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On the partial differential equation $u_x^2 u_{xx} + 2 u_x u_y u_{xy} + u_y^2 u_{yy} = 0$

By Gunnar Aronsson

1. Introduction

This paper treats various aspects of the partial differential equation

$$\left(\frac{\partial u}{\partial x}\right)^2 \frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y} + \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} = 0. \tag{1}$$

This equation was derived in [1] where an extension problem was studied, and it turned out that (1) is closely connected to this extension problem (Theorems 6, 7 and 8 in [1]). The equation is quasi-linear and parabolic ($AC - B^2 = 0$), and is not of any classical type. The results from [1] will be used very little in this paper. As far as the author knows, the equation (1) has not been treated before, apart from the paper [1].

Let $u(x, y)$ be a solution of (1) and let C be a trajectory of the vector field grad u. Then it is proved in Section 2 that C is either a convex curve or a straight line, and this result, together with a formula for the curvature of C, is fundamental for the later sections.

In Section 3 we consider two particular classes of solutions to (1).

Section 4 is devoted to a discussion of the regularity of solutions to (1). It turns out that a solution for which the trajectories of grad u are convex curves, is infinitely differentiable.

In Section 5 we consider some differential-geometric aspects of (1).

Section 6 contains an estimate for $|\text{grad}\, u|$. A consequence of this estimate is that a nonconstant solution of (1) has no stationary points.

In Section 7 we consider solutions of (1) outside a compact set and solutions in the whole plane. The latter ones turn out to be linear functions only.

In Section 8 we consider the behaviour of $|\text{grad}\, u|$ near the boundary of a region.

Section 9, finally, contains a few results on the Dirichlet problem for (1).

In this paper, we will only consider classical solutions of (1), that is, solutions in C^2. We will not discuss extensions of the results to the case of more than two independent variables.

2. Some preliminary considerations

A lemma on the curvature of a streamline

We introduce the notation

$$A(\Phi) \equiv \Phi_x^2 \Phi_{xx} + 2 \Phi_x \Phi_y \Phi_{xy} + \Phi_y^2 \Phi_{yy}.$$

It is easy to see that

$$A(\Phi) = \tfrac{1}{2} \operatorname{grad} \{(\operatorname{grad} \Phi)^2\} \cdot \operatorname{grad} \Phi. \tag{2}$$

The meaning of $A(\Phi)=0$ is therefore that $|\text{grad } \Phi|$ is constant along every trajectory of the vector field grad Φ. Such trajectories will be called *streamlines* in the sequel. Hence Φ is a linear function of the arc length along each streamline. It also follows from (2) that $A(\Phi)$ is an orthogonal invariant. Clearly, any function Φ for which $|\text{grad } \Phi|$ is constant, satisfies $A(\Phi)=0$. This particular class of solutions will be discussed in Section 3. These functions correspond to unique solutions of the extension problem in [1] (Theorems 3–5).

Consider a function $\Phi(x, y) \in C^2$ in a neighbourhood of a point P where grad $\Phi \neq 0$. Introduce curvilinear coordinates $u=\Phi$ and $v=$ a function which is constant along each streamline of Φ. Assume also that $v \in C^2$ and grad $v \neq 0$. This gives (locally) a one-to-one mapping $(x, y) \leftrightarrow (u, v)$ for which $J=d(u, v)/d(x, y) \neq 0$ and

$$u_x v_x + u_y v_y = 0. \tag{*}$$

Now we have the formulas $x_u = J_1 v_y$, $x_v = -J_1 u_y$, $y_u = -J_1 v_x$ and $y_v = J_1 u_x$, where $J_1 = 1/J$. The relation (*) can thus be written

$$x_u x_v + y_u y_v = 0.$$

Clearly, a streamline of u in the xy-plane is given by $v=$constant, and arc length along such a curve is given by

$$\int_{u_p}^{U} \sqrt{x_u^2 + y_u^2}\, du.$$

Now the condition $A(\Phi)=0$ means that this is a linear function of U which is equivalent to $\partial/\partial u(x_u^2 + y_u^2)=0$, or $x_u x_{uu} + y_u y_{uu} = 0$.

A careful analysis of the preceding reasoning leads to the result:

Let there be given two functions $x(u, v)$, $y(u, v)$ in C^2 for which $x_u^2 + y_u^2 > 0$, $x_v^2 + y_v^2 > 0$ and which satisfy the system

$$x_u x_v + y_u y_v = 0,$$

$$x_u x_{uu} + y_u y_{uu} = 0.$$

Then (each function element of) the inverse function $u=u(x, y)$ satisfies $A(u)=0$.

Example. The functions $x=v \cos u$, $y=v \sin u$ satisfy the system and $u=\text{arctg }(y/x)$ satisfies $A(u)=0$.

Consider again a function $u(x, y) \in C^2$ in a neighbourhood of a point where grad $u \neq 0$. A streamline of u is given by $u_y dx - u_x dy = 0$, where the coefficients are in C^1. Any streamline is determined by an initial point and it is a curve in C^2 with continuously varying curvature. The same holds for the level lines of u, which are governed by $u_x dx + u_y dy = 0$.

The curvature of a streamline is given by

$$\frac{1}{|\text{grad } u|}\left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y}\right) \text{arctg } \frac{u_y}{u_x} = \frac{1}{|\text{grad } u|^3}(u_x^2 u_{xy} - u_x u_y u_{xx} + u_x u_y u_{yy} - u_y^2 u_{xy})$$
$$= \frac{1}{2|\text{grad } u|^3} \text{grad } \{(\text{grad } u)^2\} \cdot (-u_y, u_x).$$

Here, the last two expressions hold also at points where $u_x = 0$.

For the level lines, we have

$$\frac{1}{|\operatorname{grad} u|}\left(-u_y\frac{\partial}{\partial x}+u_x\frac{\partial}{\partial y}\right)\operatorname{arctg}\left(-\frac{u_x}{u_y}\right)=\frac{1}{|\operatorname{grad} u|^3}(u_y^2u_{xx}-2u_xu_yu_{xy}+u_x^2u_{yy})$$
$$=\frac{1}{|\operatorname{grad} u|^3}((\operatorname{grad} u)^2\Delta u-A(u)).$$

If u is a solution of $A(u)=0$, then, clearly, the curvature of a streamline can be written

$$\frac{\pm|\operatorname{grad}\{(\operatorname{grad} u)^2\}|}{2|\operatorname{grad} u|^2}=\frac{\pm|\operatorname{grad}(|\operatorname{grad} u|)|}{|\operatorname{grad} u|}.$$

Lemma 1. *Let $u(x, y)$ satisfy $A(u)=0$ in a domain D and let* grad $u\neq0$ *in D. If C is a streamline of u in D, then there are two alternatives:*

I. *The curvature of C is $\neq 0$ at all points of C.*
II. *C is a straight line.*

Consequently, the streamlines of u are *convex curves and straight lines.*

Proof. It is sufficient to prove the following assertion: If the curvature is zero at $A\in C$, then there is a (1-dimensional) neighbourhood of A along C where the curvature is zero. Introduce a coordinate system such that $x_A=(u_y)_A=0$ and consider a neighbourhood U of A such that $u_x\neq0$ in U. We consider the streamlines as solutions of an initial-value problem $dy/dx=\Phi(x, y)$, $y(0)=y_0$, and we write $y(x_1)=y_1=y_1(x_1, y_0)$.

Since $A(u)=0$, we have, for fixed x_1,

$$\varphi(y_0)=|\operatorname{grad} u|_{(0,y_0)}=|\operatorname{grad} u|_{(x_1,y_1)}=\Psi(y_1).$$

If x_1 is fixed and $|x_1|$ is small enough, then

$$\frac{\partial}{\partial y_0}y_1(x_1,y_0)=\exp\left[\textstyle\int_0^{x_1}\Phi_y(x,y(x))\,dx\right]>0$$

([3], pp. 25–27; [9], pp. 73–74).

Clearly, we also have $y_0=y_0(y_1)$ and dy_0/dy_1 is finite. Therefore

$$d\Psi/dy_1=d\varphi/dy_0\cdot dy_0/dy_1.$$

Here, $d\varphi/dy_0=0$ for $y_0=y_A$ according to our formulas for the curvature. Hence $d\Psi/dy_1=0$ at $y_1=y_1(x_1, y_A)$ which means that $|\operatorname{grad}(|\operatorname{grad} u|)|=0$ at $(x_1, y_1(x_1, y_A))$. This completes the proof.

At the same time we have proved that if grad $(|\operatorname{grad} u|)=0$ at a point of a streamline C, then this relation holds at all points of C, which is then a straight line. If we consider the hodograph mapping $p=u_x$, $q=u_y$, it follows that $D(u)\equiv d(p, q)/d(x, y)\equiv u_{xx}u_{yy}-u_{xy}^2=0$ on C. Now let C be a curved streamline and let (r, θ) be polar coordinates in the hodograph plane. Then grad r is orthogonal to C and not zero, and grad θ has a nonzero component along C. Hence $D(u)\neq0$ on C.

Along a curved streamline C, grad $(|\operatorname{grad} u|)$ *and $D(u)$ are both nonzero, and* grad $(|\operatorname{grad} u|)$ *points to the concave side of C.*

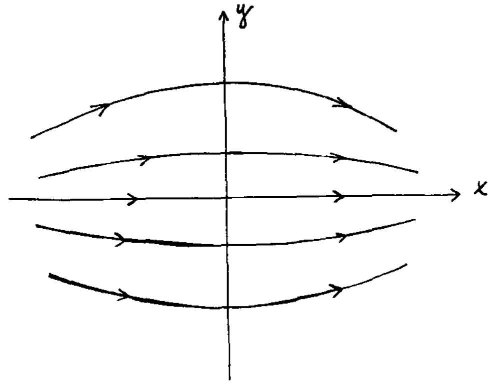

Fig. 1

If the streamline C is a straight line, then grad $(|\text{grad}\, u|)$ *and* $D(u)$ *are both zero along* C.

It should be noticed that $D(u) \equiv u_{xx} u_{yy} - u_{xy}^2 \leqslant 0$ always. This means that the surface $z = u(x, y)$ has nonpositive Gaussian curvature. It follows easily by considering $u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy}$ as a quadratic form in u_x and u_y. Finally, we observe that a streamline C cannot terminate inside D, since $|\text{grad}\, u|$ is constant on C.

Example. We shall illustrate these things by the Cauchy–Kowalewski theorem ([4], [6], [8]). We write

$$u_{xx} = -\frac{1}{u_x^2}(2u_x u_y u_{xy} + u_y^2 u_{yy})$$

and prescribe u and u_x on the y-axis, in a neighbourhood of $y = 0$. Choose, for example, $u(0, y) = 0$ and $u_x(0, y) = \varphi(y) > 0$, where $\varphi(y)$ is analytic for $|y| < \delta$. Then there is an analytic solution $u(x, y)$ in a neighbourhood of $x = y = 0$. Let $\varphi'(0) = 0$ and $\varphi'(y) \neq 0$ if $y \neq 0$. Consider three cases:

1. $\varphi(y)$ has a maximum at $y = 0$.
2. $\varphi(y)$ has a minimum at $y = 0$.
3. $\varphi'(y)$ does not change sign at $y = 0$.

In each case, the x-axis is a streamline and all other streamlines are curved. The streamlines are sketched in Figs. 1–3. In Fig. 3, it is assumed that $\varphi'(y) > 0$ if $y \neq 0$.

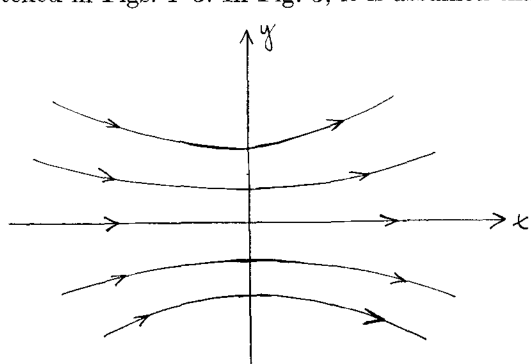

Fig. 2

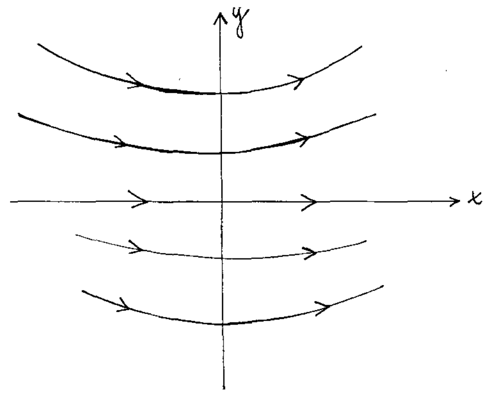

Fig. 3

3. Two particular classes of solutions to $u_x^2 u_{xx} + 2 u_x u_y u_{xy} + u_y^2 u_{yy} = 0$

In this section we will make a few remarks on the class of functions u for which $|\operatorname{grad} u|$ is constant. We will also determine all harmonic functions u which satisfy $A(u) = 0$.

It follows from the identity $A(u) \equiv \frac{1}{2} \operatorname{grad} \{(\operatorname{grad} u)^2\} \cdot \operatorname{grad} u$ that those functions $u \in C^2$, for which $|\operatorname{grad} u|$ is constant, constitute a subclass of all solutions of $A(u) = 0$. This is natural from another aspect: among all functions $u(x, y) \in C^2$, those which are absolutely minimizing, are characterized by the differential equation $A(u) = 0$ ([1], Theorem 8). A subclass of all absolute minimals are those functions u which are unique solutions of some extension problem of the type considered in [1]. And the condition for this is that $u \in C^1$ and that $|\operatorname{grad} u|$ is constant ([1], Theorems 4 and 5).

The differential equation $|\operatorname{grad} u| = \text{constant}$ is treated in [4], pp. 88–91 and [6], pp. 40–43. It is well known that if the surface $z = u(x, y)$ is the tangential developable of a helix with its axis parallel to the z-axis, then $|\operatorname{grad} u| = \text{const}$. Other basic types of solutions are linear functions and functions of the form

$$A\sqrt{(x - x_0)^2 + (y - y_0)^2} + B.$$

There also exist (even infinitely differentiable) solutions whose restrictions to different subdomains belong to different basic types (namely the first and the second).

It should be noticed that a C^1-function u, for which $|\operatorname{grad} u|$ is constant, need not be in C^2. This is shown by the function

$$u(x,y)\begin{cases} x & \text{for } x > 0, y < 0, \\ \sqrt{x^2 + y^2} & \text{for } x > 0, y \geqslant 0. \end{cases}$$

Here, $\partial^2 u / \partial y^2$ is discontinuous across the x-axis. However, we have the following result:

Theorem 1. *Let $u(x, y) \in C^1(\Omega)$ and let $|\operatorname{grad} u| = M = \text{constant}$ in Ω.*

Then $\partial u / \partial x$ and $\partial u / \partial y$ satisfy Lipschitz conditions on each compact subset of Ω.

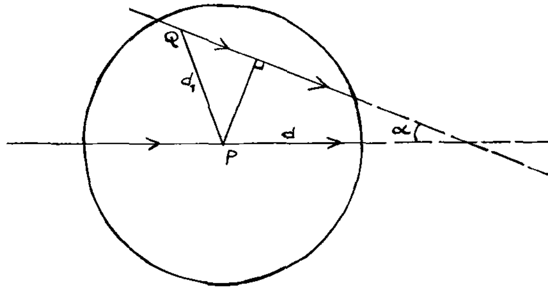

Fig. 4

Proof. We know from Lemma 1 in [1] that the streamlines of u are straight lines without common points in Ω. Consider a compact set $K \subset \Omega$ and put $d =$ the distance from K to $\partial\Omega$. Take two points $P, Q \in K$ such that $d_1 = \overline{PQ} < d$ (Fig. 4). Let P be fixed and Q variable in the circle $\overline{PQ} < d$. Clearly, grad $u(P) \cdot$ grad $u(Q) \neq 0$, and from the continuity we have grad $u(P) \cdot$ grad $u(Q) > 0$. Hence the angle between grad $u(P)$ and grad $u(Q)$ is equal to the smallest angle α between the corresponding streamlines, and we get

$$\alpha \leqslant \arcsin \frac{d_1}{d} \leqslant \frac{\pi}{2} \cdot \frac{d_1}{d}.$$

Hence $\quad |u_x(P) - u_x(Q)| \leqslant M \cdot \alpha \leqslant (M\pi/2d) \cdot d_1$. If $\overline{PQ} \geqslant d$, then

$$|u_x(P) - u_x(Q)| \leqslant 2M \leqslant (2M/d) \cdot d_1.$$

It follows that u_x satisfies a Lipschitz condition with the constant $2M/d$, and similarly for u_y.

Note that u_x and u_y need not satisfy a Lipschitz condition, or even be uniformly continuous in Ω. This can be seen from the example $u = \sqrt{x^2 + y^2}$ in Ω: $(x-1)^2 + y^2 < 1$.

The differential equation

$$A(u) \equiv u_x^2 u_{xx} + 2 u_x u_y u_{xy} + u_y^2 u_{yy} = 0 \tag{1}$$

has a formal resemblance to the differential equation of minimal surfaces,

$$(1 + u_y^2) u_{xx} - 2 u_x u_y u_{xy} + (1 + u_x^2) u_{yy} = 0 \tag{2}$$

and addition of these equations gives $[1 + (\text{grad } u)^2](u_{xx} + u_{yy}) = 0$, or

$$\Delta u = 0. \tag{3}$$

It is clear that a function $u(x, y)$, which satisfies any two of the differential equations (1), (2), and (3), also satisfies the remaining one. We will now determine these common solutions.

Theorem 2. *If $u(x, y)$ is harmonic in a domain D and if $A(u) = 0$ in D, then u is either a linear function, $u = Ax + By + C$, or it can be written*

$$u = D \operatorname{arctg} \frac{y - y_0}{x - x_0} + E$$

for some point (x_0, y_0), *such that* $\operatorname{arctg}[(y-y_0)/(x-x_0)]$ *is one-valued and continuous in* D.

(In particular, it follows that $(x_0, y_0) \notin D$.)

Proof.[1] Write $z=x+iy$ and let v be the conjugate harmonic function of u in a simply connected subdomain D_1 of D. This gives an analytic function $w(z)=u+iv$ in D_1. We have $w'(z)=\partial u/\partial x - i(\partial u/\partial y)$, which means that $|w'(z)|=|\operatorname{grad} u|$.

Put $w'(z)=\tau$. On a streamline of u we have $v=$constant and $|\tau|=$constant.

The functions $w(z)$ and $\tau=w'(z)$ map D_1 onto D_w and D_τ, respectively. We assume that D_1 is chosen such that $w'(z)\neq 0$ in D_1 and such that the mapping $D_1 \to D_w$ is one-to-one. Further, we may assume that $\theta=\arg\tau$ is one-valued and continuous in D_τ.

Now the function $\log|\tau|=\log|w'(z)|$ is harmonic in D_1. But the inverse of $w(z)$, $z=z(w)$, is analytic in D_w, and hence $h(w)=\log|\tau[z(w)]|$ is harmonic in D_w: $\partial^2 h/\partial u^2+\partial^2 h/\partial v^2=0$. Since $\partial h/\partial u=0$, we get $\partial^2 h/\partial u^2=\partial^2 h/(\partial u\,\partial v)=\partial^2 h/\partial v^2=0$ in D_w. Hence $h=c_0+c_1 v$, with real constants c_0, c_1. From the Cauchy–Riemann equations we get $\partial\theta/\partial u=-c_1$, $\partial\theta/\partial v=0$, which gives $\theta=c_2-c_1 u$, and $\log\tau=h+i\theta=(c_0+ic_2)-ic_1(u+iv)=C_0-ic_1 w$ in D_w.

Hence $\tau=C_1\exp(-ic_1 w)$, where $C_1\neq 0$. If $c_1=0$, we get $w=C_1 z+C_2$, which gives $u=Ax+By+C$.

If $c_1\neq 0$, then $dz/dw=1/C_1\exp(ic_1 w)$ and $z-z_0=C_2\exp(ic_1 w)$, $C_2\neq 0$, $z_0\notin D_1$. Taking the arguments of both members, we get $\arg(z-z_0)=c_1 u+c_3$.

This proves the result for D_1, and the general result follows by analytic continuation.

There is a consequence of this theorem that may be of some interest.

We may interpret $u(x, y)$ as a hydrodynamic potential: Consider a two-dimensional, steady potential flow of an ideal liquid. If each particle has constant speed, then the flow is either a uniform translation or otherwise the particles move in concentric circular orbits with the speed C/r, where r is the distance from the center of the circles, and the constant $C>0$ is common to all particles. (The speed of a particle is the modulus of its velocity vector.)

4. The differentiability properties of solutions to $u_x^2 u_{xx}+2u_x u_y u_{xy}+u_y^2 u_{yy}=0$.

This section contains a result on the regularity of a class of solutions to $A(u)=0$. It is proved by application of the hodograph mapping which works only if $u_{xx}u_{yy}-u_{xy}^2\neq 0$. It is shown by an example that the result is false without this restriction.

Lemma 2. *Let* $u(x, y)\in C^2$ *in a region* D *and let* $A(u)=0$ *in* D. *Assume further that* $\operatorname{grad} u\neq 0$ *in* D *and that* $u_{xx}u_{yy}-u_{xy}^2\neq 0$ *in* D.

Then $u\in C^\infty(D)$.

Proof. Apply the hodograph mapping $p=u_x$, $q=u_y$. (See [6], p. 521, or [2], p. 12.) We have $J=d(p,q)/d(x,y)=u_{xx}u_{yy}-u_{xy}^2\neq 0$. Hence the mapping $(x,y)\to(p,q)$ is one-to-one and bicontinuous in a neighbourhood of an arbitrary point in D. We restrict our attention to such a neighbourhood. The functions $x=x(p,q)$ and $y=y(p,q)$ are in C^1 and we have the well-known relations

$$q_y=Jx_p;\quad p_y=-Jx_q;\quad q_x=-Jy_p\quad\text{and}\quad p_x=Jy_q.$$

[1] This proof was suggested by Professor Bengt J. Andersson, Stockholm. The author's original proof is more complicated.

Now we introduce the function (Legendre transform) $\Psi = xp + yq - u$. Clearly, Ψ is a C^1-function of p and q, and

$$\frac{\partial \Psi}{\partial p} = x + px_p + qy_p - u_x x_p - u_y y_p = x,$$

$$\frac{\partial \Psi}{\partial q} = px_q + qy_q + y - u_x x_q - u_y y_q = y.$$

Hence Ψ is a C^2-function of p and q, and

$$\Psi_{pp} = \frac{\partial x}{\partial p} = \frac{1}{J} q_y = \frac{1}{J} u_{yy},$$

$$\Psi_{pq} = \frac{\partial x}{\partial q} = -\frac{1}{J} p_y = -\frac{1}{J} u_{xy},$$

$$\Psi_{qq} = \frac{\partial y}{\partial q} = \frac{1}{J} p_x = \frac{1}{J} u_{xx}.$$

The equation $u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$ is thus transformed into

$$q^2 \Psi_{pp} - 2pq \Psi_{pq} + p^2 \Psi_{qq} = 0 \tag{1}$$

and we get a linear differential equation in the hodograph plane. Next, we introduce polar coordinates (r, θ) in the hodograph plane. This can be done, since we have assumed grad $u \neq 0$. The equation (1) is then transformed into

$$\Psi_{\theta\theta} + r\Psi_r = 0.$$

Another substitution, $\varrho = -\log r$, gives

$$\Psi_{\theta\theta} = \Psi_\varrho,$$

which is the well-known heat equation. It is known that every solution of the heat equation is infinitely differentiable ([7], p. 314, [5], p. 74). Hence, $\Psi \in C^\infty$ as a function of (ϱ, θ), and consequently also as a function of (p, q). Thus, $x = \Psi_p$ and $y = \Psi_q$ are infinitely differentiable functions of p and q. It follows that $p = u_x$ and $q = u_y$ are infinitely differentiable with respect to x and y. This completes the proof.

Remark. It will be proved later that the condition grad $u \neq 0$ is always satisfied, unless u is constant. This condition can therefore be omitted.[1] However, the condition $u_{xx}u_{yy} - u_{xy}^2 \neq 0$ cannot be omitted. One way to show this is to construct a function $f(x, y)$ in C^2, but not in C^3, for which $|\text{grad}\, f|$ is constant. This can be done by using the geometric properties of such functions which were discussed in Section 3, but it involves some calculations and we prefer to describe a quite different example.

[1] Compare Theorems 6 and 7.

Example. Consider a Cauchy problem for $A(u)=0$. We write the equation as $u_{xx}=-1/u_x^2(2u_xu_yu_{xy}+u_y^2u_{yy})$ and prescribe u and u_x on the y-axis, in a neighbourhood of $y=0$, by $u(0,y)=0$, $u_x(0,y)=1+y^3$. From the Cauchy–Kowalewski theorem it follows that there is an analytic solution $u(x,y)$ in a neighbourhood of the origin. This corresponds to case 3 of the example in Section 2. It is clear that the x-axis is a streamline and that the other streamlines are curved. Along the x-axis, we have $u_x=1$, $u_y=0$; further, $A(u)=0$ gives $u_{xx}=0$ and from $u_{xx}u_{yy}-u_{xy}^2=0$ we get $u_{xy}=0$. Along the y-axis we have $u_x>0$, $u_y=0$, and we know that every curved streamline turns the convex side downwards. Therefore, we must have $u_y(x,y)>0$ if $x>0$, $y\neq 0$ and $u_y(x,y)<0$ if $x<0$, $y\neq 0$. Since $u_y=0$ on the x-axis we must have $u_{yy}=0$ there. Consequently, $u=x$, $u_x=1$ and $u_y=u_{xx}=u_{xy}=u_{yy}=0$ along the x-axis.

Form the function $$u_1(x,y)=\begin{cases}u(x,y) & \text{for } y\geqslant 0,\\ x & \text{for } y\leqslant 0.\end{cases}$$

It follows from above that $u_1\in C^2$ and $A(u_1)=0$. From our initial conditions it follows that $u_{yyy}=0$ and $u_{xyyy}=6$ at the origin. Therefore, $u_{yyy}(x,0)\neq 0$ if $0<|x|<\delta$ and this shows that $(u_1)_{yyy}$ does not exist at these points. Hence, u_1 does not belong to C^3 in any neighbourhood of the origin. It also follows that *a continuation of a solution of $A(u)=0$ need not be unique.*

5. Application of differential geometry

This section contains a discussion of the "intrinsic" geometric properties of a surface S: $z=\Phi(x,y)$ where $A(\Phi)=0$.

Theorem 3. *Let $\Phi(x,y)$ satisfy $A(\Phi)=0$, and* grad $\Phi\neq 0$. *Consider the surface $S: z=\Phi(x,y)$ and the projections on this surface of the streamlines of $\Phi(x,y)$.*
These image curves are both
asymptotic curves on S, and
helices with a common axis, namely the z-axis.
(For definition of a helix, see [10], p. 33).

Proof. The asymptotic directions on any surface $z=\Phi(x,y)$ are determined by

$$\Phi_{xx}(dx)^2+2\Phi_{xy}dx\,dy+\Phi_{yy}(dy)^2=0.$$

Hence, from $A(\Phi)=0$, it follows that $dy:dx=u_y:u_x$ corresponds to asymptotic curves. We also know that u is a linear function of arc length on each streamline in the xy-plane, $du/ds=C$. From this it follows that we have, on the corresponding space curve, $dz/ds_1=C/\sqrt{1+C^2}=$ const. This completes the proof.

Remark. Suppose that $\Phi_{xx}\Phi_{yy}-\Phi_{xy}^2<0$ in a domain D. Then the surface S has only hyperbolic points, and S has two distinct families of asymptotic curves. According to the theorem, one of these families consists of helices with a common axis. Further, the geodesic curvature of such a curve is not zero.

The geometric features of the case $\Phi_{xx}\Phi_{yy}-\Phi_{xy}^2=0$ (where $|\text{grad}\, u|$ is constant) were treated in Section 3 and the references cited there.

The condition grad $\Phi\neq 0$ is in fact no restriction, in view of Theorem 6.

In the converse direction we have

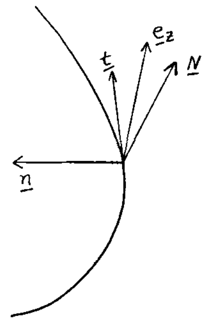

Fig. 5

Theorem 4. *Let the surface S be in C^2, and let S be simply covered by a family $\mathcal{A}$ of asymptotic curves with nonvanishing geodesic curvature. Assume now that all curves in $\mathcal{A}$ are helices with a common axis l. Introduce an orthonormal coordinate system such that the z-axis is parallel to l.*

The surface S will then define a function $z=\Phi(x, y)$, at least locally, and this function satisfies $A(\Phi)=0$.

Proof. Consider a curve $C \in \mathcal{A}$, a point $P \in C$, and let $\underline{t}$, $\underline{n}$ be respectively tangent and principal normal to C at P. (Note that $\underline{n}$ is well-defined, since the geodesic curvature is not zero.) Then $\underline{t} \cdot \underline{n}=0$, and since C is a helix with the axis parallel to the z-axis, we have $\underline{n} \cdot \underline{e}_z=0$. Further, since C is an asymptotic curve we have $\underline{N} \cdot \underline{n}=0$, where $\underline{N}$ is the surface normal. Hence, $\underline{t}$, $\underline{N}$ and $\underline{e}_z$ lie in a plane (the rectifying plane of C).

We must have $|\underline{t} \cdot \underline{e}_z|<1$ since, otherwise, C would have to be a straight line. Because of the relation $\underline{t} \cdot \underline{N}=0$, this means that $\underline{N} \cdot \underline{e}_z \neq 0$. This shows that S defines, locally, a function $z=\Phi(x, y) \in C^2$ and further, since grad Φ must lie in the intersection of the rectifying plane of C and the xy-plane, it follows that C is the image of a streamline of Φ. Finally, C is an asymptotic curve, which means that $\Phi_x^2 \Phi_{xx}+2\Phi_x \Phi_y \Phi_{xy}+\Phi_y^2 \Phi_{yy}=0$. (Compare the proof of the previous theorem.)

It may perhaps be of some interest to have another derivation of the parametric representation that was derived in Section 2.

Consider a surface $S: z=\Phi(x, y)$, where $A(\Phi)=0$, grad $\Phi \neq 0$, and $\Phi_{xx}\Phi_{yy}-\Phi_{xy}^2<0$. Then grad $(|\text{grad } \Phi|) \neq 0$ and we have locally a one-to-one mapping $(x, y) \leftrightarrow (\Phi, |\text{grad } \Phi|)$. Now we introduce parameters u, v on S by writing $u=\Phi$, $v=|\text{grad } \Phi|$. Then $F=0$, $E=1/v^2$, where the notation is that of [10], p. 58. It also follows that $\Gamma_{11}^1=0$ ([10], p. 107). From Theorem 3 we know that $v=$ constant corresponds to asymptotic curves.

Hence $e=0$ and the Gauss equation for $\underline{x}_{uu}$ takes the form $\underline{x}_{uu}=\Gamma_{11}^2 \underline{x}_v$ ([10], pp. 74–75, 107). If we write $\Gamma_{11}^2=B(u, v)$ and $x=\varphi_1(u, v)$, $y=\varphi_2(u, v)$, $z=\varphi_3(u, v) \equiv u$, we obtain

$$(\varphi_i)_{uu}-B(u, v)(\varphi_i)_v=0, \; i=1, 2, 3.$$

Further, the parametric lines are orthogonal on S, as well as their projections on the xy-plane. Hence $(\varphi_1)_u(\varphi_1)_v+(\varphi_2)_u(\varphi_2)_v=0$.

An argument in the converse direction leads to the following result:

Suppose that there are given two functions $\varphi_1(u, v)$, $\varphi_2(u, v)$ in C^2 which satisfy $(\varphi_i)_{uu} - B(u, v)(\varphi_i)_v = 0$, $i = 1, 2$, where $B(u, v)$ is an arbitrary continuous function. Suppose also that

$$(\varphi_1)_u(\varphi_1)_v + (\varphi_2)_u(\varphi_2)_v = 0$$

and

$$(\varphi_1)_u(\varphi_2)_v - (\varphi_1)_v(\varphi_2)_u \neq 0$$

in the region in question.

By the relations

$$x = \varphi_1(u, v), \quad y = \varphi_2(u, v), \quad z = u$$

z is defined, at least locally, as a function of x and y. This function $z = \Phi(x, y)$ satisfies $A(\Phi) = 0$.

It is easy to see that this is identical with the result on parametric representation that was derived in Section 2. First, if the above conditions are satisfied, then

$$(\varphi_1)_u(\varphi_1)_{uu} + (\varphi_2)_u(\varphi_2)_{uu} = B(u, v)[(\varphi_1)_u(\varphi_1)_v + (\varphi_2)_u(\varphi_2)_v] = 0.$$

And if the conditions in Section 2 are satisfied, then the vector (x_u, y_u) $(\neq 0)$ is orthogonal to (x_v, y_v) and to (x_{uu}, y_{uu}) which means that the latter ones are parallel: $x_{uu} = B(u, v)x_v$, $y_{uu} = B(u, v)y_v$.

6. An inequality for $|\operatorname{grad} u|$. Nonexistence of stationary points

An important feature of the differential equation $A(u) = 0$ is that a nonconstant solution has no stationary points. This is obtained below as a consequence of an estimate for $|\operatorname{grad} u|$. We begin with a lemma.

Lemma 3. *Consider a function $f(x) \in C^1[A, B]$ with $f(A) = 0$ and $f(B) = 1$. Then there is a sequence of open intervals $\{I_\nu\}$ on $[A, B]$, finite or denumerably infinite, such that:*

(1) *The intervals I_ν are pairwise disjoint,*
(2) *$f'(x) > 0$ on all I_ν,*
(3) *if $I_\nu = (a_\nu, b_\nu)$, then $\Sigma_\nu(f(b_\nu) - f(a_\nu)) = 1$,*
(4) *if $y \in I_\nu$ for some ν, and $x < y$, then $f(x) < f(y)$.* (In particular, $x \in I_\nu$, $y \in I_\mu$, $\nu \neq \mu$ implies that $f(x) \neq f(y)$.)
(5) *$0 < f(x) < 1$ on all I_ν.*

Proof. Form the function $\Psi(x) = \sup_{A \leqslant t \leqslant x} f(t)$. Since $f(x)$ satisfies a Lipschitz condition, so does $\Psi(x)$. Hence $\Psi(x)$ is absolutely continuous. Further, $\Psi(x)$ is non-decreasing. Write $I = [A, B]$. We may assume that $\max_I f(x) = 1$ since, otherwise, we can consider the interval $[A, X]$, where $X = \min\{x \mid f(x) = 1\}$. The result for this subinterval also gives the result for I. Hence $\Psi(A) = 0$, $\Psi(B) = 1$, $\Psi'(x) \geqslant 0$ a.e. and $\int_A^B \Psi'(x)\,dx = 1$.

Now consider $E = \{x \mid A < x < B, \Psi'(x) > 0\}$. Let $x_0 \in E$. Clearly, $\Psi(x_0) \geqslant f(x_0)$, and if we had $\Psi(x_0) > f(x_0)$, then $\Psi(x)$ would necessarily be constant in a neighbourhood of x_0. Hence $\Psi(x_0) = f(x_0)$, and if $x \geqslant x_0$, then $\Psi(x) = \sup_{x_0 \leqslant t \leqslant x} f(t)$. From this, and from $\Psi'(x_0) > 0$ it follows that $f'(x_0) > 0$. Choose a $\delta > 0$ such that

$\Psi(x_0-\delta)<\Psi(x_0)<\Psi(x_0+\delta)$ and such that $f'(x)>0$ for $x_0-\delta\leqslant x\leqslant x_0+\delta$. Choose also a δ_1, $0<\delta_1\leqslant\delta$, such that $f(x_0-\delta_1)>\Psi(x_0-\delta)$. On the interval $x_0-\delta_1\leqslant x\leqslant x_0+\delta$ we thus have

$$f(x)=\sup_{x_0-\delta\leqslant t\leqslant x} f(t)\geqslant f(x_0-\delta_1)>\Psi(x_0-\delta)=\sup_{t\leqslant x_0-\delta} f(t)$$

and hence $\Psi(x)=f(x)$.

Hence, E is an open set and $\Psi'(x)=f'(x)$ on E. Write $E=\bigcup_\nu(a_\nu,b_\nu)$. We get

$$\Psi(B)-\Psi(A)=\int_A^B\Psi'(x)\,dx=\int_E\Psi'(x)\,dx=\sum_\nu\int_{a_\nu}^{b_\nu}\Psi'(x)\,dx$$
$$=\sum_\nu\int_{a_\nu}^{b_\nu}f'(x)\,dx=\sum_\nu(f(b_\nu)-f(a_\nu)).$$

Hence, the assertion (3) is true, and it is very easy to see that (1), (2), (4), and (5) also hold. This completes the proof.

Theorem 5. *Let $u(x,y)\in C^2(D)$ and let $u_x^2u_{xx}+2u_xu_yu_{xy}+u_y^2u_{yy}=0$ in D. Let A, B be two points in D such that the closed segment AB lies in D, and such that* $\operatorname{grad}u\neq 0$ *on this segment. Then*

$$\log\frac{|\operatorname{grad}u(A)|}{|\operatorname{grad}u(B)|}\leqslant\pi\frac{\overline{AB}}{d}+\frac{\pi^2}{2},$$

where $\overline{AB}$ = the distance from A to B, and d = the distance from the segment AB to ∂D.

Proof. The idea of the proof is the following: If we put $v=|\operatorname{grad}u|$, then the curvature of a streamline can be written (apart from sign)

$$\frac{d\theta}{ds}=\frac{|\operatorname{grad}v|}{v}=\frac{dv/dn}{v},$$

where dv/dn denotes differentiation along a level line. Let us consider the function v along some fixed level line l. Suppose that the curvature of the streamlines intersecting l is bounded: $(dv/dn)/v\leqslant K$, or $dv/v\leqslant K\,dn$. Integration gives $\int(dv/v)\leqslant K\int dn$, or $\log(V_2/V_1)\leqslant K\cdot L$, where L is the length of l, and V_1, V_2 are the corresponding values of v. This is an inequality of the desired type. Now the curvature of the streamlines need not be uniformly bounded, and the vague reasoning above does not give anything. However, since the streamlines are convex curves (or straight lines), it is possible to estimate the *integrated curvature* along (part of) a streamline. Accordingly, the proof is based upon an *integration* of the formula $d\theta/ds=|\operatorname{grad}v|/v$.

Put $M=|\operatorname{grad}u(A)|$, $M_1=|\operatorname{grad}u(B)|$, and assume that $M>M_1$. Let l denote arc length along the segment AB, increasing towards B. Consider a sub-interval I of the segment, with endpoints K, L, such that $(d/dl)|\operatorname{grad}u|<0$ on I. Let I be an open interval. Let $\{\gamma\}$ be the family of streamlines which intersect I. Note that none of these streamlines can intersect I tangentially, since we would have $(d/dl)|\operatorname{grad}u|=0$ at such a point.

Suppose each streamline in $\{\gamma\}$ to be continued in both directions from I indefinitely or until it approaches ∂D. If E is the point set thus covered by these curves,

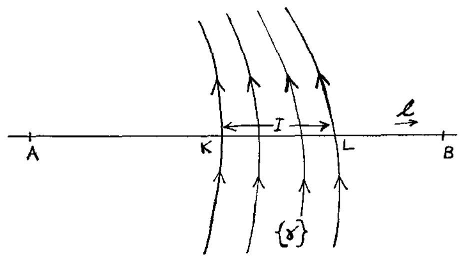

Fig. 6

then E is obviously open. Let s be arc length along the curves in $\{\gamma\}$, for each of these curves, let $s=0$ in the intersection with I, and let s and u increase in the same direction. Compare Figure 6.

Put $V_1=|\operatorname{grad} u|_L$, $V_2=|\operatorname{grad} u|_K$. Let G be the subset of E which is defined by $S_1<s<S_2$, $V_1<|\operatorname{grad} u|<V_2$. Here, $S_1<0$, $S_2>0$, and $|S_1|$, $|S_2|$ are both less than the distance from I to ∂D.

Clearly, G is open and connected. Write $v=|\operatorname{grad} u|$ and consider first the mapping $(x, y)\to(u, v)$ from G onto a set G_1 in the (u, v)-plane. Clearly, it is one-to-one, and we have

$$\frac{d(u,v)}{d(x,y)}=\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}=\pm|\operatorname{grad} u|\cdot|\operatorname{grad} v|\neq 0.$$

This follows from $A(u)=0$. Since $d(u, v)/d(x, y)$ is continuous and not zero, the same sign must hold in all of G. Further, it follows that G_1 is also open, and that the inverse mapping is also in C^1. We get

$$\frac{d(x,y)}{d(u,v)}=\frac{\pm 1}{|\operatorname{grad} u||\operatorname{grad} v|}.$$

Clearly, the sets G and G_1 are also in one-to-one correspondence with a set G_2 in the (s, v)-plane. The mapping $(s, v)\to(u, v)$ follows the formula

$$\begin{cases} u=\varphi(v)+s\cdot v \\ (v=v). \end{cases}$$

Here, $\varphi(v)$ is the value of u in the corresponding point on I. Clearly, $\varphi(v)\in C^1$. Thus

$$\frac{d(u,v)}{d(s,v)}=\begin{vmatrix} v & s+\varphi'(v) \\ 0 & 1 \end{vmatrix}=v>0.$$

Clearly, the mappings between (s,v) and (u,v) are in C^1, and we get

$$\frac{d(x,y)}{d(s,v)}=\frac{d(x,y)}{d(u,v)}\cdot\frac{d(u,v)}{d(s,v)}=\frac{\pm 1}{|\operatorname{grad} u||\operatorname{grad} v|}\cdot v=\frac{\pm 1}{|\operatorname{grad} v|}.$$

Now we have

$$mG=\iint_G 1\,dx\,dy=\iint_{V_1<v<V_2}^{S_1<s<S_2}\frac{ds\,dv}{|\operatorname{grad} v|}.$$

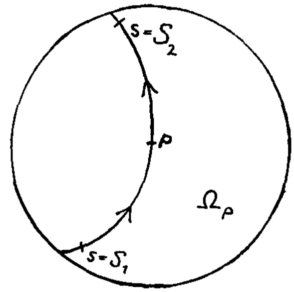

Fig. 7

Before estimating the last integral, we choose $S_1 = -S_2$ and such that, for any point P on the segment AB, the closed circle Ω_P with center at P and radius S_2 lies in D. Then the total (integrated) curvature of any curve in $\{\gamma\}$, calculated for $S_1 < s < S_2$, is not greater than 2π. This is easy to see, since $\gamma(P)$ is convex and the continuation of γ_P beyond $s = S_1$, $s = S_2$, must meet $\partial\Omega_P$ without intersecting itself. We leave the details.

Now we can estimate mG:

$$mG = \iint \frac{ds\,dv}{|\operatorname{grad} v|} = \int_{V_1}^{V_2} dv \int_{S_1}^{S_2} \frac{ds}{|\operatorname{grad} v|}.$$

From the Schwarz inequality, we get

$$\left(\int_{S_1}^{S_2} 1\, ds\right)^2 \leqslant \int_{S_1}^{S_2} |\operatorname{grad} v|\, ds \cdot \int_{S_1}^{S_2} \frac{ds}{|\operatorname{grad} v|}.$$

Now consider $\int_{S_1}^{S_2} |\operatorname{grad} v|\, ds$. The formula for curvature of streamlines, Section 2, reads $d\theta/ds = \pm |\operatorname{grad} v|/v$, and since v is constant in this integral, we get

$$\int |\operatorname{grad} v|\, ds = \int \left| v \frac{d\theta}{ds} \right| ds = v \int \left| \frac{d\theta}{ds} \right| ds \leqslant 2\pi v.$$

With the preceding inequality, we get

$$\int_{S_1}^{S_2} \frac{ds}{|\operatorname{grad} v|} \geqslant \frac{(S_2 - S_1)^2}{2\pi \cdot v} = \frac{2S_2^2}{\pi \cdot v}$$

and finally $mG \geqslant (2S_2^2/\pi) \log (V_2/V_1)$.

Now, if $dv/dl < 0$ on the whole segment AB, or on a finite number of portions of AB, then we can apply the previous reasoning to the whole segment AB or to each of these portions, respectively. However, we do not know whether $(d/dl)|\operatorname{grad} u|$ changes sign a finite or infinite number of times on AB, and therefore, we must apply Lemma 3 to the function $|\operatorname{grad} u|$ on the segment AB (with obvious modifications). This gives a sequence of intervals I_ν and we also get corresponding sets G_ν. Note that S_1, S_2 are defined independently of I. From the condition 4 in the lemma it follows that $\{G_\nu\}$ are pairwise disjoint. Write $p_\nu = \inf_{I_\nu} |\operatorname{grad} u|$,

$q_\nu = \sup_{I_\nu} |\text{grad}\, u|$. Then all p_ν, q_ν are positive and the preceding estimate gives $mG_\nu \geq (2S_2^2/\pi)\log(q_\nu/p_\nu)$ and $m(\bigcup G_\nu) \geq (2S_2^2/\pi)\,\Sigma_\nu \log(q_\nu/p_\nu)$. Now the intervals (p_ν, q_ν) are disjoint, they are sub-intervals of (M_1, M) and $\Sigma_\nu(q_\nu - p_\nu) = M - M_1$. With the aid of Levi's theorem we infer from this that $\Sigma_\nu \log(q_\nu/p_\nu) = \log(M/M_1)$. Hence we arrive at

$$\log\frac{M}{M_1} \leq \frac{\pi}{2S_2^2}\, m(\textstyle\bigcup_\nu G_\nu).$$

Now the distance from a point in $\bigcup_\nu G_\nu$ to the segment AB is at most S_2, which means that $m(\bigcup_\nu G_\nu) \leq 2S_2 \cdot \overline{AB} + \pi S_2^2$. Hence

$$\log\frac{M}{M_1} \leq \frac{\pi\overline{AB}}{S_2} + \frac{\pi^2}{2}.$$

Here, S_2 can be any positive number less than d, and if we let S_2 tend to d, then we obtain the desired estimate.

It would not be difficult to formulate and prove a corresponding estimate for any two points A, B in D that are connected by a smooth curve in D instead of a straight segment. We omit these details.

Theorem 6. *Let $u(x, y) \in C^2(D)$ and let $u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$ in D. Then* grad $u \neq 0$ *in all of D, unless u is constant in all of D.*

Proof. This is an immediate consequence of the previous theorem.

Theorem 7. *Let $u(x, y) \in C^2$ in a domain D and let $A(u) = 0$ in D. Assume also that $u_{xx}u_{yy} - u_{xy}^2 \neq 0$ in D. Then $u \in C^\infty(D)$.*

Proof. This follows immediately from Lemma 2 and the previous theorem.

7. Solutions in a neighbourhood of infinity. Global solutions

It is possible to find all solutions of $A(u) = 0$ in a neighbourhood of infinity, that is, all functions satisfying $A(u) = 0$ outside some compact set.

Theorem 8. *Let F be a compact set in the plane (R^2) and let G be the convex hull of F. Further, let $A(u) = 0$ in $R^2 - F$.*

Then $|\text{grad}\, u| =$ constant *in $R^2 - G$, and only these cases are possible:*

1. *$u = Ax + By + C$ in $R^2 - G$.*
2. *There is a simple, closed curve Γ, with continuous curvature, and enclosing a region $H \supset G$, such that*

$$u(R) = D \cdot d(R, \Gamma) + E \text{ for each } R \notin H.$$

Further, Γ is convex (not necessarily strictly).

(Here, A, ..., E are constants and $d(R, \Gamma) =$ the distance from R to Γ.)

Proof. (1) Let F be contained in an open circular disk D_0 with boundary C. Unless $u \equiv$ constant, there exist $M > 0$ and $\delta > 0$ such that $|u| \leq M$ and $|\text{grad}\, u| \geq \delta$ on C. Now let γ be a streamline of u which intersects C in (at least) two points Q, R.

Then the length of γ, measured between Q and R, is not greater than $2M/\delta$. This is clear, since $L(\gamma)\cdot\delta \leqslant |u(Q)-u(R)| \leqslant 2M$.

Let Ω be an open annulus which is concentric with C. Let r_0 be the radius of C and r_1, r_2 the inner and outer radii of Ω, respectively. Suppose that $r_1 > r_0 + M/\delta$. Consider $\max_{\bar{\Omega}} |\text{grad}\, u|$, which is taken at $P_0 \in \bar{\Omega}$. We claim that $\text{grad}(|\text{grad}\, u|)=0$ at P_0. If this is not the case, then P_0 lies on $\partial\Omega$.

(2) First, let P_0 lie on the inner boundary of Ω. It follows from our choice of r_1 that $\gamma(P_0)$ must extend to infinity at least in one direction. But then $\gamma(P_0)$ has to meet Ω, and at these points $\text{grad}\,(|\text{grad}\, u|)=0$. However, according to our results in Section 2, this contradicts the fact that $\text{grad}\,(|\text{grad}\, u|) \neq 0$ at P_0.

(3) Then assume that P_0 lies on the outer boundary of Ω, and $\text{grad}\,(|\text{grad}\, u|) \neq 0$ at P_0. Since $|\text{grad}\, u|$ is maximal at P_0, it follows that $\text{grad}\,(|\text{grad}\, u|)$ is perpendicular to $\partial\Omega$ at P_0, and it is also clear that this vector points towards the exterior of Ω. Let l be the tangent of $\partial\Omega$ at P_0. Then $\gamma(P_0)$ is a convex curve, and if U is a suitable neighbourhood of P_0, then $\gamma(P_0) \cap U$ and $\Omega \cap U$ lie on different sides of l. (Compare Section 2). However, since $\gamma(P_0)$ is convex, it also follows that $\gamma(P_0)$ does not meet l, except at P_0, and hence $\gamma(P_0)$ extends to infinity in both directions. It is not difficult to see that the total curvature of $\gamma(P_0)$ is not greater than π. Further, $\gamma(P_0)$ obviously separates the plane into two parts, D and D', one of which (D) is convex. Clearly, any streamline in D belongs to the curved type, extends to infinity in both directions and has total curvature $\leqslant \pi$.

(4) Let P be an arbitrary point in D, and let $\gamma^+(P)$ be the part of $\gamma(P)$ where $u \geqslant u(P)$. Let C be the level line of u through P and let Q, R be points on $C \cap D$ such that $V_1 = |\text{grad}\, u(Q)| < |\text{grad}\, u(P)| < |\text{grad}\, u(R)| = V_2$ and such that the part of C between Q and R belongs to D. Then $v = |\text{grad}\, u|$ is strictly monotonic along C_{QR} and v can be used as a parameter on that curve. Further, let s denote the arc length along any streamline γ of u which intersects C_{QR}. Let $s=0$ in the point of intersection, and let $du/ds > 0$ on γ.

Consider the set covered by $\gamma^+(P')$ when P' varies along C_{QR}. This set is in one-to-one correspondence with the set in the (s, v)-plane defined by $V_1 \leqslant v \leqslant V_2$, $0 \leqslant s < \infty$. Further, the mapping is in C^1 in both directions, and $d(x, y)/d(s, v) = \pm 1/|\text{grad}\, v|$ (compare the proof of Theorem 6). We want to estimate the measure of the set G in the xy-plane which corresponds to $V_1 < v < V_2$, $0 < s < S$, where $S > 0$ is arbitrary. As in Theorem 6 we have

$$mG = \iint_G 1\, dx\, dy = \iint_{0<s<S}^{V_1<v<V_2} \frac{ds\, dv}{|\text{grad}\; v|}$$

and, applying Schwarz's inequality and the formula for the curvature of a streamline, we find

$$\int_0^S \frac{ds}{|\text{grad}\; v|} \geqslant \frac{S^2}{v\theta_1(v)} > \frac{S^2}{v\theta(v)}.$$

Here, $\theta_1(v)$ is the total curvature of the corresponding streamline $(\gamma(v))$, evaluated between the limits $s=0$ and $s=S$, and $\theta(v)$ is the total curvature of $\gamma(v)$ between the limits $s=0$ and $s=\infty$. (We have $0 < \theta_1(v) < \theta(v) \leqslant \pi$.) Hence, we obtain[1]

$$mG \geqslant \int_{V_1}^{V_2} \frac{S^2\, dv}{v\cdot\theta(v)} = S^2 \int_{V_1}^{V_2} \frac{dv}{v\cdot\theta(v)}.$$

[1] The function $\theta(v)$ is measurable, since it is lower semi-continuous.

However, it is clear that G is contained in a circle with center at Q and with the radius $(d+S)$, where d is the length of C_{QR}. Hence we get

$$S^2 \int_{V_1}^{V_2} \frac{dv}{v \cdot \theta(v)} \leqslant mG \leqslant \pi(d+S)^2.$$

Here, V_1, V_2, $\theta(v)$ and d are independent of S, and S can be choosen arbitrarily large. Consequently we must have

$$\int_{V_1}^{V_2} \frac{dv}{v \cdot \theta(v)} \leqslant \pi.$$

Choose $u_1 > u(P)$, and consider the set in the (s, v)-plane defined by $s \cdot v = u_1 - u(P)$, $V_1 \leqslant v \leqslant V_2$. In the xy-plane, it corresponds to a level line $C_1 : u = u_1$ intersecting $\gamma^+(P)$ at a point P_1. Now the above argument applies to P_1 and C_1 as well as to P and C. Hence we obtain

$$\int_{V_1}^{V_2} \frac{dv}{v \cdot \theta(v, u_1)} \leqslant \pi,$$

where the meaning of the notation is obvious. Here, u_1 may be chosen arbitrarily large, and it is clear that $\lim_{u_1 \to \infty} \theta(v, u_1) = 0$, for every fixed v. Consequently,

$$\lim_{u_1 \to \infty} \int_{V_1}^{V_2} \frac{dv}{v \cdot \theta(v, u_1)} = \infty,$$

which is a contradiction to the above inequality.

(5) We may therefore conclude that grad $(|\text{grad } u|) = 0$ at P_0 where P_0 is any point which realizes $\max_{\bar{\Omega}} |\text{grad } u|$. It follows that $\gamma(P_0)$ is a straight line, and extends to infinity at least in one direction.

Let D_1 be an open circular disk with center on $\gamma(P_0)$ and such that $D_1 \supset \Omega$. Suppose that there exists a point $P \notin \bar{D}_1$, such that $\gamma(P)$ belongs the curved type. It follows from our choice of Ω that $\gamma(P)$ extends to infinity in one direction at least, for instance $\gamma^+(P)$. We may then assume that $u(P) > \max_{\bar{\Omega}} u$. Let U be a neighbourhood of P such that we have $u > \max_{\bar{\Omega}} u$ in U and such that grad $(|\text{grad } u|) \neq 0$ in U. Thus, if $Q \in U$, it follows that $\gamma(Q)$ belongs to the curved type and that $\gamma^+(Q)$ extends to infinity. Now we are in a position to apply the same reasoning as in (4). We only have to find an estimate for the total curvature of a streamline γ in $R^2 - \bar{D}_1$. However, 4π will do, and this follows easily from the fact that γ cannot meet the straight line $\gamma(P_0)$.

We arrive at a contradiction, as in 4), and may conclude that there is no curved streamline in $R^2 - \bar{D}_1$. Consequently, $|\text{grad } u| = \text{constant}$ in $R^2 - \bar{D}_1$.

(6) We have to prove that $|\text{grad } u| = \text{constant}$ in $R^2 - G$. It is sufficient to prove that grad $(|\text{grad } u|) = 0$ in $R^2 - G$. Suppose then that there is a point $P \notin G$ at which this relation does not hold. Let l be a "half-ray" from P to infinity, such that $l \cap G = \Phi$, and let Q be the point on l, closest to P, at which grad $(|\text{grad } u|) = 0$. Since $Q \notin G$, $\gamma(Q)$ must extend to infinity in one direction at least, for instance $\gamma^+(Q)$. Let Q' be a point on $\gamma^+(Q)$, such that $Q' \notin \bar{D}_1$, and put $\overline{QQ'} = S$. Let $\{R_\nu\}$ be a sequence of points on PQ, tending to Q. Then $\{\gamma(R_\nu)\}$ belong to the curved type. Further, the points $\{R'_\nu\}$ on $\gamma^+(R_\nu)$ which correspond to the arc length $s = S$, are defined for ν

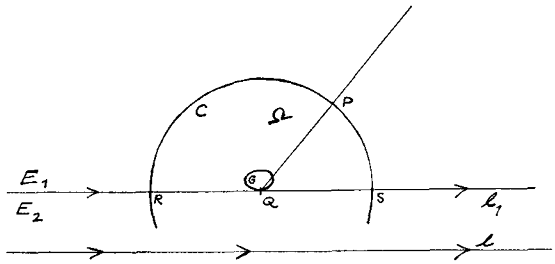

Fig. 8

large enough, and tend to Q'. (To see this, consider a streamline as a solution of a differential system with initial conditions $x = x_0$, $y = y_0$. The solution depends continuously on (x_0, y_0). Compare [9], p. 105, and Part 1 of the proof of Theorem 10.) Consequently, $R_\nu' \notin \bar{D}_1$ for ν large enough, and this contradicts the fact that $|\mathrm{grad}\, u| =$ constant in $R^2 - \bar{D}_1$. This proves that $|\mathrm{grad}\, u|$ is constant in $R^2 - G$.

(7) The next step is to find the form of the solution u.

A. First, let there exist a streamline l which does not meet G. Put $(A, B) = (\mathrm{grad}\, u)_l$. We claim that $u = Ax + By + C$ in $R^2 - G$, for some C. Let l_1 be a straight line, parallel to l, and separating the plane into half-planes E_1, E_2 such that $E_2 \cap G = \Phi$, $l_1 \cap G \neq \Phi$ (Fig. 8). It is obvious that $u = Ax + By + C$ in E_2. Further, if there is a streamline in E_1, which does not meet G, then it follows, by considering the relative positions of the streamlines, that $u = Ax + By + C$ in $R^2 - G$. Consider then the opposite case, and let Ω be a circle with center at some point $Q \in G \cap l_1$ and such that $G \subset \Omega$. Put $C = \partial\Omega \cap \bar{E}_1$. Consider the function $\Psi(P) = \mathrm{grad}\, u(P) \cdot \overrightarrow{QP}$ for a point $P \in C$. It is continuous, and since $\gamma(P)$ meets G, we must have $\Psi(P) \neq 0$. Consequently, $\Psi(R)$ and $\Psi(S)$ are of the same sign, which is an obvious contradiction (compare Fig. 8). This proves the assertion in the case A.

B. Each streamline meets G. In this case, the streamlines constitute a family of "half-rays", emerging from G and covering $R^2 - G$. The curve Γ will be obtained as a level line of u, enclosing G, and it is fairly obvious from geometric reasons that Γ must be convex. The details can be filled in as follows: Let Q be any point in G and let Ω be a circle, containing G, and with center at Q. Put $\Psi(P) = \mathrm{grad}\, u(P) \cdot \overrightarrow{QP}$ for $P \in \partial\Omega$. Clearly, $\Psi \neq 0$ and Ψ has fixed sign, for instance $\Psi > 0$. This means that each streamline is oriented from G to infinity. Let E be the smallest closed circle with center at Q such that $E \supset G$. Clearly, u may be supposed defined and continuous on ∂E. Write $M = \max_{\partial E} u$. Let M_0 be any number $> M$. If $P \in \partial\Omega$, let P_1 be the point for which $\overrightarrow{P_1 P} = (1/\lambda^2)(u(P) - M_0)\, \mathrm{grad}\, u(P)$. When P varies over $\partial\Omega$, P_1 clearly describes the set $\{P' \mid u(P') = M_0, P' \notin G\}$. It follows that this level line is a simple, closed curve Γ, twice continuously differentiable. Further, Γ encloses a region H and it is obvious that $H \supset G$. Next, we claim that $u(R) = M_0 + \lambda d(R, \Gamma)$ for any $R \notin H$. Consider any R in the exterior of H, and let $\gamma(R)$ intersect Γ in P. Clearly, $u(R) = M_0 + \lambda \overline{PR}$, and we need only prove that $\overline{PR} = d(R, \Gamma)$.

If this is not true, then there is a $P_1 \in \Gamma$ such that $\overline{RP_1} = d(R, \Gamma) < \overline{RP}$. But this means that $R \in \gamma(P_1)$ and hence $\gamma(P)$ and $\gamma(P_1)$ intersect at R, which is impossible. This proves that $u(R) = M_0 + \lambda d(R, \Gamma)$, and it also proves that $d(R, \Gamma)$ *is taken on for only one point on* C, *and this point* (P) *is the point where* $\gamma(R)$ *intersects* Γ.

Finally, fix a point $P \in \Gamma$ and let $R \in \gamma(P)$. The tangent l of Γ at P divides R^2 into half-planes E_1, E_2, and let E_2 contain the unbounded part of $\gamma(P)$. When R approaches infinity along $\gamma(P)$, it follows from the preceding statement (italicized) that $E_2 \cap \Gamma = \Phi$. Consequently, Γ is convex. This completes the proof.

Remark. If Γ is a closed, convex curve (not necessarily strictly convex) with continuous curvature, enclosing a region H, then one can verify that $v(P) = d(P, \Gamma)$ is regular (in C^2) in the exterior of H, and $|\operatorname{grad} v| = 1$.

We will not discuss the further geometric relations between Γ and G.

Finally, it should be pointed out that the assumptions in the theorem do *not* imply that $|\operatorname{grad} u|$ is constant in $R^2 - F$. This can be deduced from the example at the end of Section 4.

Theorem 9. *If $u(x, y) \in C^2(R^2)$ and $A(u) = 0$ in the whole plane, then*

$$u = Ax + By + C.$$

Proof. It follows from the previous theorem that $|\operatorname{grad} u|$ is constant. Hence the streamlines are non-intersecting straight lines, which means that they are parallel and thus $\operatorname{grad} u$ is constant. This completes the proof.

A consequence of this theorem is, for instance, that no polynomial of degree >1 can satisfy $A(u) = 0$ in any domain.

The same theorem also holds for the minimal surface equation, compare [2], p. 60.

8. An estimate for $|\operatorname{grad} u|$ in Ljapunov regions

Let $u(x, y)$ satisfy $A(u) = 0$ in a region D. Theorem 6 states that $\operatorname{grad} u \neq 0$ in D, unless $u =$ constant. This section treats the question whether $|\operatorname{grad} u|$ can be arbitrarily small near ∂D. It turns out that this cannot occur, if D is "smooth" and bounded. In the opposite case, it may happen that $\inf_D |\operatorname{grad} u| = 0$.

Lemma 4. *Let $\alpha(t)$ be defined a.e. on $T_0 < t < T_1$, and let it be bounded, positive, and measurable. Assume that*

$$\int_{T_0}^{T} \alpha(t) \frac{dt}{t} \leqslant C(\alpha(T))^p$$

for almost all $T \in (T_0, T_1)$. Here, $T_0 \geqslant 0$, T_1, $C > 0$ and $p > 1$ are constants. Then $T_0 > 0$ and[1]

$$\log \frac{T_1}{T_0} \leqslant \frac{pC}{p-1} \left(\operatorname{ess} \varliminf_{t \to T_1 - 0} \alpha(t) \right)^{p-1}.$$

Proof. Obviously, we may assume $T_0 > 0$. Define the function $\beta(t) \geqslant 0$ by

$$(\beta(T))^p = \frac{1}{C} \int_{T_0}^{T} \alpha(t) \frac{dt}{t}. \tag{1}$$

[1] $\operatorname{ess} \varliminf_{t \to T_1 - 0} \alpha(t) = \lim_{t' \to T_1 - 0} (\operatorname{ess} \inf_{t' < t < T_1} \alpha(t))$.

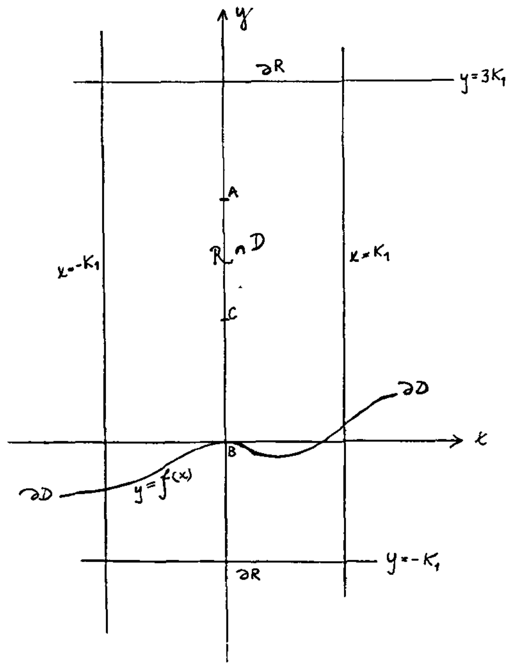

Fig. 9

Then $\beta(t)$ is increasing and $\beta(t)>0$ for $t>T_0$. Further $\beta(t)\leqslant\alpha(t)$ a.e. Differentiation of (1) gives

$$p\frac{d\beta}{dT}\cdot(\beta(T))^{p-1}=\frac{\alpha(T)}{CT}\geqslant\frac{\beta(T)}{CT}\ \text{a.e.}$$

Hence $pC\beta'(T)(\beta(T))^{p-2}\geqslant 1/T$, that is

$$\frac{d}{dT}\left(\frac{pC}{p-1}(\beta(T))^{p-1}\right)\geqslant\frac{d}{dT}(\log T)\ \text{a.e.}$$

Consequently,

$$\log\frac{T_1}{T_0}\leqslant\int_{T_0}^{T_1}\frac{d}{dT}\left(\frac{pC}{p-1}(\beta)^{p-1}\right)dT=\frac{pC}{p-1}\beta(T_1)^{p-1}$$

and since $\beta(T_1)\leqslant \operatorname{ess}\,\underline{\lim}_{t\to T_1-0}\alpha(t)$, we get the desired result.

Theorem 10. *Let $u(x, y)$ be a nonconstant solution of $A(u)=0$ in a bounded region D which satisfies the following conditions (see Fig. 9): There exist constants $K_1>0$, $K_2>0$ and λ, $0<\lambda\leqslant 1$, such that if B is an arbitrary point on ∂D, then there is a coordinate system with the origin at B such that the part of ∂D which lies in*

$$\bar{R}=\{(x, y)\mid |x|\leqslant K_1,\ -K_1\leqslant y\leqslant 3K_1\}$$

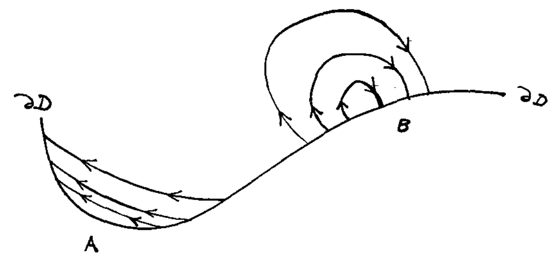

Fig. 10

can be represented as the graph of a function $y=f(x)\in C^1[-K_1, K_1]$ *for which* $f'(0)=0$ *and such that the angle between the tangents to* ∂D *at the points* $(x_1, f(x_1))$ *and* $(x_2, f(x_2))$ *is not greater than* $\min(K_2|x_1-x_2|^\lambda, \frac{1}{4}\pi)$, *finally, all points in* $\bar{R}$ *with* $y>f(x)$ *belong to* D, *and those with* $y<f(x)$ *do not belong to* D.[1] *Then* $|\operatorname{grad} u|$ *is bounded away from zero in* D, *and*

$$\log\frac{M_1}{m}\leqslant 21\pi+\frac{8K_2^2}{\lambda}(K_1)^{2\lambda},$$

where $m=\inf_D|\operatorname{grad} u|$ *and* $M_1=\inf|\operatorname{grad} u|$, *this infimum being taken over those points in* D *for which the distance to the boundary* ∂D *is at least* K_1.

This will be proved by the same type of estimates as those used in Theorem 5, but here we must study the behaviour of $|\operatorname{grad} u|$ at the boundary. Therefore, we cannot use any uniform lower bound for the length of a streamline, and we must find new estimates for the total curvature and for the measure of a set which is covered by "short" streamlines. Such estimates can be obtained using the convexity of the streamlines and the smoothness of ∂D. However, this can be done only if the streamline turns its convex side to ∂D, as suggested in Fig. 10, A. (Fig. 10, B, suggests a case that is to be avoided). Consequently, much attention must be paid to the position of the streamlines relative to ∂D.

Proof of the theorem

(1) Let B be an arbitrary point on ∂D, and introduce the corresponding coordinate system. Let A be the point $(0, 2K_1)$ and $R=\{(x,y)\,|\,|x|<K_1,\ -K_1<y<3K_1\}$. Write

$$M=|\operatorname{grad} u(A)|\geqslant \inf_{d(P,\partial D)\geqslant K_1}|\operatorname{grad} u(P)|=M_1,$$

and
$$m_1=\inf_{0<y<2K_1}|\operatorname{grad} u(0,y)|.$$

If $m_1=M$, there is nothing to prove. If $m_1<M$, take $\varepsilon>0$ such that $m_1+\varepsilon<M$, and let C be a point on the segment AB such that $|\operatorname{grad} u(C)|=m_1+\varepsilon$. Put $v(x,y)=|\operatorname{grad} u(x,y)|$. We will consider the function $v(y)=v(0,y)$ for $y_C\leqslant y\leqslant y_A=2K_1$.

According to Lemma 3[2] there is a finite or denumerable sequence of open intervals I_ν on $[y_C, y_A]$ such that

[1] A region satisfying these conditions is often called a Ljapunov region.

[2] Applied to $\varphi(t)=-v(-t)$.

(A) The intervals I_ν are pairwise disjoint.
(B) $dv/dy>0$ and $m_1+\varepsilon<v(y)<M$ on $U=\bigcup_\nu I_\nu$.
(C) If $I_\nu=(p_\nu, q_\nu)$, then $\Sigma_\nu(v(q_\nu)-v(p_\nu))=M-m_1-\varepsilon$.
(D) If $y_1\in U$ and $y_1<y_2\leqslant y_A$, then $v(y_1)<v(y_2)$.

For an arbitrary $y\in[y_C, y_A]$, let $\gamma(y)$ be the streamline of u passing through $(0, y)$, and we consider it only in $R\cap D$. Let it be continued up to the boundary of $R\cap D$ in both directions from $(0, y)$, and let $\alpha(y)$ be the length of $\gamma(y)$. The case $\alpha(y)=\infty$ is not yet excluded. However, we claim that $\alpha(y)$ is lower semicontinuous: $\alpha(y_0)\leqslant\underline{\lim}_{y\to y_0}\alpha(y)$. To see this, consider a streamline as a solution of a differential system

$$\frac{dx}{ds}=\frac{u_x}{|\text{grad } u|},$$

$$\frac{dy}{ds}=\frac{u_y}{|\text{grad } u|},$$

with initial conditions $x(s_0)=x_0$, $y(s_0)=y_0$. We know from ordinary differential equations ([9], p. 105) that the solution depends continuously on (x_0, y_0), which gives the result.

(2) Next, we claim that each streamline of u, considered in $R\cap D$, has total curvature $\leqslant 3\pi$ and finite length. Consider a streamline γ and assume that the total curvature is $>3\pi$. Then there are two successive vertical (parallel to the y-axis) tangents l_1, l_2, such that the curve $\bar\gamma$ between the points of tangency A, B, is convex downwards ($y''(s)>0$). (See Fig. 11). Let C, D be the points where l_1, l_2 intersect the line $y=3K_1$. Now $\bar\gamma$ and the segments AD, DC and CB enclose a convex domain $\Omega\subset R\cap D$, and $\bar\Omega\subset D$. The arcs of γ in Ω have finite length, since u is bounded in Ω. Therefore γ, continued beyond A and B, must have well-defined endpoints on $\partial\Omega$, and it follows easily from the convexity of γ that these endpoints are situated on CD. It also follows that the total curvature of the parts of γ between these endpoints and A, B cannot be greater than $\frac{1}{2}\pi$ for each.

Hence, the total curvature of γ in $R\cap D$ is not greater than 2π, contrary to our assumption that it is greater than 3π. So the total curvature of any streamline γ in $R\cap D$ is not greater than 3π, and it follows from this that the length of γ is finite. In particular, the function $\alpha(y)$ introduced above is always finite. Finally, every streamline in $R\cap D$ has well-defined endpoints on $\partial(R\cap D)$.

(3) Consider the set $U_0=\{y\,|\,y\in U,\ \alpha(y)<K_1\}$. (If this set is empty, then the estimates under (6) below are unnecessary.) We know that $\gamma(y)$ has well-defined endpoints on $\partial(R\cap D)=E_1\cup E_2\cup E_3\cup E_4$, where

$E_1=$ the graph of $y=f(x)$, for $-K_1\leqslant x\leqslant K_1$,

$E_2=\{x=-K_1,\ f(-K_1)<y<3K_1\}$,

$E_3=\{-K_1\leqslant x\leqslant K_1,\ y=3K_1\}$ and

$E_4=\{x=K_1,\ f(K_1)<y<3K_1\}$.

It is clear that if $y_0\in U_0$, then both endpoints of $\gamma(y_0)$ are situated on E_1. Let us study this case a little further. We know that $x'(s)\neq 0$ at $P=(0, y_0)$,[1] where $(x(s), y(s))$ are coordinates along $\gamma(y_0)$. We can assume that γ is oriented in such a way that

[1] This follows from $dv/dy\neq 0$ on $U\supset U_0$.

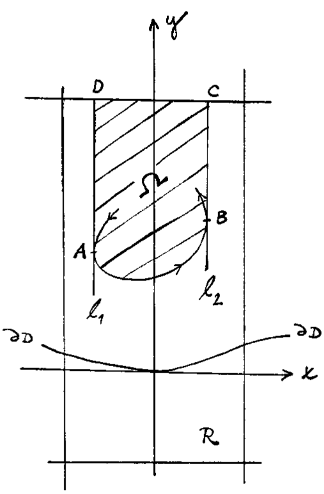

Fig. 11

$x'(s)>0$ at $(0, y_0)$. Now we want to show that *the endpoint for decreasing s is situated at a point* $(x_1, f(x_1))$ *with* $x_1<0$ *and the endpoint for increasing s is situated at* $(x_2, f(x_2))$ *for some* $x_2>0$. (See Fig. 12.) It is sufficient to prove the result for γ^+. Now v is constant along $\gamma(y_0)$, and thus it follows from property D of $\cup I_\nu$, that $\gamma^+(y_0)$ cannot meet the segment AP. Since $\alpha(y_0)<K_1$, and since the distance from any point on AD to E_1 is $>K_1$, it is clear that γ^+ cannot meet the segment AD(*). It remains to show that γ^+ cannot meet the (open) segment BP. But this follows from the convexity of $\gamma(y_0)$, if we note that $v'(y_0)>0$ implies that $\gamma(y_0)$ turns its concave side upwards at P.

This proves our assertion regarding the endpoints. Next, we will prove that *the whole curve* $\gamma(y_0)$ *can be represented by the graph of a function* $y=g(x)$, *where* $g(x)\in C^1[x_1, x_2]$, *and* $g(x_1)=f(x_1)$, $g(x_2)=f(x_2)$.

It is obvious that $\gamma(y_0)$ can be represented as $y=g(x)$ in a neighbourhood of $P=(0, y_0)$, and that $g'(x)$ is increasing. Let X be the greatest number, such that γ^+ can be represented by $y=g(x)$ on $(0, X)$. Obviously, we need only consider the case $\lim_{x\to X-0} g'(x)=+\infty$, $Y=\lim_{x\to X-0} g(x)>f(X)$. The means that γ^+ has a vertical tangent at $Q=(X, Y)$. But then the set $g(x)<y<3K_1$, $0<x<X$, is a domain Ω, containing $(x(s_Q+\varepsilon), y(s_Q+\varepsilon))$ for ε small enough, and γ^+ cannot meet the boundary of this domain (compare the reasoning above). But this contradicts the fact that the endpoint of γ^+ lies in the exterior of Ω. Hence $\lim_{x\to X-0} g'(x)<\infty$, and it is clear that $\lim_{x\to X-0} g(x)=f(X)$. Write $x_2=X$. The same reasoning as above leads to a number $x_1<0$ with analogous properties.

Now we claim that the statements regarding the endpoints of $\gamma(y_0)$ and the function representation of $\gamma(y_0)$ are valid also for $\gamma(y_0')$, if $y_0'\in U$ and $y_0'<y_0$. First of all, $\gamma(y_0')$ must be contained in $\Omega(y_0)=\{(x, y)\,|\,x_1<x<x_2, f(x)<y<g(x)\}$. Therefore, the endpoints of $\gamma(y_0')$ lie on E_1. Now, all the above arguments regarding $\gamma(y_0)$ apply to $\gamma(y_0')$ except one, which is labelled by (*). But that argument is not needed, in view of the inclusion $\gamma(y_0')\subset\Omega(y_0)$.

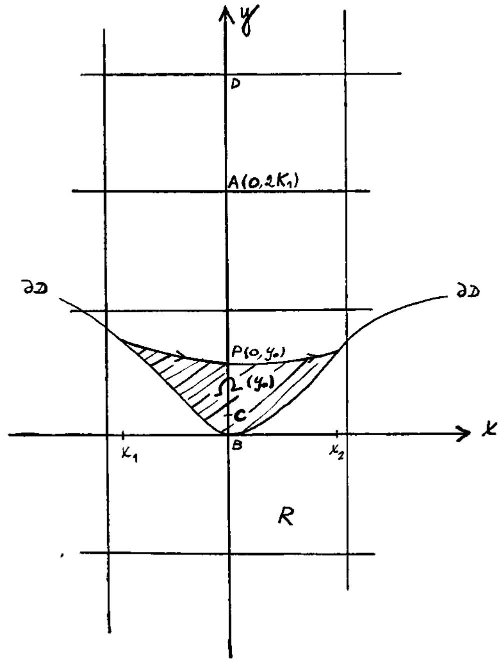

Fig. 12

Consequently, the above results are valid for $\gamma(y_0')$ if $y_0' \in \{y \mid y \in U, y < \sup U_0\}$. From now on, we denote by y_0 an arbitrary number in this set.

If we consider again the function representation of $\gamma(y_0)$, then we have

$$f'(x_1) \leqslant g'(x_1) < g'(x_2) \leqslant f'(x_2).$$

This means that the total curvature $\theta(y_0)$ of $\gamma(y_0)$ is at most $K_2(x_2 - x_1)^\lambda$, which gives

$$\theta(y_0) \leqslant K_2 \alpha(y_0)^\lambda.$$

Next, we want to estimate the measure of

$$\Omega(y_0) = \{(x, y) \mid x_1 < x < x_2, f(x) < y < g(x)\}.$$

Since $g''(x) > 0$, we have

$$f(x) \leqslant g(x) \leqslant g(x_1) + \frac{x - x_1}{x_2 - x_1}(g(x_2) - g(x_1)) = l(x).$$

Hence

$$m\Omega(y_0) = \int_{x_1}^{x_2} (g(x) - f(x))\,dx \leqslant \int_{x_1}^{x_2} (l(x) - f(x))\,dx \leqslant 2(x_2 - x_1) \max_{x_1 \leqslant x \leqslant x_2} |f(x)|.$$

But $f(0) = 0$, and if we write $\varphi = \operatorname{arctg}(f'(x))$, then

$$|f'(x)| = |tg\varphi| \leqslant 4/\pi \cdot |\varphi| \leqslant 4/\pi \cdot K_2 |x|^\lambda.$$

Hence $|f(x)| \leqslant [4/\pi(1+\lambda)] K_2 |x|^{1+\lambda}$. This gives

$$m\Omega(y_0) \leqslant \frac{8}{\pi} \cdot K_2(x_2 - x_1)^{2+\lambda},$$

and since

$$(x_2 - x_1) < \alpha(y_0) \text{ we get}$$

$$m\Omega(y_0) \leqslant \frac{8}{\pi} \cdot K_2(\alpha(y_0))^{2+\lambda}$$

(These are the estimates announced immediately after the statement of the theorem. Note that the situation is similar to that in Fig. 10, A.)

(4) Under point (1) we established a mapping $v = |\operatorname{grad} u(0, y)|$ from an open subset U of $y_C \leqslant y \leqslant y_A$ to an open subset V of $m_1 + \varepsilon \leqslant v \leqslant M$. The intervals in U and V are in pairwise correspondence, and $v(y)$ is an increasing function, considered on U. Further, $mV = M - m_1 - \varepsilon$. The set U_0 was introduced in point (3). Put $Y = \sup U_0$. If $Y \in U$, we remove it from U, and all the properties of U mentioned above remain true. Thus, in any case, Y will separate U into two parts, U_1 with $y > Y$ and U_2 with $y < Y$. At the same time, V is separated into $V_1 = v(U_1)$ with $v > v(Y)$ and $V_2 = v(U_2)$ with $v < v(Y)$. Clearly, $mV_1 = M - v(Y)$ and $mV_2 = v(Y) - m_1 - \varepsilon$.

(5) The next step is to estimate $\log [M/v(Y)]$. Let us write $U_1 = \bigcup_\nu I'_\nu$, and $I'_\nu = (a_\nu, b_\nu)$. Let G_ν be the set covered by $\gamma(y)$ for $y \in I'_\nu$. Clearly, G_ν is an open set, and hence measurable. As in the proof of Theorem 5, we change to coordinates (s, v), where s is arc length along $\gamma(y)$ ($s = 0$ at $(0, y)$) and $v = |\operatorname{grad} u|$. The set G_ν corresponds to a set of the form $v(a_\nu) < v < v(b_\nu)$, $\varphi(v) < s < \Psi(v)$. Evidently, $\Psi(v)$ is lower semicontinuous, $\varphi(v)$ is upper semicontinuous, and $\Psi(v) - \varphi(v) = \alpha(y)$, where $v = v(y)$. It follows from the semicontinuity that there are two sequences of step functions $\{\Psi_n(v)\}$, $\{\varphi_n(v)\}$, the first one increasing and the second decreasing, such that $\Psi_n(v) \nearrow \Psi(v)$ and $\varphi_n(v) \searrow \varphi(v)$ on $v(a_\nu) < v < v(b_\nu)$. Further, we may assume that φ_n and Ψ_n have the same points of discontinuity. Consider then a domain $G_{\nu,n}$ of the type $v(a_\nu) < v < v(b_\nu)$, $\varphi_n(v) < s < \Psi_n(v)$. From the reasoning in the proof of Theorem 5 it follows that

$$mG_{\nu,n} \geqslant \int_{v(a_\nu)}^{v(b_\nu)} \frac{(\Psi_n(v) - \varphi_n(v))^2}{v \cdot \theta} dv,$$

where θ is some fixed bound for the total curvature of each streamline in question. We have shown in point (2) that we can take $\theta = 3\pi$. Hence

$$mG_{\nu,n} \geqslant \frac{1}{3\pi} \int_{v(a_\nu)}^{v(b_\nu)} (\Psi_n(v) - \varphi_n(v))^2 \frac{dv}{v}.$$

But $mG_{\nu,n} \leqslant mG_\nu$, and a passage to the limit gives (by Levi's theorem)

$$mG_\nu \geqslant \frac{1}{3\pi} \int_{v(a_\nu)}^{v(b_\nu)} (\Psi(v) - \varphi(v))^2 \frac{dv}{v} = \frac{1}{3\pi} \int_{v(a_\nu)}^{v(b_\nu)} \alpha^2 \frac{dv}{v}.$$

According to the definition of U_0 and U_1, we have $\alpha(v) \geqslant K_1$ for $v \in U_1$. Hence

$$mG_\nu \geqslant \frac{K_1^2}{3\pi} \log \frac{v(b_\nu)}{v(a_\nu)}.$$

Since the sets G_ν are disjoint and contained in $R \cap D$ we get: $\Sigma_\nu\, mG_\nu \leqslant m(R \cap D) \leqslant 7K_1^2$. Finally, we arrive at

$$\log \frac{M}{v(Y)} = \sum_\nu \log \frac{v(b_\nu)}{v(a_\nu)} \leqslant 21\pi.$$

(6) The remaining difficulty is to estimate $\log [v(Y)/m_1+\varepsilon]$. This is done by means of the estimates that were derived under (3) above. There, we considered an arbitrary $y_0 \in U_2$ and proved that $\gamma(y_0)$ can be represented by the graph of $y=g(x) \in C^1[x_1, x_2]$, and $g(x_1)=f(x_1)$, $g(x_2)=f(x_2)$. We also estimated $\theta(y_0)$ and $m\Omega(y_0)$. Besides the set $\Omega(y_0)$, we introduce $\omega(y_0)$ = the set covered by all $\gamma(y)$, for $y \in U_2$ and $y<y_0$. Clearly, $\omega(y_0) \subset \Omega(y_0)$, and hence

$$m(\omega(y_0)) \leqslant \frac{8}{\pi} K_2(\alpha(y_0))^{2+\lambda}.$$

Let H_ν be the set covered by $\gamma(y)$ for $y \in I_\nu''$, where $I_\nu''=(c_\nu, d_\nu)$ is an open interval in $U_2 \cap \{y \mid y<y_0\}$. By a reasoning, similar to that used above, we have

$$mH_\nu \geqslant \int_{v(c_\nu)}^{v(d_\nu)} \frac{\alpha(v)^2}{v \cdot \theta(v)}\, dv.$$

Here, $\theta(v)$ is the total curvature of the corresponding streamline, and we have from (3): $\theta(v) \leqslant K_2 \alpha(v)^\lambda$. (Sometimes α, θ, etc. are written as functions of y and sometimes as functions of v; however, this should not cause any confusion.)

This gives

$$mH_\nu \geqslant \frac{1}{K_2} \int_{v(c_\nu)}^{v(d_\nu)} \alpha(v)^{2-\lambda} \frac{dv}{v}.$$

Now $\omega(y_1) = \cup H_\nu$, where the union is formed with all H_ν as described above. (H_ν corresponds to an interval in $U_2 \cap \{y \mid y<y_0\}$.) Since the sets H_ν are disjoint, we get

$$m(\bigcup H_\nu) \geqslant \frac{1}{K_2} \sum_\nu \int_{v(c_\nu)}^{v(d_\nu)} \alpha(v)^{2-\lambda} \frac{dv}{v}.$$

The intervals $(v(c_\nu), v(d_\nu))$ in V_2 cover $(m_1+\varepsilon, v(Y))$ except for a set of measure zero. Therefore, we can write

$$m(\omega(y_0)) \geqslant \frac{1}{K_2} \int_{m_1+\varepsilon}^{v_0} \alpha(v)^{2-\lambda} \frac{dv}{v}, \quad \text{with} \quad v_0 = v(y_0).$$

If we put $\beta(v) = \alpha(v)^{2-\lambda}$, we get

$$\int_{m_1+\varepsilon}^{v_0} \beta(v) \frac{dv}{v} \leqslant K_2 m(\omega(y_0)) \leqslant \frac{8}{\pi} K_2^2 (\beta(v_0))^{\frac{2+\lambda}{2-\lambda}}$$

for almost all v_0 on $(m_1+\varepsilon, v(Y))$. Further, $\alpha(y) < K_1\sqrt{2}$ for all $y \in U_2$. This is clear, since $|dy/dx| \leqslant 1$ on $\gamma(y)$ and $x_2 - x_1 < K_1$. Application of Lemma 4 then gives

$$\log \frac{v(Y)}{m_1+\varepsilon} \leqslant \frac{(2+\lambda)4K_2^2}{\pi\lambda}(K_1\sqrt{2})^{2\lambda}.$$

Simplification gives: $\log [v(Y)/m_1+\varepsilon] \leqslant (8K_2^2/\lambda)(K_1)^{2\lambda}$.

(7) Combination of our estimates gives, after making ε tend to zero,

$$\log \frac{M_1}{m_1} \leqslant 21\pi + \frac{8K_2^2}{\lambda}(K_1)^{2\lambda}.$$

Obviously, this holds also if we write $m = \inf_D |\text{grad } u|$ instead of m_1. This completes the proof of the theorem. (Phew.)

Corollary. *Suppose that the region D can be written as $D = \bigcup_1^N D_\nu$, where each D_ν satisfies the conditions of Theorem 10. Suppose also that u is a nonconstant solution of $A(u)=0$ in D. Then $|\text{grad } u|$ is bounded away from zero in D.*

Proof. This follows immediately from Theorem 10. The result means that Theorem 10 is extended to a class of (not all) domains with corners (or even cusps). However, in these cases, the angle of the corner (measured in D) is greater than π.

Remark. Roughly speaking, Theorem 10 says that for a nonconstant solution u on a smooth domain, $|\text{grad } u|$ is bounded away from zero. However, $|\text{grad } u|$ need not be bounded from above. This can be seen from the example $u = \text{arctg } (y/x)$ on $D : (x-1)^2 + y^2 < 1$. Further, there are domains with corners for which the theorem is not true. This is shown by the following example.

Example. Consider a Cauchy problem for $A(u)=0$. We write the equation as $u_{xx} = (-1/u_x^2)(2u_x u_y u_{xy} + u_y^2 u_{yy})$ and prescribe u, u_x on the positive y-axis by $u(0,y)=0$, $u_x(0,y)=y$.

Take an arbitrary $y_0 > 0$. According to the Cauchy–Kowalewski theorem there is a solution $u(x,y) = \sum_{m,n=0}^{\infty} a_{m,n} x^m (y-y_0)^n$, where the series converges normally in a neighbourhood U of $(0, y_0)$. We may choose U as a circle with its center at $(0, y_0)$. Now take an arbitrary y_1 on $(0, y_0)$. Put $\lambda = (y_1/y_0)\,(0<\lambda<1)$. By the transformation $x' = x/\lambda$, $y' = y/\lambda$, U is mapped onto a circle U_1 with center at $(0, y_1)$. (Fig. 13). The function $u_1(x,y) = \lambda^2 u(x/\lambda, y/\lambda)$ is analytic in U_1, satisfies $A(u_1)=0$, and we have

$$\left(\frac{\partial u_1}{\partial x}\right)_{x=0} = \lambda^2 u_1\left(0, \frac{y}{\lambda}\right)\cdot\frac{1}{\lambda} = \lambda^2\cdot\frac{y}{\lambda}\cdot\frac{1}{\lambda} = y.$$

Hence, $u_1(x,y)$ is a solution of the same Cauchy problem.

If y_1 is allowed to vary over the interval $0 < y_1 < y_0$, then the corresponding circles U_1 will form a domain with a corner at the origin. If we can show that two function elements of this type agree on their common domain of definition, then we have the desired counterexample, since $\lim_{y\to+0} |\text{grad } u(0,y)| = 0$.

Consider then two function elements (u_1, U_1) and $(\tilde{u}_1, \tilde{U}_1)$. Clearly, all derivatives of u_1 and $\tilde{u}_1$ agree on the part of the y-axis where both functions are defined, and now the result follows from the uniqueness of analytic continuation.

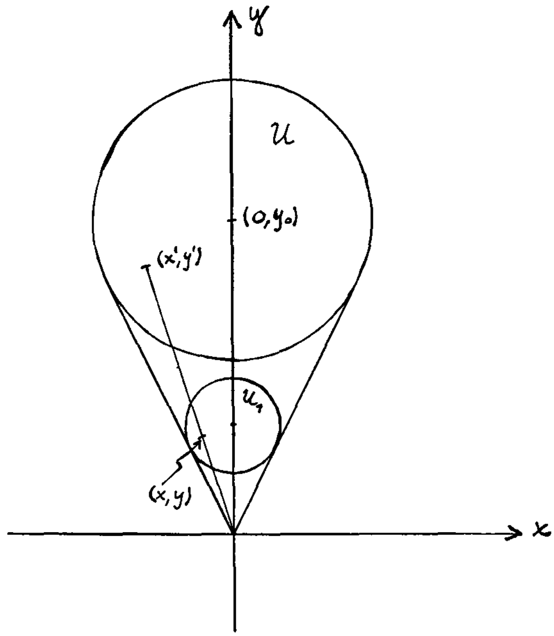

Fig. 13

9. A theorem on the boundary value problem and some consequences

This section treats the Dirichlet problem for our differential equation $A(u) = 0$. The discussion is admittedly very incomplete. To summarize, it is proved that a solution of the Dirichlet problem is unique, if there is any, and a few instances, where there are no solutions, are also described.

Theorem 11. *Let D be a bounded domain and let $u(x, y)$, $\tilde{u}(x, y)$ be functions in $C^2(D) \cap C(\bar{D})$. Assume that $A(u) = A(\tilde{u}) = 0$ in D. Then we have, for $(x, y) \in D$,*

$$\min_{\partial D} (u - \tilde{u}) \leqslant u(x,y) - \tilde{u}(x,y) \leqslant \max_{\partial D} (u - \tilde{u}).$$

Proof. Clearly, it is sufficient to prove that

$$\max_{\bar{D}} |u - \tilde{u}| = \max_{\partial D} |u - \tilde{u}|.$$

Assume then that $\max_{\bar{D}} |u - \tilde{u}|$ is taken at $P \in D$. Then $(\operatorname{grad} u)_P = (\operatorname{grad} \tilde{u})_P$, and according to Theorem 6, we may assume that both are non-zero. Consider then the case where $\operatorname{grad} (|\operatorname{grad} u|) = \operatorname{grad} (|\operatorname{grad} \tilde{u}|) = 0$ at P. Here, u and $\tilde{u}$ have the same streamline through P. It is a straight line, and we have $u - \tilde{u} = \text{constant}$ along this line. Since this line must meet ∂D, the result is proved in this case.

Therefore we may assume, for example, that $\operatorname{grad} (|\operatorname{grad} u|) \neq 0$ at P. This means that the streamlines of u in a neighbourhood of P belong to the curved type and also that $u \in C^\infty$ near P.

Now we employ the well-known method of deriving a differential equation satisfied by $\omega = u - \tilde{u}$. Compare [2], p. 54–55.

$$u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0,$$
$$\tilde{u}_x^2 \tilde{u}_{xx} + 2\tilde{u}_x \tilde{u}_y \tilde{u}_{xy} + \tilde{u}_y^2 \tilde{u}_{yy} = 0.$$

Write $A = u_x^2$, $B = u_x u_y$, $C = u_y^2$, $\tilde{A} = \tilde{u}_x^2$, etc. Subtraction then gives

$$A(u_{xx} - \tilde{u}_{xx}) + 2B(u_{xy} - \tilde{u}_{xy}) + C(u_{yy} - \tilde{u}_{yy}) + \tilde{u}_{xx}(A - \tilde{A}) + 2\tilde{u}_{xy}(B - \tilde{B}) + \tilde{u}_{yy}(C - \tilde{C}) = 0.$$

Here,
$$\begin{aligned} A - \tilde{A} &= (u_x - \tilde{u}_x)(u_x + \tilde{u}_x) = \omega_x(u_x + \tilde{u}_x), \\ B - \tilde{B} &= (u_x u_y - \tilde{u}_x u_y) + (\tilde{u}_x u_y - \tilde{u}_x \tilde{u}_y) = \omega_x u_y + \omega_y \tilde{u}_x, \\ C - \tilde{C} &= \dots = \omega_y(u_y + \tilde{u}_y). \end{aligned}$$

We get the equation

$$A\omega_{xx} + 2B\omega_{xy} + C\omega_{yy} + D\omega_x + E\omega_y = 0, \tag{1}$$

where
$$D = \tilde{u}_{xx}(u_x + \tilde{u}_x) + 2\tilde{u}_{xy} u_y$$

and
$$E = 2\tilde{u}_{xy}\tilde{u}_x + \tilde{u}_{yy}(u_y + \tilde{u}_y).$$

Now, we transform the equation (1) to canonical form in a neighbourhood of P (Compare [8], p. 49). We change to new coordinates (ξ, η), and assume that the mapping $(x, y) \to (\xi, \eta)$ is one-to-one and in C^2, as well as the inverse. To obtain the canonical form, we assume that $u_x \xi_x + u_y \xi_y = 0$. Such a function $\xi(x, y)$ is, for example, $|\text{grad}\, u|$. The equation (1) will then be transformed into (we omit the calculations),

$$\frac{\partial^2 \omega}{\partial \eta^2}(\nabla u \cdot \nabla \eta)^2 + \frac{\partial \omega}{\partial \xi}(A\xi_{xx} + 2B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y)$$
$$+ \frac{\partial \omega}{\partial \eta}(A\eta_{xx} + 2B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y) = 0. \tag{2}$$

Here, $\nabla u = (u_x, u_y)$, $\nabla \eta = (\eta_x, \eta_y)$. For $\eta(x, y)$, we can take a linear function $\eta = \alpha x + \beta y$, for which $\alpha u_x(P) + \beta u_y(P) \neq 0$, or, we can also take $\eta = u(x, y)$. Now we must analyse the coefficient for $\partial\omega/\partial\xi$ in (2), evaluated at the point P. From $u_x \xi_x + u_y \xi_y = 0$, we get

$$\begin{aligned} u_x \xi_{xx} + u_y \xi_{xy} &= -(u_{xx}\xi_x + u_{xy}\xi_y), \\ u_x \xi_{xy} + u_y \xi_{yy} &= -(u_{xy}\xi_x + u_{yy}\xi_y). \end{aligned}$$

Multiplication by u_x, u_y, respectively and addition gives

$$A\xi_{xx} + 2B\xi_{xy} + C\xi_{yy} \equiv u_x^2 \xi_{xx} + 2u_x u_y \xi_{xy} + u_y^2 \xi_{yy}$$
$$= -(u_{xx}\xi_x u_x + u_{xy}\xi_y u_x + u_{xy}\xi_x u_y + u_{yy}\xi_y u_y).$$

Since $\text{grad}\, u$ and $\text{grad}\, \xi$ are orthogonal, we may write $\xi_x = \lambda u_y$, $\xi_y = -\lambda u_x$, with $\lambda = \pm |\text{grad}\, \xi| / |\text{grad}\, u|$. Then the above expression is reduced to

$$\lambda(u_{xy}u_x^2 - u_{xy}u_y^2 + u_{yy}u_x u_y - u_{xx}u_x u_y).$$

At P, $\text{grad}\, u = \text{grad}\, \tilde{u}$, and we obtain

$$D\xi_x + E\xi_y = \lambda[Du_y - Eu_x] = 2\lambda(\tilde{u}_{xx}\tilde{u}_x\tilde{u}_y + \tilde{u}_{xy}\tilde{u}_y^2 - \tilde{u}_{xy}\tilde{u}_x^2 - \tilde{u}_{yy}\tilde{u}_x\tilde{u}_y),$$

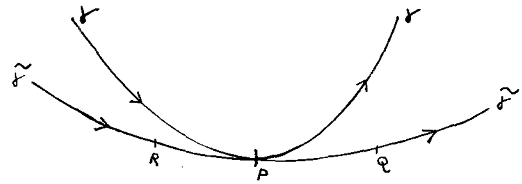

Fig. 14

which is the same expression as above, except that u is replaced by $\tilde{u}$ and it is multiplied by (-2). With the aid of the formulas for the curvature of a streamline of u or $\tilde{u}$, derived in Section 2, we get

$$A\xi_{xx}+2B\xi_{xy}+C\xi_{yy}+D\xi_x+E\xi_y=\lambda\left(|\text{grad } u|^3\frac{d\theta}{ds}-2|\text{grad } \tilde{u}|^3\frac{d\tilde{\theta}}{d\tilde{s}}\right)$$

$$=\lambda|\text{grad } u|^3\left(\frac{d\theta}{ds}-2\frac{d\tilde{\theta}}{d\tilde{s}}\right),\quad \text{which holds at } P.$$

Here $d\theta/ds$, $d\tilde{\theta}/d\tilde{s}$, are the curvatures of the corresponding streamlines.

Next, we claim that this expression is not zero.

To see that, assume that $d\theta/ds=2(d\tilde{\theta}/d\tilde{s})\neq 0$. Then the streamlines of u, $\tilde{u}$ have the same tangent vectors and parallel curvature vectors at P (See Fig. 14).

Put $M=|\text{grad } u(P)|=|\text{grad } \tilde{u}(P)|$. Along the "open" arcs RP, PQ, we have $d\tilde{u}/d\tilde{s}=M$, $du/d\tilde{s}\leqslant|\text{grad } u|<M$, and $(d/d\tilde{s})(u-\tilde{u})<0$, if R, Q are sufficiently close to P. However, this contradicts the fact that $(u-\tilde{u})$ has a local extremum at P.

This proves that the coefficient of $\partial\omega/\partial\xi$ in (2) is $\neq 0$ in a neighbourhood of P. Now we can choose the function $\xi(x, y)$ and the sign of λ such that the coefficient of $\partial\omega/\partial\xi$ is negative. Then the equation (2) takes the form

$$F(\xi,\eta)\frac{\partial^2\omega}{\partial\eta^2}-\frac{\partial\omega}{\partial\xi}+G(\xi,\eta)\frac{\partial\omega}{\partial\eta}=0, \tag{3}$$

where the functions $F(\xi,\eta)$, $G(\xi,\eta)$ are continuous and $F(\xi,\eta)>0$.

In the (ξ,η)-plane we consider a rectangle $R: \xi_P-\delta\leqslant\xi\leqslant\xi_P$, $\eta_P-\delta\leqslant\eta\leqslant\eta_P+\delta$. At the point (ξ_P,η_P), ω takes an extremum which is a positive maximum or a negative minimum. From the maximum principle for parabolic equations ([5], p. 34, Theorem 1) we infer that $\omega=\text{constant}$ in R.

Consider the streamline γ of u through P. It belongs to the curved type and it follows from the above reasoning that the subset of γ, where $|\omega|$ takes its maximum, is open. But this set is also closed (possibly after addition of the endpoints of γ). This proves that $\omega=\text{constant}$ on γ, which completes the proof.

Clearly, this theorem contains the two-dimensional version of Theorem 9 in [1].

Theorem 12. *In a bounded domain, there is at most one solution of Dirichlet's problem for* $A(u)=0$.

Proof. This is a consequence of the previous theorem.

With the aid of this uniqueness theorem and Theorem 6, we can easily construct examples for which the Dirichlet problem has no solution.

Theorem 13. *Consider the Dirichlet problem for $A(u)=0$ on a bounded domain D and with given continuous boundary values $\varphi(x, y)$. Suppose that D is symmetric with respect to the origin (that is: $(x, y) \in D$ whenever $(-x, -y) \in D$), and suppose that the origin belongs to D. Finally, we assume that $\varphi(x, y)=\varphi(-x, -y)$ for all $(x, y) \in \partial D$ and that $\varphi \not\equiv$ constant.*

Then the Dirichlet problem has no (classical) solution.

Proof. If there was a solution $u(x, y)$, then $u(-x, -y)$ would also be a solution. From the preceding theorem it follows that $u(x, y) \equiv u(-x, -y)$. Differentiate with respect to x and then put $x=y=0$. This gives $u_x(0, 0)=0$, and in the same way it follows that $u_y(0, 0)=0$.

Hence the origin is a stationary point for u, which means that $u(x, y) \equiv$ constant. This contradicts $\varphi \not\equiv$ constant, and the theorem is proved.

Example. Let D be a circle with its center at the origin, and let $\varphi(x, y)=x^{2m+1} y^{2n+1}$, where m, n are non-negative integers. Then our Dirichlet problem has no solution. The same is true for $\varphi(x, y)=x^{2m} y^{2n}$, if m, n are non-negative integers and $m+n>0$.

We mention another result of the same type:

Theorem 14. *Let D be symmetric with respect to the y-axis and let $(0, y_1)$, $(0, y_2)$ be two points on ∂D such that $(0, y) \in D$ for $y_1<y<y_2$. Further, let $\varphi(x, y)=\varphi(-x, y)$, $\varphi(0, y_1)=\varphi(0, y_2)$ and $\varphi \not\equiv$ constant.*

Then the Dirichlet problem for $A(u)=0$ has no solution.

Proof. If there was a solution $u(x, y)$, then $u(-x, y)$ would also be a solution, which means that $u(x, y) \equiv u(-x, y)$. This gives $u_x(0, y)=0$. Hence $u_y(0, y) \neq 0$, and this contradicts $\varphi(0, y_1)=\varphi(0, y_2)$.

In connection with the boundary value problem, it should be mentioned also that Theorems 1, 2 and 7 in [1] give some information on a possible solution of that problem.

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