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Minimization problems for the functional $\sup_x F(x, f(x), f'(x))$ III

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The present paper is a continuation of the papers [1] and [2]. These papers treat the problem of minimizing the functional

$$H(f) = \sup_x F(x, f(x), f'(x))$$

over the class $\mathfrak{J}$ of all absolutely continuous functions $f(x)$ which satisfy the boundary conditions $f(x_1) = y_1$ and $f(x_2) = y_2$. The discussion in [1] and [2] is mainly concerned with the existence and the properties of absolutely minimizing functions (defined in [1], p. 45) and unique minimizing functions. The question of the existence of a minimizing function is also treated in [2] and it is shown by an example ([2], p. 429) that a minimizing function *in general* need not have any of the properties proved for a.s. minimals ([2], Theorem 9'). However, if $F(x, f(x), \omega(x, f(x))) < M_0$ holds for a minimizing function $f(x)$, then $f(x)$ is a unique minimizing function (and hence $f(x)$ is smooth and $F(x, f(x), f'(x)) = M_0$). This is proved below and a few immediate consequences of this theorem are also discussed.

We assume that $F(x, y, z)$ satisfies the following conditions:

1. $F(x, y, z) \in C^1$ for $x_1 \leqslant x \leqslant x_2$ and all y, z.
2. There is a continuous function $\omega(x, y)$ such that

$$\frac{\partial F(x,y,z)}{\partial z} \text{ is } \begin{cases} >0 & \text{if } z > \omega(x,y), \\ =0 & \text{if } z = \omega(x,y), \\ <0 & \text{if } z < \omega(x,y). \end{cases}$$

3. $\lim_{|z| \to \infty} F(x, y, z) = +\infty$ if x and y are fixed.

A function $f(x)$ is admissible (belongs to $\mathfrak{J}$) if and only if $f(x)$ is absolutely continuous on $[x_1, x_2]$ and satisfies $f(x_1) = y_1$, $f(x_2) = y_2$. Put $M_0 = \inf_{f \in \mathfrak{J}} H(f)$. Thus, a function $f_0(x) \in \mathfrak{J}$ is a minimizing function if and only if $H(f_0) = M_0$.

Theorem. *Assume that $f(x)$ is a minimizing function such that*

$$F(x, f(x), \omega(x, f(x))) < M_0 \quad \textit{for} \quad x_1 \leqslant x \leqslant x_2.$$

Then $f(x)$ is the only minimizing function. Furthermore, $f(x) \in C^2[x_1, x_2]$ and $F(x, f(x), f'(x)) = M_0$ for $x_1 \leqslant x \leqslant x_2$. (Compare Theorem 6' in [2].)

Proof. 1. Since $G(x, y) \equiv F(x, y, \omega(x, y))$ is continuous, there are numbers $\delta > 0$ and $M_0' < M_0$ such that $|y - f(x)| \leqslant \delta$ implies that $G(x, y) \leqslant M_0'$. Choose M_1 such that $M_0' < M_1 < M_0$. Then the functions $\Phi(x, y, M)$ and $\psi(x, y, M)$ (the same notation as in [2]) are defined and continuously differentiable for $x_1 \leqslant x \leqslant x_2$, $|y - f(x)| \leqslant \delta$, $M_1 \leqslant M \leqslant M_0$.

Put $E = \{(x, y) \mid x_1 \leqslant x \leqslant x_2, \ |y - f(x)| \leqslant \delta\}$. Consider the differential equation

$$y' = \lambda \Phi(x, y, M) + (1 - \lambda)\psi(x, y, M), \tag{1}$$

where the parameters λ and M are assumed to satisfy $0 \leqslant \lambda \leqslant 1$ and $M_1 \leqslant M \leqslant M_0$, respectively. The differential equation is considered only in E and with the initial values $(x_0, f(x_0))$ for some arbitrary $x_0 \in [x_1, x_2]$. Since $f'(x)$ is bounded for $x_1 \leqslant x \leqslant x_2$, and Φ, ψ are bounded in E, there exists a $\delta_1 > 0$, *not depending on* x_0, λ *or* M, such that (1) has a unique solution on the interval $[x_0 - \delta_1, x_0 + \delta_1] \cap [x_1, x_2]$. Further, the solution, which we write $y(x; x_0, \lambda, M)$ depends continuously on λ and M.

2. Now we divide the interval $[x_1, x_2]$ into N sub-intervals of equal length $< \delta_1$: $x_1 = X_1 < X_2 < X_3 < \ldots < X_{N+1} = x_2$. Next, we define N numbers $\{\lambda_\nu\}_1^N$ in the following way: Consider a fixed ν, $1 \leqslant \nu \leqslant N$. Since $H(f) \leqslant M_0$, we must have[1]

$$y(X_{\nu+1}; X_\nu, 0, M_0) \leqslant f(X_{\nu+1}) \leqslant y(X_{\nu+1}; X_\nu, 1, M_0).$$

Therefore, there is a uniquely determined number λ_ν, $0 \leqslant \lambda_\nu \leqslant 1$, such that $f(X_{\nu+1}) = y(X_{\nu+1}; \ X_\nu, \lambda_\nu, M_0)$.

A. If $\lambda_\nu = 0$, then[1] $f(x) = y(x; X_\nu, 0, M_0)$ for $X_\nu \leqslant x \leqslant X_{\nu+1}$.
B. If $\lambda_\nu = 1$, then[1] $f(x) = y(x; X_\nu, 1, M_0)$ for $X_\nu \leqslant x \leqslant X_{\nu+1}$.

3. Let η be any number such that

$$y(X_{\nu+1}; X_\nu, 0, M_0) < \eta < y(X_{\nu+1}; X_\nu, 1, M_0).$$

Then there is a number $M^* < M_0$ such that

$$y(X_{\nu+1}; X_\nu, 0, M^*) \leqslant \eta \leqslant y(X_{\nu+1}; X_\nu, 1, M^*),$$

and a corresponding λ^*, $0 \leqslant \lambda^* \leqslant 1$, such that $y(X_{\nu+1}; X_\nu, \lambda^*, M^*) = \eta$.

Put $f_1(x) = y(x; \ X_\nu, \lambda^*, M^*)$. Then $F(x, f_1(x), f_1'(x)) \leqslant M^* < M_0$ for $X_\nu \leqslant x \leqslant X_{\nu+1}$, i.e. $H(f_1; X_\nu, X_{\nu+1}) < M_0$.

We may also consider the interval $[X_{\nu-1}, X_\nu]$ and formulate analogous statements if $y(X_{\nu-1}; X_\nu, 0, M_0) > \eta > y(X_{\nu-1}; X_\nu, 1, M_0)$. (Note that the inequalitites for η are reversed in this case.)

4. Next, we claim that one of these statements is true:

A. All $\lambda_\nu = 0$.
B. All $\lambda_\nu = 1$.

If A or B holds, then the assertions of the theorem follow easily (apply Theorem 6′ in [2]).

Assume now that neither A nor B holds. We will then *construct an admissible func-*

[1] Compare Theorem 6 in [1] and Theorem 6′ in [2].

tion $g_0(x)$ *on* $[x_1, x_2]$, *such that* $H(g_0) < M_0$. This will give a contradiction to the definition of M_0, and thereby prove the theorem.

We use an induction argument.

Assumption. For any system of M consecutive intervals, where $M \geqslant 2$,

$$[X_\nu, X_{\nu+1}], [X_{\nu+1}, X_{\nu+2}], \ldots, [X_{\nu+M-1}, X_{\nu+M}]$$

such that $(\sum_{k=\nu}^{\nu+M-1} \lambda_k^2) \cdot (\sum_{k=\nu}^{\nu+M-1} (\lambda_k - 1)^2) \neq 0$, there is an absolutely continuous function $g(x)$ on $[X_\nu, X_{\nu+M}]$ satisfying

$$g(X_\nu) = f(X_\nu),\ g(X_{\nu+M}) = f(X_{\nu+M}) \quad \text{and} \quad H(g;\, X_\nu, X_{\nu+M}) < M_0.$$

Consider then the intervals

$$[X_\mu, X_{\mu+1}], [X_{\mu+1}, X_{\mu+2}], \ldots, [X_{\mu+M}, X_{\mu+M+1}]$$

and assume that $(\sum_{k=\mu}^{\mu+M} \lambda_k^2) \cdot (\sum_{k=\mu}^{\mu+M} (\lambda_k - 1)^2) \neq 0$. Then the assumption can be applied to at least one of the systems of intervals

$$[X_\mu, X_{\mu+1}], \ldots, [X_{\mu+M-1}, X_{\mu+M}] \quad \text{and} \quad [X_{\mu+1}, X_{\mu+2}], \ldots, [X_{\mu+M}, X_{\mu+M+1}],$$

for instance the first. This gives a function $g(x)$ satisfying $g(X_\mu) = f(X_\mu)$, $g(X_{\mu+M}) = f(X_{\mu+M})$ and $H(g;\, X_\mu, X_{\mu+M}) < M_0$.

Put $g_\lambda(x) = g(x) + \lambda(x - X_\mu)$. It is obvious that $H(g_\lambda) < M_0$ if $|\lambda| \leqslant \lambda_0$.

Now consider the interval $[X_{\mu+M}, X_{\mu+M+1}]$. According to (3) above, there are numbers η, arbitrarily close to $f(X_{\mu+M})$, and corresponding functions $f^*(x)$ such that $f^*(X_{\mu+M}) = \eta$, $f^*(X_{\mu+M+1}) = f(X_{\mu+M+1})$ and $H(f^*;\, X_{\mu+M}, X_{\mu+M+1}) < M_0$. If η is fixed, we determine λ by the condition $g_\lambda(X_{\mu+M}) = \eta$.

Now choose η so close to $f(X_{\mu+M})$ that $|\lambda| \leqslant \lambda_0$, and consider the function

$$\varphi(x) = \begin{cases} g_\lambda(x) & \text{if} \quad X_\mu \leqslant x \leqslant X_{\mu+M}, \\ f^*(x) & \text{if} \quad X_{\mu+M} \leqslant x \leqslant X_{\mu+M+1}. \end{cases}$$

It is clear that $\varphi(x)$ is absolutely continuous, $\varphi(X_\mu) = f(X_\mu)$, $\varphi(X_{\mu+M+1}) = f(X_{\mu+M+1})$, and $H(\varphi;\, X_\mu, X_{\mu+M+1}) < M_0$.

This shows that the validity of the assumption for $M(\geqslant 2)$ intervals implies its validity for $M+1$ intervals.

Finally, the validity of the assumption for $M=1$ and $M=2$ follows easily from (3). This completes the proof.

Next, we illustrate the theorem by means of some simple examples.

Example 1. Assume that $F(x, y, z) \equiv \varphi(x, y) + \psi(x, y) z^2$, where $\varphi(x, y)$ and $\psi(x, y)$ are continuously differentiable for $x_1 \leqslant x \leqslant x_2$, $-\infty < y < \infty$. Assume also that there are constants K_1, K_2, K_3 such that $K_1 \geqslant \varphi(x, y) \geqslant K_2$, and $\psi(x, y) \geqslant K_3 > 0$. We consider the minimization problem between the points (x_1, y_1) and (x_2, y_2).

Put $t = (y_2 - y_1)/(x_2 - x_1)$.

If $K_2 + K_3 t^2 > K_1$, *then there is a unique minimizing function* $f(x)$. *Further,* $f(x) \in C^2[x_1, x_2]$, $F(x, f(x), f'(x)) = M_0$ *and* $f'(x) \neq 0$ *for* $x_1 \leqslant x \leqslant x_2$.

Proof. Since $\lim_{|z| \to \infty} F(x, y, z) = +\infty$ uniformly in x and y, there exists a minimizing function $f(x)$ (compare Chapter 1 in [2]). Further, it is obvious that $M_0 \geqslant K_2 + K_3 t^2$. Hence, $F(x, f(x), \omega(x, f(x))) = \varphi(x, f(x)) \leqslant K_1 < M_0$, and we can apply the theorem. This proves the above assertion.

Example 2. This shows an application of Theorem 1 to a "converse" problem. We assume as before that $F(x, y, z)$ satisfies the conditions 1, 2 and 3 for $x_1 \leqslant x \leqslant x_2$ and all y, z. Let there be given two numbers y_1, y_2, such that $y_1 \neq y_2$, and a number M. Here, an admissible function $g(x)$ has to be absolutely continuous on an interval $x_1 \leqslant x \leqslant \xi \leqslant x_2$ and satisfy $g(x_1) = y_1$, $g(\xi) = y_2$, and $F(x, g(x), g'(x)) \leqslant M$. We assume that the class $\mathcal{G}$ of admissible functions is not empty. For each $g(x) \in \mathcal{G}$, the functional $X(g) = \min \{x \mid g(x) = y_2\}$ is defined. The problem is to minimize $X(g)$ over $\mathcal{G}$. (This is analogous to time-optimal problems in control theory.) Hence, a minimizing function $g_0(x)$ has to satisfy $X(g_0) = \inf_{g \in \mathcal{G}} X(g)$.

Assume that $g_0(x)$ is a minimizing function such that $F(x, g_0(x), \omega(x, g_0(x))) < M$ for $x_1 \leqslant x \leqslant X(g_0)$. Then $g_0(x) \in C^2$ and $F(x, g_0(x), g_0'(x)) = M$ for $x_1 \leqslant x \leqslant X(g_0)$. Further, $g_0(x)$ is the only minimizing function.

Proof. Consider the "original" problem, to minimize $H(f)$, between the points (x_1, y_1) and $(X(g_0), y_2)$. Let $\mathfrak{F}$ be the class of admissible functions for this problem, and put $M_0 = \inf_{f \in \mathfrak{F}} H(f)$. Since $g_0 \in \mathfrak{F}$, and $H(g) \leqslant M$, we have $M_0 \leqslant M$. Assume that $M_0 < M$. Then there must be a function $f_0(x) \in \mathfrak{F}$ such that $H(f_0) < M$. Put $f_\lambda(x) = f_0(x) + \lambda(x - x_1)$. If $|\lambda| \leqslant \lambda_0$, then $H(f_\lambda, x_1, X(g_0)) < M$. Further, if $y_2 > y_1$ and $\lambda > 0$, then there is a $\xi < X(g_0)$, such that $f_\lambda(\xi) = y_2$, and the same holds if $y_2 < y_1$, and $\lambda < 0$. Consequently, λ can be chosen such that $f_\lambda(x) \in \mathcal{G}$ and $X(f_\lambda) < X(g_0)$. But this contradicts our assumptions regarding $g_0(x)$. Hence $M_0 = M$, and $g_0(x)$ is a minimizing function for both problems. Now, the results follows directly from Theorem 1.

Remark. This result can also be proved by transformation of the given problem to a control problem, and application of the Pontryagin maximum principle. It can be shown by means of examples that the result is no longer true if the condition $F(x, g_0(x), \omega(x, g_0(x))) < M$ is omitted.

Remark. Necessary conditions for minimizing functions for the "original" problem can also be derived by the following approach:[1] Let $f(x) \in C^1$ be a minimizing function and let $\Phi(x) \in C^1$ vanish at $x = x_1$ and $x = x_2$. We also assume that $F(x, y, z) \in C^1$, but no other condition on $F(x, y, z)$ is needed. Put $U = \{x \mid F(x, f(x), f'(x)) = M_0\}$. Consider a neighbouring function $f(x) + \lambda\Phi(x)$ where λ is a "small" parameter. By applying the mean-value theorem to $\varphi(t) = F(x, f + t\lambda\Phi, f' + t\lambda\Phi') - F(x, f, f')$ between $t = 1$ and $t = 0$ it is not difficult to verify that we must have $\min_{x \in U} (a(x)\Phi(x) + b(x)\Phi'(x)) \leqslant 0$, where $a(x) = F_y(x, f(x), f'(x))$ and $b(x) = F_z(x, f(x), f'(x))$. This leads to various relations between the set U and the zeros of $a(x)$ or $b(x)$. For instance, if $b(x) \neq 0$ on U, then U is the whole interval $x_1 \leqslant x \leqslant x_2$.

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[1] Compare the results in [3], pp. 14–15.

REFERENCES

1. ARONSSON, G., Minimization problems for the functional $\sup_x F(x, f(x), f'(x))$. *Arkiv för matematik 6*, 33–53 (1965).
2. ARONSSON, G., Minimization problems for the functional $\sup_x F(x, f(x), f'(x))$. (II). *Arkiv för matematik 6*, 409–431 (1966).
3. MEINARDUS, G., Approximation von Funktionen und ihre numerische Behandlung. Springer-Verlag, Berlin, 1964.

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