ARKIV FÖR MATEMATIK Band 7 nr 35

1.67 15 1 Communicated 22 November 1967 by OTTO FROSTMAN

Existence of entire functions of one variable with prescribed indicator

By C. O. KISELMAN

Let u be an entire function of one complex variable satisfying

$$\log |u(\zeta)| \leqslant A|\zeta|^\varrho + B \quad (\zeta \in \mathbf{C}) \tag{1}$$

for some constants A, B. The number ϱ is positive and fixed throughout the paper. The *indicator* of u is the function

$$p(\zeta) = \varlimsup_{t\to+\infty} \frac{\log |u(t\zeta)|}{t^\varrho} \quad (\zeta \in \mathbf{C}). \tag{2}$$

It is clear that p is positively homogeneous of order ϱ and that $p(\zeta) \leqslant A|\zeta|^\varrho$ if (1) holds. It also follows from standard theorems for subharmonic functions that the *regularized indicator* p^*, defined by

$$p^*(\zeta) = \varlimsup_{\theta\to\zeta} p(\theta),$$

is subharmonic. However, it is known that we always have $p^* = p$.

The purpose of this note is to provide a proof of the following theorem of V. Bernstein [1, 2] (see also Levin [5] and, for $\varrho = 1$, Pólya [8]).

Theorem 1. *A function defined in the complex plane* $\mathbf{C}$ *is the regularized indicator of some entire function satisfying* (1) *for some constants* A *and* B *if* (*and only if*) *it is subharmonic and positively homogeneous of order* ϱ.

As noted above, the theorem can be improved by deleting the word "regularized". We shall not prove this here.

Formulas (1) and (2) have immediate generalizations to functions of several variables; then p^* becomes a plurisubharmonic function. In [6, 7] Martineau has proved that a function in $\mathbf{C}^n$ is the regularized indicator of some entire function satisfying (1) for some constants A and B if and only if it is plurisubharmonic and positively homogeneous of order ϱ. His proof has the form of an induction on the dimension and relies on a more precise version of the same result in one variable given in Levin [5]. It might be a justification for printing the present proof of Theorem 1 that it gives a more unified proof of the characterization of regularized indicators when combined with the induction step in [6, 7]. To be precise, the induction in such a proof could start with a function satisfying the estimate (6) below which could then be extended successively in analogy with Lemma 4 of [6]. The paper also serves to illustrate the fact that the estimates for the $\bar{\partial}$ operator given by Hörmander [3] are non-trivial even in one variable.

Let F be a given subharmonic function which is positively homogeneous of order ϱ. To prove Theorem 1 we shall construct an entire function u with indicator p_u satisfying

$$p_u(1) = F(1), \qquad p_u(\zeta) \leqslant F(\zeta) \quad (\zeta \in \mathbf{C}). \tag{3}$$

Let us first observe that this implies the desired result, viz. that $p_v^* = F$ for some entire function v. (Using integral transformations one can prove that $p^* = p$ so that the regularization is unnecessary.) This is proved by a category argument which has been carried through by Martineau [6, 7] (cf. also a remark in [4]). In fact, the space of all entire functions satisfying (1) for some constants A, B and with indicator $\leqslant F$ is a Fréchet space with the topology defined by the norms

$$u \mapsto \sup_{\zeta \in \mathbf{C}} |u(\zeta)| e^{-G(\zeta)},$$

where G is an arbitrary continuous function which is positively homogeneous of order ϱ and $> F$ at every point on the unit circle. It is easy to see that F is continuous (the function $\zeta \mapsto F((a\zeta)^{1/\varrho})$ is locally convex) so it suffices to take G of the form $G(\zeta) = F(\zeta) + |\zeta|^{\varrho}/j \, (j = 1, 2, \ldots)$. Let E_F be this Fréchet space. Suppose that we have found $u \in E_F$ with $p_u(\theta) = F(\theta)$ for any preassigned $\theta \in \mathbf{C}$ (it is of course enough to do this for $\theta = 1$). Then E_G is meager in E_F by the Banach theorem provided $G \leqslant F$, $G \neq F$. Here G, as well as G_j and H below, are assumed to be continuous and positively homogeneous of order ϱ. Hence $\bigcup_{j=1}^{\infty} E_{G_j}$ is meager in E_F if $G_j \leqslant F$, $G_j \neq F \, (j = 1, 2, \ldots)$. But it is easy to find a sequence of functions $G_j \leqslant F$, $G_j \neq F$, such that $H \leqslant G_j$ for some j if $H \leqslant F$, $H \neq F$. Therefore all functions in E_F not in $\bigcup E_{G_j}$ must have regularized indicator p^* equal to F.

To find u satisfying (3) we shall use the following adoption to supremum norms of Theorem 4.4.2 in Hörmander [3].

Theorem 2. *Let G be a plurisubharmonic function in $\mathbf{C}^n$. For every form $f \in C^{\infty}_{(0,1)}(\mathbf{C}^n)$ satisfying*

$$\bar{\partial} f = 0 \quad \text{and} \quad |f(\zeta)| \leqslant e^{G(\zeta)},$$

there exists a function $u \in C^{\infty}(\mathbf{C}^n)$ with

$$\bar{\partial} u = f \quad \text{and} \quad |u(\zeta)| \leqslant e^{H(\zeta)},$$

where

$$H(\zeta) = \sup_{|\theta| \leqslant 1} G(\zeta + \theta) + a \log(1 + |\zeta|^2) + b.$$

Here a may be taken as an arbitrary number $> 1 + n/2$, and b is a constant which depends only on a and n.

As to the notation in this theorem we only mention that $C^{\infty}_{(0,1)}(\mathbf{C}^n)$ denotes the space of forms of type (0, 1):

$$f = \sum f_j \, d\bar{z}_j$$

with C^{∞} coefficients f_j; $\bar{\partial} f$ is defined by

$$\bar{\partial} f = \sum \frac{\partial f_j}{\partial \bar{z}_k} d\bar{z}_k \wedge d\bar{z}_j,$$

whereas $\bar{\partial} u$ for $u \in C^\infty(\mathbf{C}^n)$ is given by

$$\bar{\partial} u = \sum \frac{\partial u}{\partial \bar{z}_j} d\bar{z}_j,$$

a form of type (0, 1). Note that $\bar{\partial} f = 0$ is no condition when $n = 1$. For other notions we refer to Hörmander [3].

We shall take an entire function u satisfying (3) of the form

$$u = g - hv,$$

where h is the entire function

$$h(\zeta) = \prod_1^\infty (1 - 2^{-j}\zeta)$$

with zeros at 2^j ($j = 1, 2, \ldots$), and g, v are C^∞ functions to be described presently. Let $\varphi \in C_0^\infty(\mathbf{C})$ be a function which is zero when $|z| \geqslant 1$ and equal to one when $|z| \leqslant \frac{1}{2}$, $0 \leqslant \varphi \leqslant 1$. We shall define g by

$$g(\zeta) = \sum_1^\infty \varphi(\zeta - 2^j)\, e^{F(2^j)}.$$

It is then clear that $g \in C^\infty(\mathbf{C})$ and that

$$u(2^j) = g(2^j) = e^{F(2^j)}.$$

Hence, if p_u is the indicator of u, $p_u(1) \geqslant F(1)$. It remains to define $v \in C^\infty(\mathbf{C})$ so that u becomes analytic and $p_u \leqslant F$. That u is analytic means that

$$0 = \bar{\partial} u = \bar{\partial} g - h \bar{\partial} v,$$

i.e.

$$\bar{\partial} v = f,$$

where

$$f = \frac{1}{h} \bar{\partial} g \in C^\infty_{(0,1)}(\mathbf{C}).$$

It can easily be proved by estimating the factors $(1 - 2^{-j}\zeta)$ constituting h that for some constant C_1,

$$|h(\zeta)| \geqslant \frac{1}{C_1} > 0,$$

when

$$\tfrac{1}{2} \leqslant |\zeta - 2^j| \leqslant 1 \quad (j = 1, 2, \ldots).$$

Hence, if C_2 is chosen so large that

$$|\bar{\partial} \varphi| \leqslant C_2,$$

we obtain

$$|f(\zeta)| = \left| \frac{1}{h(\zeta)} \bar{\partial} g(\zeta) \right| \leqslant C_1 C_2 e^{F(2^j)}, \tag{4}$$

when $|\zeta - 2^j| \leqslant 1$. Define

$$G(\zeta) = \sup_{|\theta| \leqslant 1} F(\zeta + \theta).$$

It is easy to see that G is also continuous and subharmonic. We obtain from (4) that

$$|f(\zeta)| \leqslant C_1 C_2 e^{G(\zeta)}$$

for every $\zeta \in \mathbf{C}$, for either $f(\zeta)=0$ or else we can find a j such that $|\zeta - 2^j| \leqslant 1$ and use (4) for this j. We can therefore apply Theorem 2 to find a $v \in C^\infty(\mathbf{C})$ with

$$\bar{\partial} v = f \quad \text{and} \quad |v(\zeta)| \leqslant C_3 e^{H(\zeta)} \quad (\zeta \in \mathbf{C}),$$

where C_3 is a new constant and

$$H(\zeta) = \sup_{|\theta| \leqslant 1} G(\zeta+\theta) + a \log(1+|\zeta|^2) \leqslant \sup_{|\theta| \leqslant 2} F(\zeta+\theta) + a \log(1+|\zeta|^2). \tag{5}$$

Now $u = g - hv$ is certainly analytic, and

$$|u(\zeta)| \leqslant g(\zeta) + |h(\zeta)|\,|v(\zeta)| \leqslant e^{G(\zeta)} + |h(\zeta)|\, C_3 e^{H(\zeta)}.$$

But it is well known that h is of order zero, hence for any $\varepsilon > 0$ there are constants A and C_4 such that

$$|h(\zeta)| \leqslant C_4 e^{A|\zeta|^\varepsilon}.$$

($C_4 = 1$ will do.) We finally arrive at the inequality

$$|u(\zeta)| \leqslant C_5 e^{H(\zeta) + A|\zeta|^\varepsilon}. \tag{6}$$

It now follows in view of (5) and the continuity of F that the indicator p_u of u satisfies $p_u \leqslant F$ provided only $\varepsilon < \varrho$. The proof is complete.

Department of Mathematics, University of Stockholm, Stockholm, Sweden

REFERENCES

1. Bernstein, V., Sur les propriétés caractéristiques des indicatrices de croissance. *C. R. Acad. Sci. Paris 202*, 108–110 (1936).
2. —— Sulle proprietà caratteristiche delle indicatrici di crescenza delle trascendenti intere d'ordine finito. *Mem. R. Acc. d'Italia 7*, 131–189 (1936).
3. Hörmander, L., *An Introduction to Complex Analysis in Several Variables.* Van Nostrand, Inc., Princeton, N.J., 1966.
4. Kiselman, C. O., On entire functions of exponential type and indicators of analytic functionals. *Acta Math. 117* (1967), 1–35 (1966).
5. Левин, Б. Я., *Распределение корней целых функций.* Moscow, 1956. (English and German translations.)
6. Martineau, A., Indicatrices de croissance des fonctions entières de N-variables. *Inventiones math. 2*, 81–86 (1966).
7. —— Indicatrices de croissance des fonctions entières de N-variables (Corrections et compléments). *Inventiones math. 3*, 16–19 (1967).
8. Pólya, G., Untersuchungen über Lücken und Singularitäten von Potenzreihen. *Math. Z. 29*, 549–640 (1929).

Tryckt den 30 augusti 1968

Uppsala 1968. Almqvist & Wiksells Boktryckeri AB