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A note on the constant of Koebe

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Let S be the class of analytic functions $w(z) = a_1 z + a_2 z^2 + \cdots$ that are schlicht in the unit circle γ: $|z| < 1$. The function $w(z)$ maps γ on an open and simply connected domain D_w. We define

$$d_w = \frac{1}{|a_1|} \operatorname*{Inf}_{w \notin D_w} |w|, \qquad M_w = \frac{1}{|a_1|} \operatorname*{Sup}_{w \in D_w} |w|.$$

It is wellknown that $d_w \geq \frac{1}{4}$ (Koebe's constant), this limit being the best possible for $M_w \leq \infty$. Here we shall determine a stronger limit that depends on M_w.

Theorem. *Let* $w(z) \in S$. *If* $M_w \leq M$

$$d_w \geq 2 M^2 \left[1 - \frac{1}{2M} - \sqrt{1 - \frac{1}{M}}\right]. \tag{1}$$

It is allowed to put $w'(0) = a_1 = 1$. Let $w_0(z) = \alpha_1 z + \alpha_2 z^2 + \cdots$ be a function in S that maps γ on the circle $|w| < M$, slit along the segment (d_w, M) of the real positive axis. The inverse functions of $w(z)$ and $w_0(z)$ are $z(w)$ and $z_0(w)$: $z'(0) = 1$, $z_0'(0) = \alpha_1^{-1}$. The harmonic functions

$$\psi(w) = \log \left| \frac{w}{M z(w)} \right| \quad \text{and} \quad \psi_0(w) = \log \left| \frac{w}{M z_0(w)} \right|$$

are regular and ≤ 0 in D_w and D_{w_0} respectively. Any circle $|w| = r$, $d_w \leq r \leq M$ contains at least one point $w \notin D_w$. Further, if w approaches a point w' on the boundary of D_w we get

$$\overline{\lim}\, \psi(w) \leq \log \frac{|w'|}{M} = \psi_0(|w'|)$$

and $\psi_0(w)$ has non-negative derivatives along the inner normals of the segment (d_w, M). Then all conditions are satisfied for applying a lemma of BEURLING (1) that solves the problem. From this lemma we get $\psi_0(0) \geq \psi(0)$

and hence follows $|\alpha_1| \geq 1$. The function $w_0(z)$ is calculated by elementary methods. We obtain

$$|\alpha_1| = 4\, d_w \left[1 + \frac{d_w}{M}\right]^{-2} \geq 1.$$

Now it is easy to write this inequality in the form (1) and the lemma is proved. The lower limit in (1) is attained by $w = w_0(z)$ and is therefore the best possible.

REFERENCE

(1) Beurling, A., Etudes sur une problème de majoration, Thèse, Uppsala 1933, p. 44.

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Uppsala 1953. Almqvist & Wiksells Boktryckeri AB