x-\tau)\,p(\tau)\,d\tau\le k\,p(x)$ is satisfied for instance for $p(x)=\dfrac{1}{1+|x|^\alpha}$, $\alpha>1$, or $p(x)=e^{-|x|^\alpha}$, $0<\alpha<1$, while it is not satified for $p(x)=e^{-|x|}$.

Thus a space L_σ^r, $r>1$, is in J provided the above-named conditions on $p(x)$

are satisfied and also $\int\limits_{-\infty}^{\infty} p(x-\tau)\,p(\tau)\,d\tau \leq k\,p(x)$. A space L'_σ is in J provided $\sigma(x) \geq 1$ and $\sigma(x+y) \leq \sigma(x)\,\sigma(y)$. In particular, for σ identically 1, the ring L' of all summable functions is in J and every L in J is, on the other hand, a subring of L'.

Section 1: In [1] Beurling proved the following for the spaces L'_σ: If $\sigma(x)$ is non-decreasing and $\sigma(x+y) \leq \sigma(x)\,\sigma(y)$, then (B) holds whenever (A) holds, and (A) holds if $\int\limits_{-\infty}^{\infty} \frac{\log \sigma(x)}{1+x^2}\,dx < \infty$. Under this assumption, he showed that every proper closed ideal generated by a single element has a zero. (C) follows easily from this last fact provided that $\sigma(x)$ is an even function.

Let now L belong to J and let T_τ be the operator of translation by τ. If $\varrho(\tau) = \| T_\tau \|$, then clearly $\varrho(\tau_1+\tau_2) \leq \varrho(\tau_1)\,\varrho(\tau_2)$ and so L'_ϱ is a ring. We assert:

Lemma 3: *Properties (A) and (B) hold in L provided that they hold in L'_ϱ.*

Proof: Let F be in L'_ϱ and f in L. We assert that then $F \star f$ is in L and $\| F \star f \| \leq \| f \| \cdot \| F \|$, where $\| F \|$ is taken in L'_ϱ and $\| f \|$ and $\| F \star f \|$ are taken in L. For $\| F \star f \|$ is the supremum of $\int\limits_{-\infty}^{\infty} |\varphi(x)| \cdot |F \star f(x)|\,dx$ over all φ in $\overline{L}$ with $\| \varphi \| = 1$. But

$$\int\limits_{-\infty}^{\infty} |\varphi(x)|\,|F \star f(x)|\,dx \leq \int\limits_{-\infty}^{\infty} |F(y)|\,dy \int\limits_{-\infty}^{\infty} |f(x-y)\,\varphi(x)|\,dx \leq \int\limits_{-\infty}^{\infty} |F(y)| \cdot \| T_y f \|\,dy.$$

Hence

$$\| F \star f \| \leq \int\limits_{-\infty}^{\infty} |F(y)|\,\varrho(y)\,dy \cdot \| f \| = \| F \| \cdot \| f \|.$$

Further, the set of functions $F \star g$ where $g \in L$, $F \in L'_\varrho$, is dense in L. For suppose φ is in $\overline{L}$ and

$$\int\limits_{-\infty}^{\infty} \varphi(x)\,F \star g(x)\,dx = 0$$

for all F in L'_ϱ and g in L.

Then $\int\limits_{-\infty}^{\infty} F(x-y)\,\varphi(x)\,dx = 0$ if F is in L'_ϱ, whence at last $\varphi = 0$ which yields the assertion.

Suppose now that (A) holds in L'_ϱ and consider any real p and any positive ε. We first choose f in L with $\hat{f}(p) \neq 0$, and next we choose F in L'_ϱ with $\hat{F}(p) \neq 0$ and $\hat{F}(\lambda) = 0$ for $|\lambda - p| > \varepsilon$. Then $F \star f \in L$ and the Fourier transform of $F \star f$ vanishes outside $(p-\varepsilon,\, p+\varepsilon)$ and $\neq 0$ at p. Thus (A) holds for L.

Now suppose (B) holds in L'_ϱ. Given g in L we can find for each positive ε some G in L'_ϱ and f in L with $\| g - G \star f \| < \varepsilon$. Further, by hypothesis, there exists H in L'_ϱ with $\|G-H\|$, taken in L'_ϱ, less than $\frac{\varepsilon}{\|f\|}$ and $\hat{H}(\lambda)=0$ for $|\lambda|\geq a$ for some a. Then $\|g-H\star f\|\leq\|g-G\star f\|+\|G\star f-H\star f\|<\varepsilon+\|f\|\cdot\|G-H\|<2\varepsilon$ and the Fourier transform of $H\star f=0$, $|\lambda|\geq\alpha$.

Lemma 4: *Given $\sigma(x)\geq 1$ and $\sigma(x+y)\leq\sigma(x)\,\sigma(y)$. Then (A) holds in L'_ϱ if and only if* $\int\limits_{-\infty}^{\infty}\frac{\log\sigma(x)}{1+x^2}\,dx<\infty.$

Proof: Set $P(x)=\frac{1}{\sigma(x)(1+x^2)}$. Then $P(x)$ is real, non-negative, is in

$$L^2(-\infty,\infty)\ \text{and}\ \int\limits_{-\infty}^{\infty}\frac{|\log P(x)|}{1+x^2}\,dx<\infty.$$

By theorem 12 of Paley-Wiener, [4], there exists $F_1(x)$ in L^2 with $F_1(x)=0$, $x\geq 0$, such that $F_1(x)$ is the Fourier transform of a function $G_1(x)$ with $|G_1(x)|=P(x)$. Then $|G_1(x)|\,\sigma(x)\in L'$ and $G_1(x)\in L'$. Since $P(-x)$ obeys the same conditions as $P(x)$, there exists $F(x)$ in L^2 with $F(x)=0$, $x\geq 0$, such that $F(x)$ is the Fourier transform of $G(x)$ where $|G(x)|=P(-x)$ and so

$$|G(x)|\,\sigma(-x)=\frac{1}{1+x^2},\ \text{whence}\ |G(-x)|\,\sigma(x)\in L',\ G(-x)\in L'.$$

Set $G_2(x)=G(-x)$. Then $F_2(x)=F(-x)$ is the Fourier transform of $G_2(x)$ and $F_2(x)=0$, $x\leq 0$.

We can now choose numbers α_1, α_2 corresponding to a given point p on the real line and a given positive ε so that the Fourier transform of $e^{i\alpha_1 x}G_1(x)\neq 0$ for $\lambda=p$ and $=0$ for $\lambda>p+\varepsilon$, while the transform of $e^{i\alpha_2 x}G_2(x)\neq 0$ for $\lambda=p$ and $=0$, $\lambda<p+\varepsilon$. Then if $H(x)=e^{i\alpha_1 x}G_1(x)\star e^{i\alpha_2 x}G_2(x)$, we have $\hat{H}(p)\neq 0$ and $\hat{H}(\lambda)=0$, $|\lambda-p|>\varepsilon$. Also $H\in L'_\sigma$ since it is the convolution of two functions in this space. Thus (A) holds.

Conversely, suppose $\int\limits_{-\infty}^{\infty}\frac{\log\sigma(x)\,dx}{1+x^2}=\infty$. Given F in L'_ϱ we have

$$\int\limits_{-N}^{N}\log|F(x)|\frac{dx}{1+x^2}=\int\limits_{-N}^{N}\log|F(x)\,\sigma(x)|\frac{dx}{1+x^2}-\int\limits_{-N}^{N}\frac{\log|\sigma(x)|\,dx}{1+x^2}$$

$$\leq\log\left\{\int\limits_{-\infty}^{\infty}\frac{|F(x)|\,\sigma(x)}{1+x^2}\,dx\right\}-\int\limits_{-N}^{N}\log\sigma(x)\frac{dx}{1+x^2}.$$

Thus $\int\limits_{-\infty}^{\infty}\frac{\log|F(x)|\,dx}{1+x^2}=-\infty$. By the theorem of Paley-Wiener quoted above, it follows that if $\hat{F}$ vanishes outside a finite interval, then $F=0$ a. e. Thus (A) fails in this case.

Lemma 5: *Given* $\sigma(x)$ *as in Lemma 4. If there exists* $\sigma_1(x) \geqq \sigma(x)$ *with* $\int\limits_{-\infty}^{\infty} \frac{\log \sigma_1(x)\, dx}{1+x^2} < \infty$ *such that* $\sigma_1(x)$ *is even and increasing on* $(0, \infty)$, *then (B) holds in* L'_σ.

Proof: Under the hypotheses, we can find a non-null function $h(x)$ with $\int\limits_{-\infty}^{\infty} |h(x)|\, \sigma_1(x)\, dx < \infty$ and $\hat{h}(\lambda)=0$, $|\lambda|>\alpha$. (Cf. Levinson, [5].) We may suppose that $\int\limits_{-\infty}^{\infty} h(x)\, dx = 1$, for else we take $k e^{ipx} h(x)$ which has the same properties as h but for suitable k, p has the integral $=1$. For $n=1, 2, \ldots$ we then set $h_n(\tau) = n h(n\tau)$. Then

$$\int\limits_{-\infty}^{\infty} |h_n(\tau)|\, \sigma(\tau)\, d\tau = \int\limits_{-\infty}^{\infty} |h(\tau)|\, \sigma\left(\frac{\tau}{n}\right) d\tau \leq \int\limits_{-\infty}^{\infty} |h(\tau)|\, \sigma_1(\tau)\, d\tau < \infty .$$

Let now F belong to L'_σ. We have

$$\|h_n \star F - F\| = \sup_\varphi \left| \int\limits_{-\infty}^{\infty} h_n(\tau)\, d\tau \int\limits_{-\infty}^{\infty} F(x-\tau)\, \varphi(x)\, dx - \int\limits_{-\infty}^{\infty} h(\tau)\, d\tau \int\limits_{-\infty}^{\infty} F(x)\, \varphi(x)\, dx \right|,$$

where φ is in the conjugate space of L'_σ and $\|\varphi\|=1$,

$$= \sup \left| \int\limits_{-\infty}^{\infty} h(\tau) \left\{ \int\limits_{-\infty}^{\infty} \left(F\left(x - \frac{\tau}{n}\right) - F(x) \right) \varphi(x)\, dx \right\} d\tau \right|$$

$$\leq \int\limits_{-\infty}^{\infty} |h(\tau)|\, \|T_{\frac{\tau}{n}} F - F\|\, d\tau .$$

Now

$$|h(\tau)| \cdot \|T_{\frac{\tau}{n}} F - F\| \leq |h(\tau)|\, \|F\| \left(\sigma\left(\frac{\tau}{n}\right) + 1 \right) \leq \|F\| \left(\sigma_1(\tau)\, h(\tau) + h(\tau) \right)$$

for all τ. Also $\lim\limits_{n=\infty} \|T_{\frac{\tau}{n}} F - F\| = 0$ for all τ, whence by the theorem of dominated convergence $\|h_n \star F - F\|$ converges to 0. Since $\hat{h}_n \cdot \hat{F}(\lambda) = 0$ when $\hat{h}_n(\lambda) = \hat{h}\left(\frac{\lambda}{n}\right) = 0$ and so when $|\lambda| \geq n\alpha$, (B) holds.

It follows from the general theory of commutative normed rings that if (A) and (B) hold in a semi-simple ring, then (C) also holds there. (See [8], Th. 38, p. 114.) Our rings are semi-simple, since f in L and $\hat{f}(\lambda)=0$ for all λ implies $f=0$. The preceding Lemmas thus yield:

Theorem 1: *Let L be in J. Then (A) holds in L provided that*

$$\int_{-\infty}^{\infty} \frac{\log \|T_\tau\|\, d\tau}{1+\tau^2} < \infty$$

and (B) and (C) hold in L if there exists $\sigma_1(\tau)$ even, increasing on $(0, \infty)$ with $\int_{-\infty}^{\infty} \frac{\log \sigma_1(\tau)\, d\tau}{1+\tau^2} < \infty$ and such that $\|T_\tau\| \le \sigma_1(\tau)$.

Lemma 5′: *Let L be in J. For any f in L and positive ε, there is some g in L with $\|f \star g - f\| < \varepsilon$.*

Proof: Set $\varrho_1(\tau) = \sup_{|x| \le |\tau|} \|T_x\|$. Then $\varrho_1(\tau)$ is non-decreasing and $\|T_\tau\| \le \varrho_1(\tau)$. Let $h(x)$ be the characteristic function of $(-\frac{1}{2}, \frac{1}{2})$ and let $h_n(x) = n\,h(nx)$. Then $\int_{-\infty}^{\infty} h(x)\,dx = 1$ and each h_n is in L. Also

$$\int_{-\infty}^{\infty} |h_n(x)|\, \varrho_1(x)\, dx \le \int_{-\infty}^{\infty} |h(x)|\, \varrho_1(x)\, dx < \infty.$$

An argument like that of Lemma 5 then yields that $\lim_{n=\infty} \|h_n \star f - f\| = 0$ which proves our assertion.

Section 2: In this Section, we study the problem of approximation of functions in $\bar{L}$ by trigonometric polynomials, the approximation being in the weak topology of $\bar{L}$ induced in $\bar{L}$ by L. If L is reflexive, e. g. if L is a space L_p^s with $s > 1$, weak and strong closure is equivalent for subspaces of $\bar{L}$, and so the approximation here will actually be in the norm of $\bar{L}$.

We shall not assume that L is a ring or even that L is invariant under translation but shall only suppose that L satisfies conditions (1) through (4).

The weakly closed subspace of $\bar{L}$ spanned by trigonometric polynomials $\sum_1^N c_r e^{i\lambda_r x}$ coincides with all of $\bar{L}$, since if f is in L and if f is orthogonal to all $e^{i\lambda x}$ in $\bar{L}$, this means that $\hat{f}(\lambda)$ vanishes for all λ and so $f=0$. Given now any $\varphi(x)$ in $\bar{L}$, there thus certainly exist closed sets Λ of real numbers such that $\varphi(x)$ is in the closure of trigonometric polynomials with frequencies in Λ. If a set Λ has this property for a certain φ, we shall say that Λ "synthesizes" φ.

We now ask: Can we assign to each φ a set Λ which consists of those and only those frequencies involved in synthesizing φ? In precise language, our question is: Given $\varphi \ne 0$. If S_φ denotes the intersection of all closed sets which synthesize φ, is S_φ non-empty?

Theorem 2: *Let $h(\sigma)$ be a function unbounded at the origin, even, decreasing on $(0, \infty)$ and with $\int_0 \log^+ h(\sigma)\, d\sigma < \infty$. If for each positive σ the norm in L of the function $e^{-\sigma|x|}$ satisfies the inequality*

$$\text{(I)} \qquad \| e^{-\sigma|x|} \| \leq k e^{h(\sigma)}$$

then for each φ in $\bar{L}$, $\varphi \neq 0$, S_φ exists as a non-empty set.

Corollary: *If (I) holds in L, then (A) and (B) hold there.*

Proof: Suppose (A) fails in L. Then there is a point p and a positive ε such that if $f \in L$ and f annihilates all $e^{i\lambda x}$ with $|\lambda - p| \geq \varepsilon$, i. e. $\int_{-\infty}^{\infty} f(x) e^{i\lambda x} dx = 0$ for these λ, then also f annihilates e^{ipx}. But that means that the set of λ with $|\lambda - p| \geq \varepsilon$ synthesizes e^{ipx}. On the other hand, the set consisting of p alone synthesizes e^{ipx}. Thus for $\varphi(x) = e^{ipx}$, S_φ is empty.

Suppose now (B) fails in L. Then there is some $\varphi \neq 0$ in $\bar{L}$ with φ annihilating all f in L with $\hat{f}$ vanishing outside some finite interval. Thus for any positive α, if f in L annihilates all $e^{i\lambda x}$ with $|\lambda| \geq \alpha$, then also f annihilates φ. Hence the set of λ with $|\lambda| \geq \alpha$ synthesizes φ. Since this holds for each a, S_φ is empty.

But by Theorem 2, S_φ is non-empty for $\varphi \neq 0$. Hence both (A) and (B) hold.

We do not consider the difficult question of finding when S_φ synthesizes φ. We shall only show the following:

Theorem 2': *Let S be any set which contains in its interior both S_φ and ∞, i.e. contains the complement of some finite interval. Then S synthesizes φ.*

In the proofs of theorems 2 and 2' we shall make use of the theory of the spectrum of a function developed by Beurling. Let $\varphi(x)$ be any function with $\int_{-\infty}^{\infty} |\varphi(x)| e^{-\sigma|x|} dx < \infty$ if $\sigma > 0$. We set:

$$\Phi^+(s) = \int_0^{\infty} \varphi(x) e^{-sx} dx, \ \sigma = Re(s) > 0; \ \Phi^-(s) = \int_0^{-\infty} \varphi(x) e^{-sx} dx, \ \sigma < 0.$$

Then Φ^+ is analytic in the right half-plane and Φ^- is analytic in the left half-plane. If the functions Φ^+ and Φ^- do not continue each other analytically over any interval of the imaginary axis, we say that φ has as its spectrum the whole real line. Otherwise, there is a function Φ analytic in both half-planes and on some subset of the imaginary axis, with $\Phi(s) = \Phi^+(s)$ if $\sigma > 0$ and $\Phi(s) = \Phi^-(s)$ if $\sigma < 0$. The set of singularities of $\Phi(s)$ is then a closed set of points it on the imaginary axis. The corresponding set of real numbers t is defined as the spectrum of φ and denoted $\sum_\varphi$. In the study of the spectrum, the following Lemma is useful:

Lemma: *Let $h(x)$ be a function unbounded at the origin, even, decreasing and with $\int_0 \log^+ h(x)\, dx < \infty$. Then for each rectangle $|x| \leq a$, $|y| \leq b$ and each $b' < b$, there exists a constant M such that if $u(x, y)$ is subharmonic and less* *than or equal to $h(x)$ in the rectangle, then $u(x, y) \leq M$ if $|x| \leq a$, $|y| \leq b'$. (Cf. Sjöberg, [3], for a proof of this Lemma.)*

Let now $\varphi(x)$ be a function such that $\int_{-\infty}^{\infty} |\varphi(x)| e^{-\sigma(x)} dx \leq k e^{h(\sigma)}$ where $h(\sigma)$ satisfies the conditions of the Lemma and k is a constant. Beurling showed in [2] that then $\sum_\varphi$ empty implies $\varphi(x) = 0$ a. e.

Lemma 6: *If for all φ in $\bar{L}$ $\int_{-\infty}^{\infty} |\varphi(x)| e^{-\sigma|x|} dx \leq k \|\varphi\| e^{h(\sigma)}$ where $h(\sigma)$ is as in the Lemma above, then for any closed set Λ of real numbers, the set C_Λ of φ with $\sum_\varphi$ included in Λ is a subspace of $\bar{L}$ closed in the weak topology of $\bar{L}$.*

Beurling has proved this result in the norm-topology for a class of function-spaces. Our proof is a modification of his.

Proof: It follows at once from the definition of spectrum that the spectrum of a linear combination of two functions is included in the union of their spectra. Hence C_Λ is a subspace. It remains to prove closure and, since L is separable, to prove sequential closure. (Cf. Banach, [6], Th. 8, p. 131). Given a sequence $\varphi_1, \varphi_2, \ldots$ in $\bar{L}$ with $\sum_{\varphi_n}$ included in Λ for all n, and with φ_n converging to φ, we must show that $\sum_\varphi$ is included in Λ.

Take any s_0 not in Λ. Then there is a circle γ around $s_0 = i t_0$ the interior of which does not meet any $\sum_\varphi$ and hence such that $\Phi_n(s)$ is regular inside γ for all n. Since φ_n converges to φ, $\|\varphi_n\|$ is bounded and $\int_{-\infty}^{\infty} \varphi_n(x) f(x) dx$ converges to $\int_{-\infty}^{\infty} \varphi(x) f(x) dx$ for each f in L. In particular, for any $s = \sigma + it$ with $\sigma > 0$, we can let f be the function which $= e^{-st}$ if $x \geq 0$, and which vanishes identically for $x < 0$. Thus:

$$\Phi_n(s) = \int_0^{\infty} \varphi_n(x) e^{-sx} dx = \int_{-\infty}^{\infty} \varphi_n(x) f(x) dx.$$

Then $\Phi_n(s)$ converges to $\Phi(s)$ for each s inside γ with $\sigma > 0$, and similarly for $\sigma < 0$. By hypothesis, we have for s inside γ:

$$|\Phi_n(s)| \leq k \|\varphi_n\| e^{h(\sigma)} \leq k' e^{h(\sigma)}$$

and hence the function $u_n(\sigma, t) = \log |\Phi_n(\sigma + i(t + t_0))|$ is subharmonic in a rectangle $|\sigma| \leq a$, $|t| \leq b$ and satisfies there the inequality

$$u_n(\sigma, t) \leq \log k' + h(\sigma).$$

By the above Lemma, then, $|\Phi_n(s)|$ is uniformly bounded for all s in some rectangle centered at s_0. Hence some subsequence $\Phi_{n_i}(s)$ convergences uniformly in each proper subregion to a function $\psi(s)$ holomorphic in the subregion. But we saw that $\Phi_n(s)$ converges to $\Phi(s)$ pointwise except on the imaginary axis.

Hence $\Phi(s) = \psi(s)$ and so $\Phi(s)$ is regular in a neighborhood of s_0. Thus $s_0 \notin \sum_\varphi$ and so $\sum_\varphi$ is included in Λ. Lemma 6 thus is proved.

Proof of Theorem 2: Since $\int_{-\infty}^{\infty} |\varphi(x)| e^{-\sigma|x|} dx \leq \| e^{-\sigma|x|} \| \cdot \|\varphi\|$, our assumption on $\| e^{-\sigma(x)}$ yields the hypothesis of Lemma 6. Let Λ be a set which synthesizes φ. If $\varphi_\lambda = e^{i\lambda x}$, $\Phi_\lambda(s) = \int_0^\infty e^{i\lambda x} e^{-sx} dx = \frac{1}{s - i\lambda}$ and so the spectrum of φ_λ consists of the point λ. Hence each φ_λ with λ in Λ belongs to the set of functions whose spectrum is included in Λ. By Lemma 6, this set is a weakly closed subspace and since it contains each $e^{i\lambda x}$ with λ in Λ, it also contains φ, since by assumption Λ synthesizes φ. Thus $\sum_\varphi$ is included in Λ. Also we saw above that $\sum_\varphi$ is not empty if $\varphi \neq 0$, under our hypothesis. Hense S_φ contains the non-empty set $\sum_\varphi$ and so is non-empty, as claimed.

Proof of Theorem 2′: Let S be a set which contains S_φ in its interior and which contains, for some a, all λ with $|\lambda| \geq a$. We assert that if for an f in L $\hat{f}(\lambda) = 0$ for all λ in S, then $\int_{-\infty}^{\infty} f(x)\varphi(x)\,dx = 0$. Let A be the set of λ where $\hat{f}(\lambda) \neq 0$. Then $\bar{A}$, the closure of A is compact and is disjoint from S_φ. Since $\hat{f}(\lambda) = 0$ if $|\lambda| \geq a$, $\frac{1}{2\pi} f(x) = \int_{-a}^{a} \hat{f}(\lambda) e^{-i\lambda x} d\lambda$ and so we have, for positive σ:

$$\int_{-\infty}^{\infty} f(x)\varphi(x) e^{-\sigma|x|} dx = \int_{-\infty}^{\infty} \varphi(x) e^{-\sigma|x|} \int_{-a}^{a} 2\pi \hat{f}(\lambda) e^{-i\lambda x} d\lambda\, dx =$$

$$2\pi \int_{-a}^{a} \hat{f}(\lambda) \int_{-\infty}^{\infty} \varphi(x) e^{-\sigma|x| - i\lambda x} dx\, d\lambda = 2\pi \int_{-a}^{a} \hat{f}(\lambda) (\Phi(\sigma + i\lambda) - \Phi(-\sigma + i\lambda))\, d\lambda$$

$$= 2\pi \int_{\bar{A}} \hat{f}(\lambda) (\Phi(\sigma + i\lambda) - \Phi(-\sigma + i\lambda))\, d\lambda.$$

Then

$$\int_{-\infty}^{\infty} f(x)\varphi(x)\,dx = \lim_{\sigma=0} \int_{-\infty}^{\infty} f(x)\varphi(x) e^{-\sigma(x)} dx = 0.$$

For

$$\lim_{\sigma=0} |\Phi(\sigma + i\lambda) - \Phi(-\sigma + i\lambda)| = 0$$

uniformly for λ in $\bar{A}$ because $\bar{A}$ is a compact set disjoint from the set of singularities of $\Phi(s)$, and also $|\hat{f}(\lambda)|$ is bounded, and finally $\hat{f}(\lambda)$ vanishes outside of the finite interval $(-a, a)$. The assertion is thus proved.

Section 3: In this Section we return to the class J of rings L satisfying (1) through (6), and shall discuss problem (D) for a subclass of J.

Def.: If I is an ideal in L, $h(I)$ is the set of zeros of I, i. e. the set of λ with $\hat{f}(\lambda)=0$ for all f in I.

Def.: If φ is in $\bar{L}$, I_φ is the closed ideal of f in L with $f\star\varphi=0$.

Theorem 3: *Let L be in J and let (A) and (B) hold in L. Then if φ is in $\bar{L}$, $S_\varphi = h(I_\varphi)$.*

Proof: Consider any open set O with $h(I_\varphi)$ included in O. Let now f be in L and $\hat{f}$ vanishes identically on O. We can find g in L with $\|f\star g-f\|<\varepsilon$, by use of Lemma 5'. Further, since by hypothesis (B) holds, we can find a g_1 in L with $\|g-g_1\|<\dfrac{\varepsilon}{\|f\|}$ and $\hat{g}_1(\lambda)=0$ for $|\lambda|\geq a$ for some a. Then

$$\|f-f\star g_1\|<2\varepsilon \text{ and } \hat{f}\hat{g}_1(\lambda)=0$$

for λ in O and also $=0$ for $|\lambda|\geq\alpha$. Then $f\star g_1$ is in I_φ, by a general theorem on normed rings in which (A) holds. (Cf. MACKEY, [8], pp. 111–12.) Thus $f\in I_\varphi$ and so $f(\lambda)$ vanishing on O implies $\int\limits_{-\infty}^{\infty} f(x)\,\varphi(x)\,dx=0$. Hence O synthesizes φ and since this holds for any open set O which includes $h(I_\varphi)$, S_φ is included in $h(I_\varphi)$.

Conversely, let Λ be any set which synthesizes φ. If now $\hat{f}(\lambda)=0$ for all λ in Λ, then $\int\limits_{-\infty}^{\infty} f(x-y)\,e^{i\lambda x}\,dx=\hat{f}(\lambda)\,e^{i\lambda y}=0$ for all y and λ in Λ, and so $\int\limits_{-\infty}^{\infty} f(x-y)\,\varphi(x)\,dx=0$ for all y, whence f is in I_φ.

For any λ_0 in $h(I_\varphi)$, if $\hat{f}(\lambda)=0$ for all λ in Λ, then $\hat{f}(\lambda_0)=0$. By (A), then, λ_0 is in Λ. Thus $h(I_\varphi)$ is included in Λ. Since Λ was any set which synthesizes φ, it follows that $h(I_\varphi)$ is included in S_φ. Thus $h(I_\varphi)=S_\varphi$, as asserted.

We next give a criterion, in terms of $\|T_\tau\|$, which assures that condition (I) is satisfied for a given L in J, and so allows us to apply to L the results of Section 2. The criterion turns out to be very close to that of Theorem 1.

Lemma 7: *Let L be in J. If there exists a function $\sigma_1(\tau)$ which is even, increasing on $(0,\infty)$ while $\dfrac{\log\sigma_1(\tau)}{\tau}$ decreases on $(0,\infty)$, and with*

$$\int_{-\infty}^{\infty}\frac{\log\sigma_1(\tau)}{1+\tau^2}\,d\tau<\infty,$$

such that $\|T_\tau\|\leq\sigma_1(\tau)$, then ($I$) holds.

Proof: Set $\varrho(\tau)=\|T_\tau\|$ and let $\chi(x)$ be the characteristic function of the interval $(0, 1)$. Then $\chi\in L$ and we have for $n=0, \pm1, \pm2, \ldots$

$$\int_n^{n+1} |\varphi(x)|\,dx = \int_0^1 |\varphi(x+n)|\,dx = \int_{-\infty}^{\infty} \chi(x)\,|\varphi(x+n)|\,dx \le \|\chi\|\cdot\|\varphi\|\cdot\varrho(n).$$

Hence

$$\int_{-\infty}^{\infty} |\varphi(x)|\,e^{-\sigma(x)}\,dx \le \sum_{-\infty}^{\infty}\int_n^{n+1} |\varphi(x)|\,e^{-\sigma|x|}\,dx \le C\,\|\varphi\|\sum_{-\infty}^{\infty} e^{-\sigma|x|}\,\varrho(n).$$

Let $u(t) = \log\sigma_1(\tau)$. We define $N = N(\sigma)$ as the first positive integer with $\frac{u(N)}{N} \le \frac{\sigma}{2}$. Since $\frac{u(t)}{t}$ decreases to 0 by hypothesis, $N(\sigma)$ is well-defined for $\sigma > 0$ and $N(\sigma)$ grows to ∞ as σ approaches 0. Let now $0 < \sigma < 1$ and set $N = N(\sigma)$. Then

$$\sum_{-\infty}^{\infty} e^{-\sigma|n|}\,\varrho(n) \le 2\sum_0^{\infty} e^{-\sigma n}\,\sigma_1(n) = 2\sum_0^{\infty} e^{u(n)-\sigma n} = 2\sum_0^{\infty} e^{n\left(\frac{u(n)}{n}-\sigma\right)}.$$

Since $u(n)$ increases with n by hypothesis, we have

$$\sum_{-\infty}^{\infty} e^{-\sigma|n|}\,\varrho(n) \le 2\left(\sum_0^{N-1} e^{u(n)} + \sum_N^{\infty} e^{-\frac{\sigma}{2}n}\right) \le 2\,N\,e^{u(N)} + 0\left(\frac{1}{\sigma}\right) \le 2\,N\,e^{N\frac{\sigma}{2}} + 0\left(\frac{1}{\sigma}\right).$$

Let now $h(\sigma) = \log\sum_{-\infty}^{\infty} e^{-\sigma|n|}\,\varrho(n)$. Then

$$h(\sigma) \le C_1 + \log N + N\cdot\frac{\sigma}{2} + \log\frac{1}{\sigma} \le C_1 + 2\,N + \log\frac{1}{\sigma}$$

where C_1 is a constant.

Therefore $\int_0 \log^+ h(\sigma)\,d\sigma < \infty$ if $\int_0 \log N(\sigma)\,d\sigma < \infty$.

Choose now for each positive σ the number $z(\sigma)$ with $\frac{u(e^{z(\sigma)})}{e^{z(\sigma)}} = \sigma$. Then $N(\sigma) \le e^{z\left(\frac{\sigma}{2}\right)} + 1$ and so

$$\log N(\sigma) < c_2 + z\left(\frac{\sigma}{2}\right)$$

where C_2 is a constant. But

$$\int_0 z(\sigma)\,d\sigma = \int^{\infty} \frac{u(e^z)}{e^z}\,dz = \int^{\infty} \frac{u(y)}{y^2}\,dy < \infty$$

by hypothesis, and so $\int_0 z\left(\frac{\sigma}{2}\right) d\sigma < \infty$. This implies that $\int_0 \log N(\sigma)\,d\sigma < \infty$ which in turn gives $\int_0 \log^+ h(\sigma)\,d\sigma < \infty$. Clearly also $h(\sigma)$ decreases on $(0, \infty)$ and so $h(\sigma)$ has the properties required in (I). Since $\|e^{-\sigma|x|}\| \le C\,e^{h(\sigma)}$, (I) thus holds, as asserted.

In particular, if, for some finite n, $\|T_\tau\| = 0(|\tau|^n)$ as $|\tau|$ approaches ∞, Lemma 7 tells us that (I) holds.

Lemma 8: *Let* $|\varphi(\tau)| = 0\,(|\tau|^n)$ *as* $|\tau|$ *approaches* ∞. *Then if* $\sum_\varphi$ *consists of the single point* 0, φ *is a polynomial of order not exceeding* n.

Proof: Beurling has shown in [2] that if a function φ has compact spectrum and if $\int_{-\infty}^{\infty} |\varphi(x)|\, e^{-\sigma|x|}\, dx \le k\, e^{h(\sigma)}$ where $h(\sigma)$ is as in Section 2, then for any contour Γ in the complex plane which surrounds the set of singularities of Φ, we have

$$\varphi(x) = \frac{1}{2\pi i} \int_\Gamma \Phi(s)\, e^{s x}\, ds.$$

In particular for the φ of this Lemma we can choose Γ as a circle around the origin of arbitrarily small radius.

Letting x take on complex values in the preceding formula, we see that φ is an entire function and that for each positive ε

$$|\varphi(z)| = 0\,(e^{\varepsilon|z|})$$

for all complex z. Thus φ has order 1 and type 0.

We now assert that any entire function $\varphi(z)$ of order 1 and type 0 with $|\varphi(x)| = 0\,(|x|^k)$ for real x is a polynomial. For $k=0$ this is a well-known result. Suppose further that the assertion holds for $k = n-1$ and consider any φ satisfying our hypotheses for $k=n$. Then $\varphi_1(z) = \dfrac{\varphi(z) - \varphi(0)}{z}$ satisfies the hypotheses for $k = n-1$ and hence by assumption φ_1 is a polynomial. Hence $\varphi(z) = \varphi(0) + z\,\varphi_1(z)$ also is a polynomial. Thus by induction the assertion is proved for all k. In particular the φ of this Lemma is a polynomial. Since $|\varphi(x)| = 0\,(|x|^k)$ for real x, φ is a polynomial of order not exceeding k. Lemma 8 is thus established.

Theorem 4: *Let* L *be in* J *and assume* $\|T_\tau\| = 0\,(|\tau|^k)$ *as* $|\tau|$ *tends to* ∞, *for some positive* k. *Let* I *be a closed ideal in* L *with precisely one zero, the point* p. *Then there exists an integer* $n \le k$ *so that* I *consists of all functions in* L *whose Fourier transforms vanish at* p *together with their first* n *derivatives.*

Proof: Without loss of generality we may suppose that $p=0$. We claim that I is invariant under translation. For else there is some f in I with $f(x+\tau_0)$ not in I for some τ_0. Then we can choose χ in $\bar{L}$ with $\chi(g) = 0$ if g is in I, $\chi(f(x+\tau_0)) \ne 0$. By continuity, this implies that $\chi(f(x+\tau)) \ne 0$ if $|\tau - \tau_0| < \varepsilon$ for some positive ε. Then we can choose h in L with

$$\int_{-\infty}^{\infty} \chi(x)\, f \star h(x)\, dx = \int_{-\infty}^{\infty} h(-\tau)\, d\tau \int_{-\infty}^{\infty} \chi(x)\, f(x+\tau)\, dx \ne 0.$$

Since $f \star h$ is in I, this is a contradiction. Hence $f(x+\tau)$ is in I for all τ.

Let I' denote the set of ψ in $\bar{L}$ with $\int_{-\infty}^{\infty} \psi(x)\, f(x)\, dx = 0$ for all f in I. Consider now any ψ in I'. Take any h in L and set $\varphi = h \star \psi$. Then

$$|\varphi(x)| \leq \int_{-\infty}^{\infty} |h(y)|\,|\psi(x+y)|\,dy \leq \|h\| \cdot \|\psi\| \cdot \|T_x\| = 0(|x|^k).$$

Suppose $\lambda_0 \in \sum_\varphi$. In the proof of Theorem 2 we saw that $\sum_\varphi$ is included in S_φ. Hence by Theorem 3, $\sum_\varphi$ is included in $h(I_\varphi)$. If f is in I, then

$$\int_{-\infty}^{\infty} f(x-\tau)\,\psi(x)\,dx = 0$$

for all τ. Thus $f \star \psi = 0$ and hence $f \star \varphi = 0$. Thus f in I implies $f \in I_\varphi$ and, since $\lambda_0 \in h(I_\varphi)$, this implies $\lambda_0 \in h(I)$. Since $h(I)$ contains by hypothesis only the point 0, we conclude that either $\sum_\varphi$ is empty or $\sum_\varphi$ consists of 0 alone. In the first case $\varphi = 0$, while in the second case Lemma 8 yields that φ is a polynomial of degree not exceeding k.

Choose now any f in L with $\int_{-\infty}^{\infty} f(x)\,x^\nu\,dx = 0$, $\nu = 1, 2, \ldots k$. Then

$$\int_{-\infty}^{\infty} f(x)\,h \star \psi(x)\,dx = 0$$

for all h in L. Choose h in L with $\|h \star f - f\| < \varepsilon$. This is possible by Lemma 5′. Then

$$\left|\int_{-\infty}^{\infty} f(x)\,\psi(x)\,dx\right| = \left|\int_{-\infty}^{\infty} f(x)\,\psi(x)\,dx - \int_{-\infty}^{\infty} f(x)\,h \star \psi(x)\,dx\right| \leq \|f - f \star h\| \cdot \|\psi\|.$$

Hence $\int_{-\infty}^{\infty} f(x)\,\psi(x)\,dx = 0$. It follows that $\psi(x)$ is a polynomial of degree not exceeding k. Then there is an $n \leq k$ such that all ψ in I' are polynomials of degree less than or equal n but at least one ψ_0 is in I' having degree n. Since I and hence I' is invariant under translation, I' contains all functions $\psi_0(x+\tau)$ and hence I' contains the functions x^ν, $\nu = 0, 1, \ldots n$. Hence I' is the set of all polynomials of degree not exceeding n.

It follows that f is in I if and only if $\int_{-\infty}^{\infty} f(x)\,x^\nu\,dx = 0$, $\nu = 0, 1, \ldots n$ and so if and only if $f^{(\nu)}(0) = 0$, $\nu = 0, 1, \ldots n$. The assertion is thus proved.

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