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Real algebras with a Hilbert space structure

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1. Introduction

In the theory of Banach algebras, it would seem natural to give special attention to such algebras that are also Hilbert spaces. With one notable exception (the H^*-theory, [1]) there are rather few interesting results known along these lines. This must, at least in part, be due to the fact that if one assumes identity and the conventional axioms for a normed algebra, there are only trivial realizations [2]. Some further general remarks on algebras with a Hilbert space structure are given in this paper (sec. 3) and in [4].

The now classical paper by W. Ambrose [1] established very definitive structural results for complex H^*-algebras. The interest in these was chiefly motivated by the L^2-algebras of a locally compact group. The main objective of this paper is to give a structure theory for real H^*-algebras. It is shown that a real H^*-algebra with sufficiently non-degenerate multiplication is the Hilbert space direct sum of matrix algebras, each consisting of all matrices with real, complex or quaternion entries and whose sums of squares of the absolute values of elements are finite (sec. 4).

It is not obvious that the complex H^*-result should extend to the case of real scalars (cf. the related case of B^*-algebras [3, p. 265], which does not). The fact that it does makes it possible to weaken the assumptions of a complex H^*-algebra, so that relations are required to hold essentially only for "real parts" (Theorem 4.3). We have not found reason to extend the closely related theories of [9] and [10] to the real case. With the H^*-theory known, however, this should be easy.

The most interesting consequence of the result is that the real L^2-algebra of a compact group is the Hilbert space direct sum of finite-dimensional algebras, each consisting of all matrices with either real, complex or quaternion entries (Theorem 5.1). In this connection it is also pointed out that real group algebras in general give rather more structural information than the corresponding complex ones.

2. Preliminaries on H^*-algebras

Let A be an algebra over the real numbers (R). An *involution* on A is a linear map $x \to x^*$, of A into A, satisfying $x^{**}=x$ and $(xy)^*=y^*x^*$. An algebra with involution (*-algebra) is called *proper* if $x^*x \neq 0$ whenever $x \neq 0$.

An H^*-*algebra* is a *-algebra that is also a Hilbert space and in which involution, multiplication and inner product are linked to each other by the identity

$$(xy, z) = (y, x^*z) = (x, zy^*). \tag{1}$$

It is not assumed that either the algebra multiplication or the involution are continuous in the topology defined by the Hilbert space norm. We will notice that both these are in fact consequences of (1). The following result, which is valid for complex as well as real scalars, has been obtained by Saworotnow [8] (cf. also [6]).

Theorem 2.1. *Let A be a Hilbert space which is also an associative algebra. On A are defined two functions, $x \to x'$, $x \to x''$, so that*

$$(xy, z) = (y, x'z) = (x, zy'')$$

for all x, y, z. Then multiplication on A is continuous.

Proof. The uniform boundedness principle shows that multiplication with a fixed element is continuous. But in a Banach space, this implies (simultaneous) continuity of multiplication.

Corollary 2.2. *In an H^*-algebra, multiplication is continuous.*

Hence, if necessary after adjusting by a constant, we can assume

$$\|xy\| \leqslant \|x\| \cdot \|y\|.$$

Granted this Corollary, we have [1, Theorem 2.3]:

Theorem 2.3. *In a proper H^*-algebra $\|x\| = \|x^*\|$ and so involution is continuous.*

It is clear that the real numbers (R) themselves form an H^*-algebra over R. Together with R, the complex numbers (C) and the quaternion algebra (Q) are building blocks in the theory. For Q we denote a typical element $\alpha + \bar{u}$, $\alpha \in R$, $\bar{u} \in R^3$. Then 1 ($= 1 + 0$) is identity element and the product of two "vectors" $\bar{u}$, $\bar{v}$ is defined

$$\bar{u}\bar{v} = -\langle \bar{u}, \bar{v} \rangle + \bar{u} \times \bar{v};$$

here the products are respectively the usual inner product on R^3 and the vector cross-product.

We make some elementary observations regarding C and Q.

Proposition 2.4. *The only involutions on C are the identity map and complex conjugation. Only the latter makes C a real H^*-algebra.*

Proposition 2.5. *The involutions on Q are all of the form $\alpha + \bar{u} \to \alpha + T\bar{u}$ where T is either a reflexion in a plane through the origin or minus the identity. Only the latter makes Q a real H^*-algebra.*

The verifications are left to the reader.

From here on we let R, C and Q carry with them the usual inner product and the H^*-involution. For Γ any given set and $K = R$, C or Q define

$$l^2(\Gamma, K) = \{f; f: \Gamma \to K, \sum_{\alpha \in \Gamma} |f(\alpha)|^2 < \infty\},$$

which is a Hilbert space of functions, with the inner product

$$(f, g) = \tfrac{1}{2} \sum_{\gamma \in \Gamma} f^*(\gamma)\, g(\gamma) + g^*(\gamma) f(\gamma).$$

If Γ is a set of pairs, $\Gamma = \Lambda \times \Lambda$, $l^2(\Lambda, K)$ can be made into an algebra by defining "matrix" multiplication

$$fg(\alpha, \beta) = \sum_{\gamma \in \Lambda} f(\alpha, \gamma)\, g(\gamma, \beta).$$

With involution $f \to f^*$ defined by

$$f^*(\alpha, \beta) = (f(\beta, \alpha))^*,$$

it is not hard to verify that $l^2(\Lambda \times \Lambda, K)$ is an H^*-algebra. We will prove that every real H^*-algebra is a Hilbert space direct sum of algebras of this type.

3. Hilbert spaces as algebras

It is well known that any real Hilbert space H is congruent to $l^2(\Lambda, R)$, where Λ is a maximal orthonormal set in H. But for every infinite cardinal number $\varkappa$ it is true that $\varkappa^2 = \varkappa$, in other words, every infinite set Σ is in one-to-one correspondence with $\Sigma \times \Sigma$. Hence any Hilbert space of infinite dimension is congruent to $l^2(\Lambda \times \Lambda, R)$. But then it can be given the "matrix" multiplication of sec. 2 and we have

Proposition 3.1. *Any Hilbert space of infinite dimension can be given a multiplication that makes it a topologically simple (i.e. without closed two-sided ideals) associative Banach algebra, the norm satisfying* $\|xy\| \leqslant \|x\| \cdot \|y\|$.

In the direction of this observation it is natural to ask what Banach spaces can be given an associative multiplication which makes them topologically simple Banach algebras. We have shown that this is possible for all Hilbert spaces of infinite dimension; it is quite clear, however, that these are not the only ones, and that it is not possible for all Banach spaces. In finite dimension it follows from the Wedderburn structure theory that the only possible dimensionalities and the corresponding algebras are

$$\begin{array}{lll} n^2 \begin{cases} n \text{ odd} & \mathfrak{M}_n(R) \\ n \text{ even} & \mathfrak{M}_n(R) \quad \text{and} \quad \mathfrak{M}_{n/2}(Q) \end{cases} \\ 2n^2 \qquad\qquad\quad \mathfrak{M}_n(C). \end{array}$$

Here n is an integer and $\mathfrak{M}_n(K)$ the algebra of all $n \times n$ matrices with entries from K. (For $K = C$ the only possibility is dimension n^2 and the algebra $\mathfrak{M}_n(C)$.)

In a slightly different direction we can ask for Hilbert spaces H that can be made into topological (not necessarily associative) algebras H_A *with identity*, satisfying certain additional assumptions and whose norm satisfies

$$\|e\| = 1,$$

$$\|xy\| \leqslant \theta \|x\| \cdot \|y\|.$$

For $\theta = 1$ the problem is completely solved in [2] and [4]: if alternativity is assumed there are just four non-isomorphic cases (R, C, Q and the Cayley algebra D), but if no assumption stronger than power associativity is made, every infinite-dimensional Hilbert space is possible.

For H_A associative and $\theta > 2/\sqrt{3}$ again every H of infinite dimension is eligible. First notice that $H \cong R \oplus H'$ and introduce a multiplication on H' according to Proposition 3.1. Then we regard H as H' with "adjoined identity"; its norm is given by

$$|||\alpha + a|||^2 = \alpha^2 + \eta^2 \|a\|^2, \quad a \in H', \quad \eta \geqslant 1.$$

We notice that $\max_{0 \leqslant \xi,\, \eta \leqslant 1} \xi + \eta - \xi\eta + 2\sqrt{\xi - \xi^2}\sqrt{\eta - \eta^2} = \frac{4}{3}$.

For $x = \alpha + a$, $y = \beta + b$ we have $xy = \alpha\beta + \alpha b + \beta a + ab$ and

$$\begin{aligned} |||xy|||^2 &= \alpha^2\beta^2 + \eta^2 \|\alpha b + \beta a + ab\|^2 \leqslant \\ &\leqslant \alpha^2\beta^2 + \eta^2(\alpha^2\|b\|^2 + \beta^2\|a\|^2 + 2|\alpha\beta|\,\|a\|\,\|b\|) + \\ &\qquad + \eta^2\|ab\|\,(\|ab\| + 2|\alpha|\,\|b\| + 2|\beta|\,\|a\|) \leqslant \\ &\leqslant \left(\frac{4}{3} + \frac{5}{\eta}\right) |||x|||^2 \cdot |||y|||^2 \leqslant \theta^2 |||x|||^2\, |||y|||^2 \end{aligned}$$

if η is chosen big enough.

Other examples in the same direction are found in [4].

4. Structure of H^*-algebras

The Ambrose structure theory of complex H^*-algebras is given in two steps, the first of which goes over without change to the real case [1, Theorem 4.1]:

A proper H^*-algebra is the Hilbert space direct sum of its minimal closed two-sided ideals.

Hence we can concentrate on the structure of H^*-algebras that are topologically simple (i.e. have no closed two-sided ideals). The key notion is that of a self-adjoint idempotent (*sai*). A *sai* e is called *reducible* if there exist *sai*'s $e_1, e_2 \neq 0$ so that $e = e_1 + e_2$; otherwise e is called *irreducible*.

Lemma 4.1. *Let e be an irreducible sai in a proper real H^*-algebra A. Then eAe is isomorphic to R, C or Q.*

Proof. If $0 \neq x \in eAe$ then $Ax \subset Ae$ and $Ax = Ae$ since Ae is a minimal left ideal. Hence there exists a $y \in A$ such that $yx = e$. But $(eye)(exe) = e$, and eAe is a division algebra. Since the only real normed division algebras are R, C and Q the conclusion follows.

For the formulation of the main theorem, we let $\{e_\alpha\}_{\alpha \in \Lambda}$ be a maximal collection of mutually orthogonal irreducible *sai*'s. The algebra $l^2(\Lambda \times \Lambda, K)$ is defined in sec. 2.

Theorem 4.2. *A topologically simple proper real H^*-algebra is homeomorphically *-isomorphic to the algebra $l^2(\Lambda\times\Lambda, K)$ with $K=R$, C or Q.*

Proof. Λ is not empty [1, Th. 3.1] and we let e_0 denote a certain element of Λ. Then $A_0=e_0Ae_0$ is isomorphic to R, C or Q according to Lemma 4.1. Take an arbitrary $\alpha\in\Lambda$. The linear span of $Ae_\alpha A$ is an ideal in A, hence it is dense in A since A is topologically simple. It then follows that $(e_0Ae_\alpha)(e_0Ae_\alpha)^*=e_0(Ae_\alpha A)e_0$ is non-zero and e_0Ae_α contains an element $\neq 0$, say $e_{0\alpha}$. But $e_{0\alpha}e_{0\alpha}^*$ is a self-adjoint element of A_0 and since A_0 is isomorphic to R, C or Q it follows from Proposition 2.4 or 2.5 that $e_{0\alpha}e_{0\alpha}^*$ is a positive multiple of e_0, we can take

$$e_{0\alpha}\,e_{0\alpha}^*=e_0.$$

We now define $e_{\alpha 0}=e_{0\alpha}^*$ and $e_{\alpha\beta}=e_{\alpha 0}e_{0\beta}$ and have ($\delta_{\alpha\beta}$ is the Kronecker δ-symbol)

$$\begin{aligned}&e_{\alpha\beta}=e_{\beta\alpha},\\&e_{\alpha\beta}e_{\gamma\delta}=\delta_{\beta\gamma}e_{\alpha\beta},\\&e_{\alpha\alpha}=e_\alpha,\\&(e_{\alpha\beta},e_{\gamma\delta})=\delta_{\alpha\gamma}\,\delta_{\beta\delta}\|e_0\|^2.\end{aligned}$$

Let $S_{\alpha\beta}=e_\alpha Ae_\beta$. For every pair α,β the vector space $S_{\alpha\beta}$ is homeomorphically isomorphic to $A_0=S_{00}$ under the map

$$h_{\alpha\beta}\colon x\to e_{0\alpha}x\,e_{\beta 0}$$

For any $x\in A$ we have [1, Th. 4.1].

$$x=\sum_\alpha e_\alpha\,x=\sum_\beta x\,e_\beta=\sum_{\alpha,\beta}e_\alpha\,x\,e_\beta$$

and since all $S_{\alpha\beta}$ are orthogonal

$$\|x\|^2=\sum_{\alpha,\beta}\|e_\alpha\,x\,e_\beta\|^2.$$

Let $k\colon A_0\to K$ ($K=R$, C or Q) be the mapping of Lemma 4.1 and take

$$x_{\alpha\beta}=k(h_{\alpha\beta}(e_\alpha\,x\,e_\beta)).$$

The mapping of the theorem is

$$h\colon x\to(x_{\alpha\beta})$$

and we first demonstrate that h maps into $l^2(\Lambda\times\Lambda,K)$ and is continuous (we assume for simplicity that $\|xy\|\leqslant\|x\|\,\|y\|$, cf. Corollary 2.2):

$$\begin{aligned}\|h(x)\|^2=\sum|x_{\alpha\beta}|^2&\leqslant\sum\|k\|^2\,\|h_{\alpha\beta}\|^2\,\|e_\alpha\,x\,e_\beta\|^2\\&\leqslant\|k\|^2\,\|e_0\|^2\sum\|e_\alpha\,x\,e_\beta\|^2=\|k\|^2\,\|e_0\|^4\,\|x\|^2.\end{aligned}$$

In the same manner we can show that h is an isomorphism and has continuous inverse. It is routine to verify that h is a real *-algebra homomorphism, and so the theorem is proved.

The fact that a successful H^* structure theory exists for real scalars makes it possible to weaken somewhat the assumptions of the complex theory.

Theorem 4.3. *Let A be a complex Hilbert space which is also a complex associative algebra. On A there is defined a mapping $x \to x^*$ which satisfies*

$$(x+y)^* = x^* + y^*,$$

$$(\alpha x)^* = \alpha x^* \quad \textit{for real scalars } \alpha,$$

$$x^{**} = x,$$

$$(xy)^* = y^* x^*,$$

$$\mathrm{Re}\,(xy, z) = \mathrm{Re}\,(y, x^*z) = \mathrm{Re}\,(x, zy^*).$$

If A is proper it is the Hilbert space direct sum of algebras $l^2(\Lambda \times \Lambda, C)$.

Proof. By restricting scalar multiplication to R and defining $\langle x, y\rangle = \mathrm{Re}\,(x, y)$ A becomes a real H^*-algebra. The conclusion then follows from Theorem 4.2 and the fact that the possibility of reextension to C of scalar multiplication on $l^2(\Lambda \times \Lambda, K)$ rules out all but $K = C$.

In direct analogy with [1, Corr. 4.1] we also have

Corollary 4.4. *An abelian proper real H^*-algebra is the Hilbert space direct sum of real and complex fields.*

5. On the real L^2-algebra of a compact group

The main importance of the Ambrose H^*-theory is that it yields a complete structure theory for complex L^2-algebras of a compact group G, i.e. the square-integrable (in the Haar measure) complex functions with convolution multiplication, $L^2_C(G)$. For the corresponding algebra of real-valued functions, $L^2_R(G)$, we easily verify that these are real H^*-algebras and get (cf. [5, p. 158]):

Theorem 5.1. *For a compact group G, the algebra $L^2_R(G)$ is homeomorphically isomorphic to the Hilbert space direct sum of finite-dimensional full matrix algebras with real, complex or quaternion entries.*

As is fairly widely realized, the real group algebras can yield more structural information about the group than the complex. This is seen already in finite, low orders. Let Z_k denote the cyclic group of order k.

Example 1. We have

$$L^2_R(Z_2 \oplus Z_2) \simeq R^4, \quad L^2_C(Z_2 \oplus Z_2) \simeq C^4,$$

$$L^2_R(Z_4) \simeq R^2 \oplus C, \quad L^2_C(Z_4) \simeq C^4,$$

i.e. $Z_2 \oplus Z_2$ and Z_4 can be distinguished by their real, but not by their complex, group algebras.

It is true, however, that $L^2_C(G)$ is the complexification [7, p. 6] of $L^2_R(G)$, which illustrates the (sometimes overlooked) fact that a good deal of interesting structure may vanish in complexification. This is even more striking in the non-commutative L^2-theory. The building-blocks of Theorem 5.1 are thus transformed in complexification:

$$\begin{aligned} \mathfrak{M}_n(R) &\to \mathfrak{M}_n(C), \\ \mathfrak{M}_n(C) &\to \mathfrak{M}_n(C) \oplus \mathfrak{M}_n(C), \\ \mathfrak{M}_n(Q) &\to \mathfrak{M}_{2n}(C). \end{aligned}$$

The quaternions, which are a new but not wholly unexpected feature of the real theory, show up already in finite order.

Example 2. Let Q_8 be the quaternion group, i.e. $\{\pm e, \pm i, \pm j, \pm k\}$ where e, i, j, k are the basis elements of the quaternion algebra. Then

$$L^2_R(Q_8) \simeq R^4 \oplus Q,$$

whereas

$$L^2_C(Q_8) \simeq C^4 \oplus \mathfrak{M}_2(C).$$

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