Communicated 9 March 1949 by F. CARLSON

On the constant in Hölder's inequality

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With 5 figures in the text

It is well known that the '=' in the so-called Hölder's inequality in one of its forms

$$\sum a_n b_n \leq \left(\sum a_n^p\right)^{\frac{1}{p}} \left(\sum b_n^q\right)^{\frac{1}{q}} \qquad \left(\frac{1}{p}+\frac{1}{q}=1\right) \tag{1 a}$$

or

$$\int_0^1 f(x)^r g(x)^{1-r} dx \leq \left(\int_0^1 f(x)\,dx\right)^r \left(\int_0^1 g(x)\,dx\right)^{1-r} \qquad (0 < r < 1) \tag{1 b}$$

occurs only if $a_n^p = A \cdot b_n^q$ for all n

or

$$f(x) \equiv g(x)$$

If $\{a_n\}$ and $\{b_n\}$ (or $f(x)$ and $g(x)$) are subjected to some restrictive conditions that exclude the proportionality just mentioned, then one can have only '<'.

We shall consider the most general form of Hölder's inequality, which includes (1 a) and (1 b), and study the value of the Lebesgue-Stieltjes integral

$$I_r = \int_E (f(x))^r (g(x))^{1-r}\, d\varphi(x) \qquad (0 \leq r \leq 1) \tag{1}$$

where $f(x)$ and $g(x)$ are non-negative functions and $\varphi(x)$ an increasing function on the set E. Hölder's inequality then takes the form

$$I_r \leq \theta_r \cdot I_1^r \cdot I_0^{1-r} \qquad (0 \leq \theta_r \leq 1) \tag{2}$$

In the following we shall assume throughout that the functions $f(x)$ and $g(x)$ are normalized in such a way that

$$I_1 = I_0 = 1$$

which is no real restriction of the study.

Then we have

$$I_r \leq \theta_r \qquad (0 < r < 1)$$

and $I_r = 1$, if and only if $f(x) \equiv g(x)$.

In this note we are going to determine the best possible value of θ_r (i. e. the least upper bound of I_r) under various assumptions that exclude the possibility $f(x) \equiv g(x)$.

Searching the best value of θ_r one can *a priori* suppose that $g(x) > 0$ on E. For the addition to E of a set where $g(x) = 0$ will leave the values of I_r and I_0 unchanged, while I_1 may increase; thus the quotient $I_r / I_0 I_1$ will certainly not attain lower maximal values if we suppose that $g(x) \neq 0$ everywhere on E.

If $g(x) > 0$, on can define $u(x) = \dfrac{f(x)}{g(x)}$ and $d\psi(x) = g(x)\, d\varphi(x)$ and write

$$I_r = \int_E u(x)^r\, d\psi(x) \qquad (0 < r < 1)$$

Then $I_r = 1$ if and only if $u(x) \equiv 1$.

We now start with a hypothesis that excludes this possibility: we suppose that $u(x)$ does not take any values in the interval (a, b), where $a < 1 < b$. Let us regard

$$F(u) = \begin{vmatrix} u & a & b \\ u^r & a^r & b^r \\ 1 & 1 & 1 \end{vmatrix}$$

Apparently $F(a) = F(b) = 0$ and $F''(u) = -r(1-r)(b-a)\, u^{r-2} < 0$, hence

$$F(1) > 0$$

and $F(u) \leq 0$ for values of u outside the interval (a, b). Hence

$$\begin{vmatrix} I_1 & a & b \\ I_r & a^r & b^r \\ I_0 & 1 & 1 \end{vmatrix} \leq 0$$

which can be written

$$\begin{vmatrix} 1 & a & b \\ 1 & a^r & b^r \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 0 & a & b \\ I_r - 1 & a^r & b^r \\ 0 & 1 & 1 \end{vmatrix} \leq 0$$

or

$$I_r \leq 1 - \frac{F(1)}{b-a}$$

Here '$=$' can occur, that is, we have really

$$(*) \qquad \theta_r = 1 - \frac{F(1)}{b-a}$$

as is shown by the example

$$u(x) = a \qquad \psi(x) = \frac{b-1}{b-a} \cdot x \qquad \text{when } -1 \le x \le 0$$

$$= b \qquad\qquad = \frac{1-a}{b-a} \cdot x \qquad \text{when } \quad 0 \le x \le 1$$

giving

$$\int_{-1}^{1} u(x)\, d\psi(x) = \int_{-1}^{1} d\psi(x) = 1 \quad \text{and}$$

$$\int_{-1}^{1} u(x)^r\, d\psi(x) = \frac{a^r (b-1) + b^r (1-a)}{b-a} = 1 - \frac{F(1)}{b-a}$$

This result is essentially due to F. Carlsson, in his paper Sur le module maximum d'une fonction analytique uniforme, Arkiv f. mat., astr. o. fys. Bd 26 A, nr 9, pp 4—5 (1938).

The method just used can hardly be generalized to problems based on other assumptions about the way in which $f(x)$ differs from $g(x)$. A more general method is as follows.

Let E_1 be the set where $f(x) \ge g(x)$ and E_2 the set where $f(x) < g(x)$. We suppose *a priori* that E does not include any set where $f(x) = g(x) = 0$, since such a set has no importance on the value of I_r.

Then we have $E_1 + E_2 = E$, $f > 0$ in E_1, $g > 0$ in E_2, and we can define

$$\text{in } E_1 \begin{cases} h(x) = \dfrac{g(x)}{f(x)} \\ d\alpha(x) = f(x)\, d\varphi(x) \end{cases} \qquad \text{and in } E_2 \begin{cases} k(x) = \dfrac{f(x)}{g(x)} \\ d\beta(x) = g(x)\, d\varphi(x) \end{cases}$$

where $\alpha(x)$ and $\beta(x)$ become increasing functions on E_1 and E_2, and

$$0 \le h(x) \le 1, \qquad 0 \le k(x) < 1$$

We get

$$(3) \qquad I_r = \int_{E_1} h(x)^{1-r}\, d\alpha(x) + \int_{E_2} k(x)^r\, d\beta(x)$$

and thus, since the expressions in (3) are convex functions of r

$$(4) \quad I_r \le \left(\int_{E_1} h(x)\, d\alpha(x)\right)^{1-r} \left(\int_{E_1} d\alpha(x)\right)^r + \left(\int_{E_2} k(x)\, d\beta(x)\right)^r \left(\int_{E_2} d\beta(x)\right)^{1-r}$$

(Hölder's inequality)

Let us put

$$u = \int_{E_1} d\,\alpha(x), \quad v = \int_{E_2} d\,\beta(x), \quad z = \int_{E_1} h(x)\,d\,\alpha(x), \quad w = \int_{E_2} k(x)\,d\,\beta(x)$$

Then

$$I_0 = z + v = 1, \qquad I_1 = u + w = 1$$

and (4) takes the form

$$I_r \leq u^r (1-v)^{1-r} + (1-u)^r v^{1-r} \tag{5}$$

In (4) and (5) the '=' holds only if $h(x)$ and $k(x)$ both are constants. Let

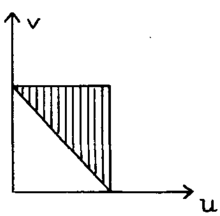

Fig. 1.

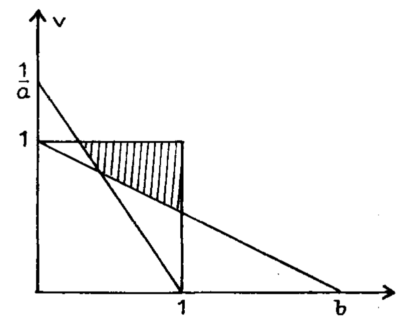

Fig. 2.

us call the right-hand side in (5) $H(u, v)$. We get the domain in which $H(u, v)$ is defined from the inequalities $0 \leq z \leq u$ and $0 \leq w \leq v$.

It is apparently a triangle in the uv-plane defined by $u + v - 1 \geq 0$, $1 \geq u \geq 0$, $1 \geq v \geq 0$. We shall call this triangle the fundamental triangle. (Fig. 1.) The maximum of $H(u, v)$ is $= 1$, and is obtained only on the boundary $u + v = 1$ (independently of r, by the way). By restrictive conditions on $f(x)$ and $g(x)$ the domain of $H(u, v)$ is diminished, so that

$$\theta_r = \text{Max}\, H(u, v)$$

may become < 1. Assuming e. g. the aforesaid hypothesis that the quotient $f(x)/g(x)$ does not take values in the interval (a, b), where $a \leq 1 \leq b$, we have

$$0 \leq h(x) \leq \frac{1}{b}, \quad 0 \leq k(x) \leq a,$$

hence

$$z = 1 - v \leq \frac{u}{b}, \quad w = 1 - u \leq a v,$$

which means that $H(u, v)$ is defined in a domain bounded by the straight lines

$$\frac{u}{b} + v = 1, \quad u + av = 1, \quad u = 1, \quad v = 1. \qquad \text{(Fig. 2.)}$$

That $H(u, v)$ attains its maximum in the lower left-hand corner of that domain can be shown in the following way.

Put

$$\left.\begin{aligned} s &= \frac{1-u}{v} \\ t &= \frac{u}{1-v} \end{aligned}\right\} \text{ which gives } \left\{\begin{aligned} u &= \frac{t(1-s)}{t-s} \\ v &= \frac{t-1}{t-s} \end{aligned}\right.$$

Then

$$H(u, v) = H = \frac{1-s}{t-s} \cdot t^r + \frac{t-1}{t-s} \cdot s^r$$

and since

$$s \leq a \leq 1 \leq b \leq t$$

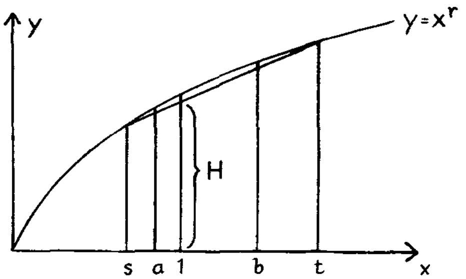

Fig. 3.

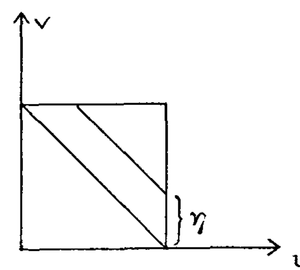

Fig. 4.

it is geometrically evident (see Fig. 3), that

$$\text{(*)} \qquad \operatorname{Max} H = \theta_r = \frac{(1-a)\,b^r + (b-1)\,a^r}{b-a},$$

in accordance with the previous result.

Now let us replace the hypothesis that $f(x)/g(x)$ does not take values in an interval including 1, with the assumption

$$\text{(6)} \qquad \Gamma_p = \int_E |f(x) - g(x)|^p \, d\varphi(x) = 2\eta > 0.$$

Then we have, with the same abbreviations as before, in the case $p = 1$

$$\int_{E_1} (f(x) - g(x))\, d\varphi(x) + \int_{E_2} (g(x) - f(x))\, d\varphi(x) = u - z + v - w = 2\eta$$

hence

$$u + v - 1 = \eta$$

and thus $H(u, v)$ is defined only on that part of the straight line

$$u + v = 1 + \eta$$

which lies inside the fundamental triangle. (Fig. 4.)

Accordingly we have

$$\operatorname{Max} H = \theta_r = \operatorname*{Max}_{\eta \leqq u \leqq 1} [u^r (u-\eta)^{1-r} + (1-u)^r (1-u+\eta)^{1-r}]$$

To determine this maximum is a problem of elementary analysis. [One finds that the maximum occurs for a u that is the real root of the equation

$$\frac{1 - r \cdot \dfrac{\eta}{u}}{\left(1 - \dfrac{\eta}{u}\right)^r} = \frac{1 + r \cdot \dfrac{\eta}{1-u}}{\left(1 + \dfrac{\eta}{1-u}\right)^r}$$

and this root can be obtained graphically by constructing a horizontal chord of length 1 in the curve (see Fig. 5)

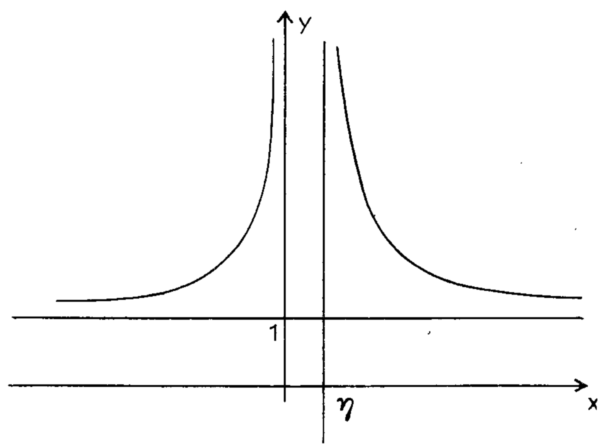

Fig. 5.

$$y = \frac{x - r\eta}{x^{1-r}(x-\eta)^r}$$

whereby the root in question is equal to the abscissa of the right-hand endpoint of the chord.]

We note that the condition (6) for $p \neq 1$ only leads to the conclusion $\theta_r = 1$, if no further assumptions are made. For in this case we can always make Γ_1 arbitrarily small, as is shown by the following two examples.

A. If $1 < p < \infty$, let E be the interval $(0, 1)$ and put

$$\begin{aligned}
\varphi(x) \equiv \frac{x^{k+1}}{k+1},\ g(x) \equiv x^{-k},\ f(x) &= x^{-k} && \text{in } (0, a) \text{ and } (a + 2\varepsilon, 1) \\
&= 0 && \text{in } (a, a+\varepsilon) \\
&= 2x^{-k} && \text{in } (a+\varepsilon, a+2\varepsilon)
\end{aligned}$$

where $k = \dfrac{1}{p-1}$ and $a = \dfrac{\varepsilon \cdot e^{-\eta}}{\sinh \eta}$

B. If $0 < p < 1$, let E be the interval $(1, \infty)$ and put

$$\varphi(x) \equiv x, \quad g(x) \equiv (h-1)\,x^{-h}, \quad f(x) = \begin{cases} (h-1)\,x^{-h} & \text{in } (1, P) \text{ and } (R, \infty) \\ 0 & \text{in } (P, Q) \\ 2\,(h-1)\,x^{-h} & \text{in } (Q, R) \end{cases}$$

where $h = \dfrac{1}{p}$,
$$P^{h-1} = \frac{1}{2\,\varepsilon}\left(1 - \exp\left(-2\,\eta\,(h-1)^{1-p}\right)\right)$$
$$R = P \cdot \exp\left(2\,\eta\,(h-1)^{-p}\right)$$
$$Q^{1-h} = \tfrac{1}{2}\left(P^{1-h} + R^{1-h}\right)$$

Then in both cases A. and B. we have $I_0 = I_1 = 1$, $\Gamma_p = 2\,\eta$ and $\Gamma_1 = 2\,\varepsilon$.

In general, every condition that excludes the possibility $f(x) \equiv g(x)$ leads to a restriction of the domain D of $H(u, v)$ from the original fundamental triangle to some part of it. If any part of the boundary of D coincides with the hypothenuse of the fundamental triangle, then $\theta_r = 1$ (even if I_r is always < 1). If, on the contrary, the boundary of D is situated completely above the hypothenuse, then $\theta_r < 1$, and we have always

$$\theta_r = \operatorname*{Max}_{u,\, v \in D} [H(u, v)]$$

The determination of θ_r is thus in all cases a problem of elementary mathematics — in principle!

It is also possible to estimate I_r by other means than by Hölder's inequality. Regard, e. g., the identity

$$\int_E f^{\frac{1}{2}}\, g^{\frac{1}{2}}\, d\varphi \equiv 1 - \tfrac{1}{2} \int_E \left(f^{\frac{1}{2}} - g^{\frac{1}{2}}\right)^2 d\varphi$$

which tells us how $I_{\frac{1}{2}}$ differs from 1, when $f \not\equiv g$. One can get a generalization from $r = \frac{1}{2}$ to arbitrary values of r in the interval $(0, \frac{1}{2})$ by the well-known inequalities

$$\frac{1 - u^r - r(1-u)}{1-r} \le (1 - u^r)(1 - u^{1-r}) \le \frac{1 - u^r - r(1-u)}{r} \qquad (0 < r < \tfrac{1}{2})$$

Put $u = \dfrac{f}{g}$ and multiply with $r(1-r)\,g$, then one has

$$r\left(f + g - f^r g^{1-r} - f^{1-r} g^r\right) \le r f + (1-r)\,g - f^r g^{1-r} \le$$
$$\le (1-r)\left(f + g - f^r g^{1-r} - f^{1-r} g^r\right)$$

Here, on the left, '=' holds only if $g = 0$ or $g = f$, and,

on the right '=' holds only if $f = 0$ or $f = g$.

After further multiplication with $d\varphi(x)$ and integration over E one obtains

$$r(2 - I_r - I_{1-r}) \leq 1 - I_r \leq (1 - r)(2 - I_r - I_{1-r})$$

or, with the abbreviation

$$1 - I_r = w_r$$

$$(**) \qquad \frac{r}{1-r} w_{1-r} \leq w_r \leq \frac{1-r}{r} w_{1-r} \qquad (0 < r < \tfrac{1}{2})$$

giving a connection between the deviations of I_r from the value 1 for conjugate pairs of r. The sign '$=$' occurs only when $f \equiv g$.

Tryckt den 21 juni 1949

Uppsala 1949. Almqvist & Wiksells Boktryckeri AB