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A set of uniqueness for functions, analytic and bounded in the unit disc

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1. Introduction

The purpose of this note is to establish a uniqueness theorem, similar to the well-known result of F. and M. Riesz. Before we state the theorem, let us introduce some notation.

Throughout this note let $\mathfrak{J}$ be the class of all functions, analytic and bounded in the open unit disc C. We will also consider the subclass $\mathfrak{J}_0 \subset \mathfrak{J}$ of functions, with only a finite number of zeros in C.

If ζ is a point on the boundary of C (henceforth denoted by ∂C) and α is a real number, $0 \leqslant \alpha < 1$, let $S(\zeta, \alpha)$ denote the Stolz domain with vertex $\zeta \in \partial C$ and angle arcsin α; i.e.

$$S(\zeta, \alpha) = \{z \,|\, |z| < 1,\ |z-\zeta| < \sqrt{1-\alpha^2},\ |\arg(1-\bar{\zeta}z)| \leqslant \arcsin\alpha\}.$$

Moreover, if $\zeta \in \partial C$ and φ is a function, defined on C, such that

$$\lim_{\substack{z\to\zeta\\ z\in S(\zeta,\alpha)}} \varphi(z) = A \quad \text{for all} \quad \alpha,\ 0 \leqslant \alpha < 1,$$

we write

$$\lim_{z\to\zeta}{}^{S} \varphi(z) = A \quad \text{or} \quad \varphi(z) \xrightarrow{S} A \quad \text{as} \quad z \to \zeta.$$

We will use the first notation exclusively when A is a (proper) complex number, while the second notation will be used not only when A is a proper complex number but also in the case of a real-valued function φ and $A = \pm\infty$.

For $f, g \in \mathfrak{J}$ consider the set

$$D_S(f,g) = \{\zeta \,|\, \zeta \in \partial C,\ \lim_{z\to\zeta}{}^{S} f^{(k)}(z) = \lim_{z\to\zeta}{}^{S} g^{(k)}(z),\ k = 0, 1, 2, \ldots\}.$$

An immediate consequence of F. and M. Riesz's theorem ([2], p. 209) is the following result:

If $D_S(f,g)$ has positive Lebesgue measure, then $f = g$.

The main result to be proved in this note can be stated as follows:

If $D_S(f,g)$ has positive Hausdorff measure where the Hausdorff measure is determined by the function h, given by

$$h(t)=\left.\begin{array}{lll}0 & & t=0,\\ -t\log t & \text{if} & 0<t<e^{-2},\\ e^{-2}+t & & t\geqslant e^{-2},\end{array}\right\} \tag{1.1}$$

then $f=g$.

Since $D_S(f,g)\subset D_S(f-g,0)$, this statement is equivalent to the following statement:

If $f\in\mathfrak{J}$ and $f\neq 0$, then $D_S(f)=D_S(f,0)$ is of Hausdorff measure[1] *zero.*

We will prove this last statement by proving, first, that the set

$$D_S(f)=D_S(f,0)=\{\zeta\,|\,\zeta\in\partial C,\ \lim_{z\to\zeta}{}^S f^{(k)}(z)=0,\ k=0,1,2,\ldots\}$$

is equal to the set

$$L_S(f)=\{\zeta\,|\,\zeta\in\partial C,\ (\log|f(z)|)/\log|\zeta-z|\overset{S}{\to}+\infty,\quad\text{as}\quad z\to\zeta\}$$

and, secondly, that the set $L_S(f)$ is of Hausdorff measure zero. If $f\in\mathfrak{J}_0$, we will also prove that the two sets

$$D(f)=\{\zeta\,|\,\zeta\in\partial C,\ \lim_{r\to1-0}f^{(k)}(r\zeta)=0,\ k=0,1,2,\ldots\}$$

and $$L(f)=\{\zeta\,|\,\zeta\in\partial C,\ (\log|f(r\zeta)|)/\log(1-r)\to+\infty\quad\text{as}\quad r\to1-0\}$$

are both equal to the set $D_S(f)=L_S(f)$, and therefore we have:

If $f\in\mathfrak{J}_0$ the set $D(f)$ is of Hausdorff measure zero.

The proofs of the equalities, $L_S(f)=D_S(f)$ if $f\in\mathfrak{J}$, and, $L(f)=L_S(f)=D_S(f)=D(f)$ if $f\in\mathfrak{J}_0$, are carried out in Section 2, while Section 3 is devoted to proving that $L_S(f)$ is of Hausdorff measure zero. This latter proof is based on the following result:

If u is harmonic in C and

$$\int_0^{2\pi}|u(re^{ix})|\,dx=O(1)\quad as\quad r\to1-0,$$

then $$u(r\zeta)=O(-\log(1-r))\quad as\quad r\to1-0$$

for all $\zeta\in\partial C$ except possibly for a set of Hausdorff measure zero.

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[1] Throughout this note we will exclusively consider the Hausdorff measure determined by the function h, given by (1.1).

2. Three lemmas

The following lemma relates the sets $L_S(f)$, $L(f)$, $D_S(f)$ and $D(f)$, introduced in Section 1.

Lemma 2.1. *If* $f \in \mathfrak{J}$, *then*

$$L_S(f) = D_S(f) \subset D(f) \subset L(f);$$

and if $f \in \mathfrak{J}_0$, *then* $$L_S(f) = D_S(f) = D(f) = L(f).$$

The proof of this lemma is given in three steps.

(i) $L_S(f) \subset D_S(f)$

Suppose that $\zeta \in L_S(f)$ and let $S(\zeta, \alpha)$ be any Stolz domain with vertex ζ. Choose ε, such that $0 < \varepsilon < 1 - \alpha$. Then if $z \in S(\zeta, \alpha)$, the circle C_z with center z and radius $\varepsilon|\zeta - z|$ is a subset of $S(\zeta, \alpha + \varepsilon)$ for all z sufficiently close to ζ, and as z approaches ζ in $S(\zeta, \alpha)$, the points on the circle C_z approach ζ within $S(\zeta, \alpha + \varepsilon)$. Using Cauchy's formula

$$f^{(k)}(z) = k!\,(2\pi i)^{-1} \int_{C_z} (t - z)^{-k-1} f(t)\,dt,$$

it is readily seen that

$$|f^{(k)}(z)| \leqslant C(k, \varepsilon) \sup_{t \in C_z} |(t - \zeta)^{-k} f(t)|, \quad C(k, \varepsilon) = k!\,(1 + \varepsilon^{-1})^k.$$

Hence, since $\zeta \in L_S(f)$ obviously implies that

$$\lim_{\substack{t \to \zeta \\ t \in S(\zeta, \alpha + \varepsilon)}} (t - \zeta)^{-k} f(t) = 0, \quad k = 0, 1, 2, \ldots,$$

we have $$\lim_{\substack{z \to \zeta \\ z \in S(\zeta, \alpha)}} f^{(k)}(z) = 0, \quad k = 0, 1, 2, \ldots,$$

and thus $L_S(f) \subset D_S(f)$.

(ii) $D_S(f) \subset L_S(f)$, $D(f) \subset L(f)$

Suppose $\zeta \in D_S(f)$. If $z \in S(\zeta, \alpha)$, let L_z be the line segment joining z and ζ. Then for $t \in L_z$, we have

$$|f^{(k-1)}(t)| = \left| \int_{L_t} f^{(k)}(\tau)\,d\tau \right| \leqslant |t - \zeta| \sup_{\tau \in L_t} |f^{(k)}(\tau)| \leqslant |z - \zeta| \sup_{\tau \in L_z} |f^{(k)}(\tau)|$$

and therefore

$$\sup_{t \in L_z} |f^{(k-1)}(t)| \leqslant |z - \zeta| \sup_{t \in L_z} |f^{(k)}(t)|, \quad k = 1, 2, 3, \ldots.$$

Repeated use of these inequalities yields

$$|f(z)| \leqslant |z-\zeta|^n \sup_{t \in L_z} |f^{(n)}(t)| \leqslant |z-\zeta|^n \sup_{t \in S_z(\zeta,\alpha)} |f^{(n)}(t)|, \quad n=0,1,2,\ldots, \tag{2.1}$$

where $$S_z(\zeta,\alpha) = S(\zeta,\alpha) \cap \{t \,|\, |t-\zeta| \leqslant |z-\zeta|\}.$$

It follows from (2.1) that

$$(\log|f(z)|)/\log|\zeta - z| \to +\infty \quad \text{as} \quad z \to \zeta,\ z \in S(\zeta,\alpha)$$

and therefore, since α, $0 \leqslant \alpha < 1$, is arbitrarily chosen, we have $D_S(f) \subset L_S(f)$. Also, by the first part of (2.1), with $z = r\zeta$, we have $D(f) \subset L(f)$.

(iii) $L(f) \subset L_S(f)$ if $f \in \mathfrak{J}_0$

In the proof of this step, we will use the following lemma, which follows easily from Harnack's inequalities (cf. [3], p. 295).

Lemma 2.2. *Let u be a nonnegative function harmonic in the open unit disc and let C_z be a circle with center z, $|z|<1$ and radius $\alpha(1-|z|)$, $0<\alpha<1$. Then*

$$\frac{1-\alpha}{1+\alpha} u(z) \leqslant u(t) \leqslant \frac{1+\alpha}{1-\alpha} u(z)$$

for every $t \in C_z$.

We will also use the mappings (cf. [3], p. 295)

$$T_{\zeta,\alpha} : S(\zeta,\alpha) \to \{r\zeta \,|\, 0<r<1\},\ \zeta \in \partial C,\ 0<\alpha<1$$

defined in the following way: if $z \in S(\zeta,\alpha)$ let $T_{\zeta,\alpha}z$ be the point closest to ζ, such that

$$\arg T_{\zeta,\alpha}z = \arg\zeta \quad \text{and} \quad |z - T_{\zeta,\alpha}z| = \alpha(1-|T_{\zeta,\alpha}z|).$$

Obviously $$(1-\alpha)(1-|T_{\zeta,\alpha}z|) \leqslant |\zeta - z| \leqslant (1+\alpha)(1-|T_{\zeta,\alpha}z|) \tag{2.2}$$

and therefore $z \to \zeta$, $z \in S(\zeta,\alpha)$ if and only if $T_{\zeta,\alpha}z \to \zeta$.

Now let $f \in \mathfrak{J}_0$. Then $f = \|f\| \cdot B \cdot E$, where $\|f\|$ denotes the supremum norm, B is the normalized, finite Blaschke product of f and E is analytic and zero-free in C. Moreover, $\|E\| \leqslant 1$. Obviously $L_S(f) = L_S(E)$ and $L(f) = L(E)$. Thus it suffices to prove (iii) when $f = E$. Suppose that $\zeta \in L(E)$. Then by Lemma 2.2, with $u = -\log|E|$ and by (2.2)

$$\frac{\log|E(z)|}{\log|z-\zeta|} \geqslant \frac{1}{2} \cdot \frac{1-\alpha}{1+\alpha} \cdot \frac{\log|E(T_{\zeta,\alpha}z)|}{\log(1-|T_{\zeta,\alpha}z|)}$$

for all $z \in S(\zeta,\alpha)$ such that $|T_{\zeta,\alpha}z| \geqslant \alpha$, and thus $L(E) \subset L_S(E)$. These three steps, together with the obvious inclusion $D_S(f) \subset D(f)$, prove Lemma 2.1.

Let $f = \|f\| \cdot B \cdot E$ be the decomposition of a function in $\mathfrak{J}$. Then if $B(f)$ is the set of $\zeta \in \partial C$ with the property: there exists a $\delta > 0$, such that

$$|B(z)| \leqslant 2^{-\frac{1}{2}} |z-\zeta| \quad \text{for all} \quad z \in S(\zeta, 2^{-\frac{1}{2}}) \quad \text{with} \quad |z-\zeta| < \delta,$$

and if $\bar{L}(f)$ is the set defined by

$$\bar{L}(f) = \left\{ \zeta \,|\, \zeta \in \partial C, \overline{\lim_{r \to 1-0}} \frac{\log |E(r\zeta)|}{\log (1-r)} = +\infty \right\}$$

we have the following result.

Lemma 2.3. *If $f \in \mathfrak{J}$ and $f \neq 0$, then*

$$L_S(f) \subset B(f) \cup \bar{L}(f).$$

To prove this, suppose that $\zeta \in L_S(f)$ but $\zeta \notin B(f)$. Then there is a sequence $\{z_v\}_1^\infty$, $z_v \in S(\zeta, 2^{-\frac{1}{2}})$, $z_v \to \zeta$ as $v \to +\infty$, such that

$$0 \leqslant \lim_{v \to \infty} \frac{\log |B(z_v)|}{\log |z_v - \zeta|} \leqslant 1,$$

and therefore $\qquad (\log |E(z_v)|)/\log |z_v - \zeta| \to \infty \quad \text{as} \quad v \to +\infty.$

However, by Lemma 2.2 with $u(z) = -\log |E(z)|$ and $\alpha = 2^{-\frac{1}{2}}$ and by (2.2), we have

$$\frac{\log |E(T_{\zeta,\alpha} z_v)|}{\log (1 - |T_{\zeta,\alpha} z_v|)} \geqslant \frac{1}{2} \cdot \frac{1-\alpha}{1+\alpha} \cdot \frac{\log |E(z_v)|}{\log |z_v - \zeta|}$$

if $|z_v - \zeta| \leqslant (1+\alpha)^{-1}$. Hence

$$\overline{\lim_{r \to 1-0}} \frac{\log |E(r\zeta)|}{\log (1-r)} = +\infty$$

and $\zeta \in \bar{L}(f)$. This proves Lemma 2.3.

3. A uniqueness theorem

In this section we prove that for all f in $\mathfrak{J}$, such that $f \neq 0$ the set $D_S(f)$ is of Hausdorff measure zero. By Lemma 2.1 and Lemma 2.3 it suffices to prove that the two sets $B(f)$ and $\bar{L}(f)$ are of Hausdorff measure zero. The fact that $\bar{L}(f)$ is of Hausdorff measure zero is an immediate consequence of the following theorem.

Theorem 3.1. *If u is harmonic in the open unit disc C and*

$$\int_0^{2\pi} |u(re^{ix})| \, dx = O(1) \quad as \quad r \to 1-0,$$

then $\qquad u(r\zeta) = O(-\log(1-r)) \quad as \quad r \to 1-0$

for all $\zeta \in \partial C$, except possibly for a set of Hausdorff measure zero.

Under the hypotheses of the theorem there is a function μ of bounded variation on $[0, 2\pi]$, such that ([2], p. 198)

$$u(re^{ix}) = \int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(x-t)}\, d\mu(t), \quad 0 \leqslant r < 1.$$

Since
$$|u(re^{ix})| \leqslant \int_0^{3\pi} \frac{1-r^2}{1+r^2-2r\cos(x-t)}\, d|\mu|(t), \quad 0 \leqslant r < 1,$$

where $|\mu|(t)$ is the total variation of μ on $[0, t]$, it suffices to prove the theorem for a nonnegative harmonic function u, i.e. the corresponding function μ is nondecreasing.

In [3], p. 290, we proved the inequality

$$\overline{\lim_{r\to 1-0}} \frac{u(re^{ix})}{-\log(1-r)} \leqslant \pi \overline{\lim_{t\to+0}} \frac{\mu(x+t)-\mu(x-t)}{h(t)}, \quad 0 < x < 2\pi,$$

where h is the function given by (1.1), and therefore it suffices to prove that the set

$$M = \left\{ e^{ix} \,\middle|\, 0 < x < 2\pi,\ \overline{\lim_{t\to+0}} \frac{\mu(x+t)-\mu(x-t)}{h(t)} = +\infty \right\}$$

is of Hausdorff measure zero. We will prove this using a covering principle due to Besicovitch ([1]).

Definition (Besicovitch): If G is a set in the plane and Γ is a class of discs, such that to each point z in G there correspond discs in Γ, with center z and arbitrarily small radii, then we call Γ a covering of G in the Vitali narrow sence.

Theorem (Besicovitch): Let G be a bounded set of the plane and Γ a covering of G in the Vitali narrow sense. Then there is a subcovering $\bar{\Gamma}$ of G, where $\bar{\Gamma}$ can be split into 22 countable subclasses Γ_k $(k = 1, 2, \dots, 22)$, such that no pair of discs in the same subclass meet.

Let ε and ϱ be two positive numbers and consider for each $e^{ix} \in M$ those open discs $C(e^{ix}, t)$, with center e^{ix} and radius $t \leqslant \varrho$, such that

$$h(t) \leqslant \frac{\varepsilon}{22} \cdot \frac{\mu(x+t)-\mu(x-t)}{\mu(2\pi)-\mu(0)} \quad \text{and} \quad (x-t, x+t) \subset (0, 2\pi).$$

This class of discs in then a covering of M in the Vitali narrow sense, and by Besicovitch's covering principle, there is a subcovering $\Gamma = \bigcup_1^{22} \Gamma_k$, such that no pair of discs in the same subclass Γ_k meet. Then, if

$$\Gamma_k = \bigcup_v C(e^{ix_{k,v}}, t_{k,v}), \quad k = 1, 2, \dots, 22,$$

the corresponding intervals $(x_{k,v} - t_{k,v}, x_{k,v} + t_{k,v})$ are disjoint and

$$\sum_v h(t_{k,v}) \leqslant \frac{\varepsilon}{22} \cdot \frac{\sum_v (\mu(x_{k,v}+t_{k,v}) - \mu(x_{k,v}-t_{k,v}))}{\mu(2\pi)-\mu(0)} \leqslant \frac{\varepsilon}{22}.$$

Thus
$$\sum_{k=1}^{22} \sum_v h(t_{k,v}) \leqslant \varepsilon$$

and, since ε is arbitrarily chosen, we have proved that M is of Hausdorff measure zero. This completes the proof of Theorem 3.1.

Corollary 3.1. *If $f \in \mathfrak{J}$ and $f \neq 0$, then $\bar{L}(f)$ is of Hausdorff measure zero.*

Proof. Apply Theorem 3.1 to the nonnegative, harmonic function

$$u(z) = -\log |E(z)|.$$

For a function f in $\mathfrak{J}_0$ the set $B(f)$ is empty. Therefore, by Corollary 3.1, Lemma 2.3 and Lemma 2.1 we have the following theorem.

Theorem 3.2′. *If $f \in \mathfrak{J}_0$, then $D(f)$ is of Hausdorff meazure zero.*

The corresponding result for a function $f \in \mathfrak{J}$ is given in the following theorem.

Theorem 3.2. *If $f \in \mathfrak{J}$ and $f \neq 0$, then $D_S(f)$ is of Hausdorff measure zero.*

The proof of this theorem follows immediately from Corollary 3.1, Lemma 2.3, Lemma 2.1 and the following lemma.

Lemma 3.1. *If $f \in \mathfrak{J}$ and $f \neq 0$, then $B(f)$ is of Hausdorff measure zero.*

Proof. Let $\{r_n\}_1^\infty$ be a sequence of real numbers, such that $1/\sqrt{2} < r_n < 1$ and $\lim_{n\to\infty} r_n = 1$. Then $B(f) = \bigcup_1^\infty B_n$, where

$$B_n = \left\{ e^{ix} \left| \frac{|B(z)|}{|z-e^{ix}|} < \frac{1}{\sqrt{2}} \quad \text{for all} \quad z \in S\left(e^{ix}, \frac{1}{\sqrt{2}}\right) \quad \text{with} \quad |z| \geqslant r_n \right. \right\}.$$

Obviously it suffices to prove that B_n is of Hausdorff measure zero for $n = 1, 2, 3, \ldots$. Choose any ϱ such that $1 > \varrho > \max\{r_n, 1-e^{-2}\}$. Then, since $|e^{ix} - \varrho e^{it}| \leqslant \sqrt{2}(1-\varrho)$ for $|x-t| \leqslant 1-\varrho$, we have for $e^{ix} \in B_n$

$$h(1-\varrho) = -(1-\varrho)\log(1-\varrho) \leqslant -\tfrac{1}{2}\int_{x-(1-\varrho)}^{x+(1-\varrho)} \log \frac{|e^{ix}-\varrho e^{it}|}{\sqrt{2}}\, dt$$

$$\leqslant -\tfrac{1}{2}\int_{x-(1-\varrho)}^{x+(1-\varrho)} \log |B(\varrho e^{it})|\, dt.$$

Cover each point $e^{ix} \in B_n$ by an open disc with center e^{ix} and radius $1-\varrho$. From this cover we can extract a finite subcovering, such that each point in B_n is covered by at most two discs. Therefore, if $N(\varrho)$ is the number of discs in this subcovering,

$$N(\varrho)\cdot h(1-\varrho) \leqslant -\int_0^{2\pi} \log|B(\varrho e^{it})|\,dt$$

and, since the limit of this integral is zero as ϱ approaches 1 ([2], p. 207), we conclude that B_n is of Hausdorff measure zero. This completes the proof.

Combining Lemma 3.1 with the inequality (2.1.) we have:

Theorem 3.3. *If B is a Blaschke product, then the set*

$$\{\zeta \mid \zeta \in \partial C, \lim_{z\to\zeta}{}^S B(z) = \lim_{z\to\zeta}{}^S B'(z) = 0\}$$

is of Hausdorff measure zero.

Proof. By (2.1) the set in Theorem 3.3 is a subset of $B(B)$.

We are now able to prove the uniqueness theorem.

Theorem 3.4. *If $f, g \in \mathfrak{J}$ and*

$$\lim_{z\to\zeta}{}^S f^{(k)}(z) = \lim_{z\to\zeta}{}^S g^{(k)}(z), \quad k=0,1,2,\ldots,$$

for a set of points $\zeta \in \partial C$ of positive Hausdorff measure, then $f=g$.

Proof. Suppose that $h=f-g\neq 0$. Then $D_S(h)$ is of Hausdorff measure zero (Theorem 3.2), violating the assumption of Theorem 3.4. Therefore, $h=f-g=0$.

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