A note on recurring series

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We wish to prove the following theorem, which extends to any field of characteristic 0 a result, proved by SKOLEM [4] in the field of rational numbers and by MAHLER [2] in the field of algebraic numbers.

Theorem. *In a field of characteristic 0, let a sequence*

$$c_\nu \qquad\qquad \nu = 0,\ 1,\ 2,\ \dots$$

satisfy a recursion formula of the type

$$c_\nu = \alpha_1\, c_{\nu-1} + \alpha_2\, c_{\nu-2} + \cdots + \alpha_n\, c_{\nu-n} \qquad \nu = n,\, n+1,\, n+2,\, \dots.$$

If $c_\nu = 0$ for infinitely many values of ν, then those c_ν that are equal to zero occur periodically in the sequence from a certain index on.

It will be shown by an example that the restriction for the characteristic is essential (section 6).

From the theorem can be deduced a characterization of those sequences $\{c_\nu\}$ that contain 0 (or: any number) an infinity of times (see MAHLER [2]). In particular, only a finite number of the c_ν can be equal to zero if the quotient of two different roots of the equation $1 = \alpha_1\, t + \alpha_2\, t^2 + \cdots + \alpha_n\, t^n$ is never a root of unity.

SKOLEM and MAHLER used for their proofs a p-adic method, due to SKOLEM [3]. Our proof will closely follow that of MAHLER, and is partly built on it.

1. A sequence $\{c_\nu\}$ in any field may be considered as the TAYLOR-coefficients of a rational function if it satisfies a linear recursion formula as above. By resolving this rational function into partial fractions it is possible to get an explicit expression for the c_ν. In a field of characteristic 0 we get

$$c_x = \sum_{j=1}^{m} A_j^x\, P_j\,(x) \qquad\qquad x = 0,\ 1,\ 2,\ \dots,$$

where the $P_j\,(x)$ are polynomials whose coefficients, together with the A_j, are algebraic over the field that is generated by the c_ν. Therefore, to prove the theorem, it is sufficient to prove the following lemma.

Lemma. *Let the function $F\,(x)$ be defined by*

$$F\,(x) = \sum_{j=1}^{m} A_j^x\, P_j\,(x),$$

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where the $P_j(x)$ are polynomials whose coefficients, together with the A_j, belong to a field K of characteristic 0. If $F(x) = 0$ for an infinity of rational integral values of x, then there is a natural number r and different numbers $r^{(1)}, r^{(2)}, \ldots, r^{(\varrho)}$ of the set $0, 1, \ldots, r-1$, so that all the values

$$r^{(1)} + rz, \quad r^{(2)} + rz, \quad \ldots, \quad r^{(\varrho)} + rz \qquad\qquad z = 0, \pm 1, \pm 2, \ldots$$

and, in addition, at most a finite number of rational integral values of x make $F(x)$ vanish.

Without loss of generality we can suppose that K is just the field generated by the A_j and the coefficients of the $P_j(x)$. Denote by P the field of rational numbers.

2. First assume that K is algebraic over P. This is the case treated by MAHLER. We give a sketch of his proof. Choose a prime ideal $\mathfrak{p}$ of K, dividing neither numerator nor denominator of any A_j. Denote by $|\alpha|_{\mathfrak{p}}$ ($\alpha \in K$) the corresponding valuation of K, normed so that $|p|_{\mathfrak{p}} = \dfrac{1}{p}$, where p is the natural prime divided by $\mathfrak{p}$. (In the sequel this special norming of a valuation will be tacitly assumed.) Applying a generalization of FERMAT's theorem we can find a natural number r, so that

$$(1) \qquad\qquad |A_j^r - 1|_{\mathfrak{p}} < p^{-\frac{1}{p-1}} \qquad\qquad j = 1, 2, \ldots, m.$$

As a consequence of these inequalities, the functions

$$A_j^{rx} \qquad\qquad j = 1, 2, \ldots, m,$$

where x is restricted to rational integral values, can be represented by $\mathfrak{p}$-adic power series of x. Instead of $F(x)$ consider now the r functions

$$F_\sigma(z) = F(\sigma + rz) = \sum_{j=1}^{m} A_j^\sigma A_j^{rz} P_j(\sigma + rz) \qquad\qquad \sigma = 0, 1, \ldots, r-1.$$

For rational integral values of z these functions can all be represented by $\mathfrak{p}$-adic power series. If any of them has an infinity of rational integral zeros, it vanishes identically. In fact, the zeros then have a $\mathfrak{p}$-adic point of accumulation in the region of convergence of the power series, and, as in the case of a power series of a complex variable, this is impossible unless the series vanishes identically. From the connection between $F(x)$ and the r functions $F_\sigma(z)$ we get the desired result about the rational integral zeros of $F(x)$.

3. Now assume that K is not algebraic over its prime field P. Observe that if we were able to find a valuation of K, continuating a p-adic valuation of P, so that a precise analogue of the inequalities (1) could be proved, then the reasoning of the preceding section could be carried through. Actually we shall construct such a valuation of K. Denoting it by $\varphi(\alpha)$ ($\alpha \in K$) and meaning by p the corresponding natural prime $\left(\varphi(p) = \dfrac{1}{p}\right)$, we have to prove that for some natural number r

(2) $$\varphi(A_j^r - 1) < p^{-\frac{1}{p-1}} \qquad\qquad j = 1, 2, \ldots, m.$$

4. Since K is generated by a finite number of elements, it must be of finite transcendence-degree k over P. It may therefore be considered as an algebraic extension of a field $P(\varkappa_1, \ldots, \varkappa_k)$, where $\varkappa_1, \ldots, \varkappa_k$ are algebraically independent over P. This algebraic extension must be finite and hence simple, as the characteristic is 0. Thus every element of K may be written in the form

(3) $$\frac{1}{Q(\varkappa_1, \ldots, \varkappa_k)}\left(P_0(\varkappa_1, \ldots, \varkappa_k) + P_1(\varkappa_1, \ldots, \varkappa_k)\lambda + \cdots + P_{l-1}(\varkappa_1, \ldots, \varkappa_k)\lambda^{l-1}\right),$$

where λ satisfies an equation, irreducible in $P(\varkappa_1, \ldots, \varkappa_k)$,

$$\varphi(\varkappa_1, \ldots, \varkappa_k; \lambda) \equiv \lambda^l + C_1(\varkappa_1, \ldots, \varkappa_k)\lambda^{l-1} + \cdots + C_l(\varkappa_1, \ldots, \varkappa_k) = 0,$$

$Q(\varkappa_1, \ldots, \varkappa_k)$, the $P_\nu(\varkappa_1, \ldots, \varkappa_k)$ and the $C_\nu(\varkappa_1, \ldots, \varkappa_k)$ being polynomials with rational integral coefficients.

Let $A_j(\varkappa_1, \ldots, \varkappa_k; \lambda)$ $(j = 1, 2, \ldots, m)$ be expressions of the form (3) for the numbers A_j and denote by $D(\varkappa_1, \ldots, \varkappa_k)$ the discriminant of $\varphi(\varkappa_1, \ldots, \varkappa_k; \lambda)$, regarded as a polynomial in λ.

After these preliminaries we proceed to the construction of the valuation.

5. Choose k rational integers $a_1, \ldots, a_k$, so that when these numbers are substituted for $\varkappa_1, \ldots, \varkappa_k$, then $D(\varkappa_1, \ldots, \varkappa_k)$ and the denominators of the $A_j(\varkappa_1, \ldots, \varkappa_k; \lambda)$ do not vanish. Fix in $P[\lambda]$ one irreducible factor of the polynomial $\varphi(a_1, \ldots, a_k; \lambda)$ and let ϑ be a root of this factor, thus $\varphi(a_1, \ldots, a_k; \vartheta) = 0$. According to section 2 there are a valuation of the field $P(\vartheta)$, corresponding to some prime ideal $\mathfrak{p}$, and a natural number r, such that

(4) $$|(A_j(a_1, \ldots, a_k; \vartheta))^r - 1|_\mathfrak{p} < p^{-\frac{1}{p-1}} \qquad\qquad j = 1, 2, \ldots, m,$$

p denoting the natural prime divided by $\mathfrak{p}$ $\left(|p|_\mathfrak{p} = \dfrac{1}{p}\right)$.

We denote by $\overline{P(\vartheta)}$ the perfect $\mathfrak{p}$-adic extension of $P(\vartheta)$. — Let $\varepsilon_1, \ldots, \varepsilon_k$ be elements of $\overline{P(\vartheta)}$ which are algebraically independent over P[1]. We have

$$\lim_{n=\infty} |(a_\nu + p^n \varepsilon_\nu) - a_\nu|_\mathfrak{p} = 0 \qquad\qquad \nu = 1, 2, \ldots, k,$$

which means that in the sense of the valuation of $\overline{P(\vartheta)}$ the numbers $a_1 + p^n \varepsilon_1, \ldots, a_k + p^n \varepsilon_k$ tend to $a_1, \ldots, a_k$ respectively when n tends to infinity. We assert that for large values of n there is also in $\overline{P(\vartheta)}$ a root $\lambda^{(n)}$

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[1]) The existence of such elements $\varepsilon_1, \ldots, \varepsilon_k$ follows from the fact that $\overline{P(\vartheta)}$ has the cardinality of the continuum. To obtain an effective construction, let $p_\nu(x_1, \ldots, x_k)$ $(\nu = 1, 2, 3, \ldots)$ be an enumeration of all polynomials in k variables with rational integral coefficients. Write $\varepsilon_1, \ldots, \varepsilon_k$ as infinite sums of rational numbers. Define these sums by steps, every step consisting in the addition of one term to each sum, and the νth step having no influence on $|p_\mu(\varepsilon_1, \ldots, \varepsilon_k)|_\mathfrak{p}$ $(\mu < \nu)$ but assuring that $p_\nu(\varepsilon_1, \ldots, \varepsilon_k) \neq 0$. This construction can be precised by some univocal law.

of the equation $\varphi(a_1 + p^n \varepsilon_1, \ldots, a_k + p^n \varepsilon_k; x) = 0$, such that $\lambda^{(n)}$ tends to ϑ when n tends to infinity. For it is clear that the polynomial congruence

$$\varphi(a_1 + p^n \varepsilon_1, \ldots, a_k + p^n \varepsilon_k; x) \equiv \varphi(a_1, \ldots, a_k; x) \pmod{\mathfrak{p}^n}$$

is satisfied. Furthermore, for large values of n the discriminant of $\varphi(a_1 + p^n \varepsilon_1, \ldots, a_k + p^n \varepsilon_k; x)$ will have the same valuation as the number $D(a_1, \ldots, a_k)$, which is different from zero. According to HENSEL ([1], pp. 68–70, 153–156) we can therefore deduce, since there is a factorization in $P(\vartheta)$

$$\varphi(a_1, \ldots, a_k; x) = (x - \vartheta) \psi(x),$$

that, for large n, there is a factorization in $\overline{P(\vartheta)}$

$$\varphi(a_1 + p^n \varepsilon_1, \ldots, a_k + p^n \varepsilon_k; x) = (x - \lambda^{(n)}) \psi_n(x)$$

with

$$\lim_{n=\infty} |\lambda^{(n)} - \vartheta|_{\mathfrak{p}} = 0.$$

As the rational functions in a valued field are continuous in the sense of the valuation, we can find an n for which

$$(5) \qquad \left| (A_j(a_1 + p^n \varepsilon_1, \ldots, a_k + p^n \varepsilon_k; \lambda^{(n)}))^r - (A_j(a_1, \ldots, a_k; \vartheta))^r \right|_{\mathfrak{p}} < p^{-\frac{1}{p-1}}$$

$$j = 1, 2, \ldots, m.$$

Fixing this value of n, we have, by (4) and (5),

$$(6) \qquad \left| (A_j(a_1 + p^n \varepsilon_1, \ldots, a_k + p^n \varepsilon_k; \lambda^{(n)}))^r - 1 \right|_{\mathfrak{p}} < p^{-\frac{1}{p-1}} \qquad j = 1, 2, \ldots, m.$$

The valuation of $K = P(\varkappa_1, \ldots, \varkappa_k; \lambda)$ is now obtained by an embedding of this field in the valued field $\overline{P(\vartheta)}$ in conformity to the formulas

$$\varkappa_\nu \to a_\nu + p^n \varepsilon_\nu$$

$$\lambda \to \lambda^{(n)}.$$

It is indeed evident from our choice of the ε_ν and $\lambda^{(n)}$ that the field $P(a_1 + p^n \varepsilon_1, \ldots, a_k + p^n \varepsilon_k; \lambda^{(n)})$ $(< \overline{P(\vartheta)})$ is isomorphic to K. The validity of the inequalities (2) is expressed by (6). Our proof is thus completed.

6. There are fields of prime characteristic in which the statement of the theorem (and of the lemma) does not hold. We shall show this by an example. Let e be the unity element of the prime field with p elements and let $\varkappa$ be an indeterminate over the same field. The sequence

$$c_\nu = (e + \varkappa)^\nu - e - \varkappa^\nu \qquad\qquad \nu = 0, 1, 2, \ldots$$

satisfies the recursion formula

$$c_\nu = (2e + 2\varkappa) c_{\nu-1} - (e + 3\varkappa + \varkappa^2) c_{\nu-2} + (\varkappa + \varkappa^2) c_{\nu-3}.$$

It is seen that $c_\nu = 0$ if and only if ν is an integral power of p.

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