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On a diophantine equation in two unknowns

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§ 1.

The purpose of this paper is to examine the solvability in integers x and y of the equation

$$x^2 + x + \tfrac{1}{4}(D + 1) = y^q \tag{1}$$

where D is a positive integer $\equiv 3 \pmod{4}$ and q denotes an odd prime.

The special case $D = 3$ has already been treated by T. NAGELL, who showed that the equation[1]

$$x^2 + x + 1 = y^n$$

is impossible in integers x and y, when $y \neq \pm 1$, for all whole exponents $n\,(> 2)$ not being a power of 3.

W. LJUNGGREN completed this result by proving that the equation[2]

$$x^2 + x + 1 = y^3$$

has the only solutions $y = 1$ and $y = 7$. Thus it is sufficient in (1) to take $D \geqq 7$. *We furthermore suppose that D has no squared factor* > 1.

According to a theorem of AXEL THUE the equation (1) has only a finite number of solutions in integers x and y, when D and q are given.[3]

§ 2.

If we put $\varrho = \tfrac{1}{2}(-1 + \sqrt{-D})$ and $\varrho' = \tfrac{1}{2}(-1 - \sqrt{-D})$, the equation (1) can be written

$$(x - \varrho)(x - \varrho') = y^q. \tag{1'}$$

ϱ and ϱ' are conjugate integers in the quadratic field $K(\sqrt{-D})$. The numbers $1, \varrho$ form a basis of the field.

The two principal ideals

$$(x - \varrho) \quad \text{and} \quad (x - \varrho')$$

are relatively prime. To show it we denote by $\mathfrak{j}$ their highest common divisor. The number $2x + 1 = 2x - (\varrho + \varrho')$ is contained in $\mathfrak{j}$ and also the number $D = -(\varrho - \varrho')^2$. If we write the equation (1) in the form

$$(2x+1)^2 + D = 4y^q,$$

we see that the numbers $2x+1$ and D are relatively prime since D has no squared factor > 1. Hence we have $\mathfrak{j} = (1)$. Therefore we get from (1′) the ideal equation

$$(x - \varrho) = \mathfrak{a}^q \tag{2}$$

where $\mathfrak{a}$ is an ideal. *Let us for the present suppose that* $\mathfrak{a}$ *is a principal ideal.* Then (2) can be written

$$x - \varrho = \varepsilon (a + b\varrho)^q \tag{2'}$$

where a and b are relatively prime integers and ε is a unit in $K(\sqrt{-D})$. D being > 3, the only units are ± 1. Thus the unit ε is a q-th power, and we can replace ε by 1.

Hence we get from (2′)

$$x - \varrho = (a + b\varrho)^q = \left(a - \frac{b}{2} + \frac{b}{2}\sqrt{-D}\right)^q = \frac{(c + b\sqrt{-D})^q}{2^q} \tag{3}$$

with

$$c = 2a - b.$$

From (3) we get

$$2^q \sqrt{-D} = (c - b\sqrt{-D})^q - (c + b\sqrt{-D})^q \tag{3'}$$

or developed

$$2^{q-1} = -\sum_{r=0}^{\frac{1}{2}(q-1)} \binom{q}{2r+1} c^{q-2r-1} b^{2r+1} (-D)^r. \tag{4}$$

From (4) we get $b = \pm 2^m$. Here $m = 0$ is the only possibility, for otherwise c would be even too by (3) and the right member of (4) would be divisible by 2^q. Hence $b = \pm 1$.

From (4) we get modulo q

$$1 \equiv -b(-D)^{\frac{1}{2}(q-1)} \pmod{q}.$$

Hence

$$b = -\left(\frac{-D}{q}\right).$$

Then the equation (4) is transformed into

$$2^{q-1}\left(\frac{-D}{q}\right) = \sum_{r=0}^{\frac{1}{2}(q-1)} \binom{q}{2r+1} c^{q-2r-1} (-D)^r, \tag{5}$$

which is an algebraic equation in c^2 of degree $\frac{1}{2}(q-1)$ and with integral coefficients. To every integral solution $\pm c$ of the equation (5) corresponds one integral solution y of the equation (1) given by

$$y = N(\mathfrak{a}) = \tfrac{1}{4}(D + c^2). \tag{6}$$

In this way we can have at most $\frac{1}{2}(q-1)$ solutions y of (1), when D and q are given.

The right member of (5) is a binary form of degree $n = \frac{1}{2}(q-1)$ in c^2 and D. This form is irreducible; to see it we regard the polynomial in z

$$f(z) = \sum_{r=0}^{\frac{1}{2}(q-1)} \binom{q}{2r+1} z^r = \sum_{r=0}^{n} a_r z^r,$$

which has the following properties: $a_n \not\equiv 0 \pmod{q}$; $a_i \equiv 0 \pmod{q}$ for all $i < n$; $a_0 \not\equiv 0 \pmod{q^2}$.

Hence $f(z)$ is irreducible according to the theorem of EISENSTEIN. Using the wellknown theorem of AXEL THUE[4] on the corresponding form

$$f(x, y) = y^n f\left(\frac{x}{y}\right)$$

we see that the equation (5) has only a finite number of solutions in integers c^2 and D, when q is given and $\geqq 7$.

§ 3.

Let us denote by $h(\sqrt{-D})$ the number of ideal classes in the field $K(\sqrt{-D})$. We shall prove the following proposition:

Theorem 1. *If D is a positive integer $\equiv 3 \pmod{4}$ having no squared factor > 1 and if $h(\sqrt{-D})$ is not divisible by the prime q, the equation*

$$x^2 + x + \tfrac{1}{4}(D+1) = y^q$$

is solvable in integers x and y only for a finite number of integers D for a given $q \geqq 7$. The equation has at most $\frac{1}{2}(q-1)$ solutions y, when D and q are given.

In consequence of the results in the preceding paragraph the theorem is proved, when we can prove that in (2) the ideal $\mathfrak{a}$ is a principal ideal if $h = h(\sqrt{-D})$ is not divisible by q.

For if $h \not\equiv 0 \pmod{q}$ there are two integers f and g so that

$$fq - gh = 1.$$

Hence we get from (2) the equivalence

$$\mathfrak{a} \sim \mathfrak{a}^{fq} \sim (1).$$

From the relation (6) we see that if the equation (1) has a solution $y < \frac{1}{4}(D+1)$, we must have $h(\sqrt{-D}) \equiv 0 \pmod{q}$. So we get the following result:

Theorem 2. *Let x and y be any integers so that*

$$y^q - y > x^2 + x$$

and so that the number

$$D = 4y^q - (2x+1)^2$$

has no squared factor > 1. Then the number $h(\sqrt{-D})$ is divisible by the odd prime q.

Numerical examples

1. If $q = 3$, $y = 2$ and $x = 1$ we get $D = 4 \cdot 2^3 - 3^2 = 23$ with $2^3 - 2 > 1^2 + 1$. Hence $h(\sqrt{-23}) \equiv 0 \pmod{3}$.
2. If $q = 5$, $y = 2$ and $x = 4$ we get $D = 4 \cdot 2^5 - 9^2 = 47$ with $2^5 - 2 > 4^2 + 4$. Hence $h(\sqrt{-47}) \equiv 0 \pmod{5}$.
3. If $q=7$, $y=2$ and $x=10$ we get $D = 4 \cdot 2^7 - 21^2 = 71$ with $2^7 - 2 > 10^2 + 10$. Hence $h(\sqrt{-71}) \equiv 0 \pmod{7}$.
4. If $q=11$, $y=2$ and $x=44$ we get $D=4 \cdot 2^{11} - 89^2 = 271$ with $2^{11} - 2 > 44^2 + 44$. Hence $h(\sqrt{-271}) \equiv 0 \pmod{11}$.

§ 4.

We shall determine an upper limit for the solutions y of the equation (1), when the number $h(\sqrt{-D})$ is not divisible by q. As was shown in the preceding paragraph the ideal $\mathfrak{a}$ in (2) is then a principal ideal.

Let us write (3′) as a product. From (3′) we get

$$2^q\sqrt{-D} = \alpha^q - \alpha'^q \quad \text{with} \quad \alpha = c - b\sqrt{-D}.$$

$$2^q\sqrt{-D} = -2b\sqrt{-D}\prod_{r=1}^{q-1}\left(\alpha - \alpha' e^{\frac{2\pi i}{q}r}\right);$$

$$2^{q-1}\left(\frac{-D}{q}\right) = \prod_{r=1}^{q-1} e^{\frac{\pi i}{q}r}\left(\alpha e^{-\frac{\pi i}{q}r} - \alpha' e^{\frac{\pi i}{q}r}\right) =$$

$$= (-1)^{\frac{1}{2}(q-1)}\prod_{r=1}^{q-1} 2\left(-ic\sin\frac{\pi}{q}r - b\sqrt{-D}\cos\frac{\pi}{q}r\right) =$$

$$= 2^{q-1}\prod_{r=1}^{q-1}\left(c\sin\frac{\pi}{q}r + b\sqrt{D}\cos\frac{\pi}{q}r\right).$$

Hence

$$\left(\frac{-D}{q}\right) = \prod_{r=1}^{q-1}\left(c\sin\frac{\pi}{q}r + b\sqrt{D}\cos\frac{\pi}{q}r\right).$$

Hence we get because $\sin\varphi = \sin(\pi - \varphi)$, $\cos\varphi = -\cos(\pi - \varphi)$ and $b^2 = 1$

$$\left(\frac{-D}{q}\right)=\prod_{r=1}^{\frac{1}{2}(q-1)}\left(c^2\sin^2\frac{\pi}{q}r-D\cos^2\frac{\pi}{q}r\right).$$

From (6) we get

$$\left(\frac{-D}{q}\right)=\prod_{r=1}^{\frac{1}{2}(q-1)}\left(4\,y\sin^2\frac{\pi}{q}r-D\right).$$

If

$$4\,y\sin^2\frac{\pi}{q}-D\geqq 4\sin^2\frac{\pi}{q},$$

we have, since $\sin x$ is increasing in the interval $0<x<\frac{\pi}{2}$

$$\prod_{r=1}^{\frac{1}{2}(q-1)}\left(4\,y\sin^2\frac{\pi}{q}r-D\right)\geqq\prod_{r=1}^{\frac{1}{2}(q-1)}\left(2\sin\frac{\pi}{q}r\right)^2=q>1.$$

Hence we must have

$$4\,y\sin^2\frac{\pi}{q}-D<4\sin^2\frac{\pi}{q}$$

i. e.

$$y<\tfrac{1}{4}D\operatorname{cosec}^2\frac{\pi}{q}+1$$

and we get the following result:

Theorem 3. *When $h(\sqrt{-D})$ is not divisible by q, the integral solutions y of the equation* (1) *are all less than the number*

$$\tfrac{1}{4}D\operatorname{cosec}^2\frac{\pi}{q}+1.$$

§ 5.

We shall prove the following proposition:

Theorem 4. *The equation*

$$x^2+x+\tfrac{1}{4}(q+1)=y^q$$

is unsolvable in integers x and y, if $q>3$ is a prime $\equiv 3$ (mod. 4).

In an imaginary quadratic field $K(\sqrt{d})$ with the discriminant d ($d<-4$) the number of ideal classes $h(\sqrt{d})$ is given by the formula[5]

$$h(\sqrt{d})=-\frac{1}{|d|}\sum_{n=1}^{|d|-1}n\cdot\left(\frac{d}{n}\right),$$

where the characters $\left(\frac{d}{n}\right)$ can only have the values 0, ± 1. From this formula we get the following inequality

$$h(\sqrt{d}) < \frac{1}{|d|} \sum_{n=1}^{|d|-1} n = \frac{|d|-1}{2} < |d|.$$

In the present case we have $d = -q$ and hence $h(\sqrt{-q})$ is not divisible by q. As was shown in § 3 this involves $\mathfrak{a} \sim (1)$ in the equation (2). But the equation (5) is impossible, when $D = q$, since every term in the right member is divisible by q, while the left member is not divisible by q.

§ 6.

We next consider the special case $q = 3$ supposing that $h(\sqrt{-D})$ is not divisible by 3. From (5) and (6) we then get

$$4\left(\frac{-D}{3}\right) = 3c^2 - D$$

and

$$y = \tfrac{1}{4}(D + c^2) = \tfrac{1}{3}\left(D - \left(\frac{D}{3}\right)\right).$$

We get the following result:

Theorem 5. *If D is positive integer $\equiv 3$ (mod. 4) having no squared factor > 1 and if $h(\sqrt{-D})$ is not divisible by 3, the equation*

$$x^2 + x + \tfrac{1}{4}(D+1) = y^3$$

has the only solution $y = \frac{1}{3}(D-1)$ if D is of the form $3c^2 + 4$, and the only solution $y = \frac{1}{3}(D+1)$ if D is of the form $3c^2 - 4$, and has no integral solutions for other values of D.

Remark. There are infinitely many integers without squared factor > 1 of the form $3c^2 + 4$ and $3c^2 - 4$.[6]

*

Let us now suppose that the class number $h(\sqrt{-D})$ is divisible by 3. In this case the equation

$$x^2 + x + \tfrac{1}{4}(D+1) = y^3$$

can have other solutions than those given in the theorem 5. As examples we treat the cases $D = 23, 31, 59, 83$ where the class number has the value 3.

I. $D = 23$. The ideal classes in $K(\sqrt{-23})$ can be represented by the ideals (1), $(2, \varrho)$ and $(2, \varrho')$ where $\varrho = \frac{1}{2}(-1 + \sqrt{-23})$.

We have

$$(2, \varrho) \cdot (2, \varrho') = (2) \quad \text{and} \quad (2, \varrho')^3 = (1 - \varrho).$$

The equation

$$x^2 + x + 6 = y^3 \tag{7}$$

gives as in the general case

$$(x - \varrho) = \mathfrak{a}^3 \tag{8}$$

where $\mathfrak{a}$ is an ideal.

If $\mathfrak{a} \sim (1)$ we get as in theorem 5 $y = \frac{1}{3}(23 + 1) = 8$, since $23 = 3 \cdot 3^2 - 4$.

If $\mathfrak{a} \sim (2, \varrho)$ we get from (8)

$$(2, \varrho')^3 (x - \varrho) = (2, \varrho')^3 \mathfrak{a}^3$$

$$(1 - \varrho)(x - \varrho) = (a + b\varrho)^3 \tag{9}$$

where a and b are integers.

$$x - 6 - \varrho(x + 2) = a^3 - 18ab^2 + 6b^3 + \varrho(3a^2b - 3ab^2 - 5b^3).$$

Hence we get the system

$$x - 6 = a^3 - 18ab^2 + 6b^3$$

$$-x - 2 = 3a^2b - 3ab^2 - 5b^3.$$

Hence after elimination of x

$$-8 = a^3 + 3a^2b - 21ab^2 + b^3 \tag{10}$$

From (9) we get

$$2y = N(a + b\varrho) = a^2 - ab + 6b^2.$$

The equation (10) has the solutions

$$\begin{cases} a = -2 \\ b = 0 \end{cases}; \quad \begin{cases} a = 3 \\ b = 1 \end{cases} \quad \text{and} \quad \begin{cases} a = 0 \\ b = -2 \end{cases}$$

which give the solutions $y = 2$, $y = 6$ and $y = 12$ respectively of (7).

The case $\mathfrak{a} \sim (2, \varrho')$ leads to (10) too. We see that by replacing x by $-1 - x$, a by $-a$ and b by $-b$ in (8) and (9).

II. $D = 31$. As representatives of the ideal classes we choose (1), $(2, \varrho)$ and $(2, \varrho')$ with $\varrho = \frac{1}{2}(-1 + \sqrt{-31})$.

We have

$$(2, \varrho) \cdot (2, \varrho') = (2) \quad \text{and} \quad (2, \varrho')^3 = (\varrho')$$

The equation

$$x^2 + x + 8 = y^3 \tag{11}$$

gives

$$(x - \varrho) = \mathfrak{a}^3. \tag{12}$$

If $\mathfrak{a} \sim (1)$ we get as in theorem 5 $y = \frac{1}{3}(31 - 1) = 10$, since $31 = 3 \cdot 3^2 + 4$.
If $\mathfrak{a} \sim (2, \varrho)$ we get from (12)

$$\varrho' (x - \varrho) = (a + b\varrho)^3$$

where a and b are integers.

$$-x - 8 - \varrho x = a^3 - 24ab^2 + 8b^3 + \varrho(3a^2b - 3ab^2 - 7b^3).$$

Hence

$$-8 = a^3 - 3a^2b - 21ab^2 + 15b^3 \tag{13}$$

with

$$2y = a^2 - ab + 8b^2.$$

The equation (13) has the solutions

$$\begin{cases} a = -2 \\ b = 0 \end{cases}; \quad \begin{cases} a = 1 \\ b = 1 \end{cases} \quad \text{and} \quad \begin{cases} a = 4 \\ b = 6 \end{cases}$$

which give the solutions $y = 2$, $y = 4$ and $y = 140$ respectively of (11). $\mathfrak{a} \sim (2, \varrho')$ gives the same solutions.

III. $D = 59$. As representatives of the ideal classes we choose (1), $(3, \varrho)$ and $(3, \varrho')$ with $\varrho = \frac{1}{2}(-1 + \sqrt{-59})$.
We have

$$(3, \varrho) \cdot (3, \varrho') = (3) \quad \text{and} \quad (3, \varrho')^3 = (\varrho' - 3)$$

The equation

$$x^2 + x + 15 = y^3 \tag{14}$$

gives

$$(x - \varrho) = \mathfrak{a}^3. \tag{15}$$

$\mathfrak{a} \sim (1)$ gives no solution of (14).
If $\mathfrak{a} \sim (3, \varrho)$ we get from (15)

$$(\varrho' - 3)(x - \varrho) = (a + b\varrho)^3$$

$$-4x - 15 + \varrho(3 - x) = a^3 - 45ab^2 + 15b^3 + \varrho(3a^2b - 3ab^2 - 14b^3)$$

Hence

$$-27 = a^3 - 12a^2b - 33ab^2 + 71b^3 \tag{16}$$

with

$$3y = a^2 - ab + 15b^2.$$

The equation (16) has the solutions

$$\begin{cases} a = -3 \\ b = 0 \end{cases} \quad \text{and} \quad \begin{cases} a = -1 \\ b = -1 \end{cases}$$

which give the solutions $y = 3$ and $y = 5$ respectively of (14).

IV. $D = 83$. As representatives of the ideal classes we choose (1), $(3, \varrho)$ and $(3, \varrho')$ with $\varrho = \frac{1}{2}(-1 + \sqrt{-83})$.

We have

$$(3, \varrho)\cdot(3, \varrho') = (3) \quad \text{and} \quad (3, \varrho')^3 = (\varrho' + 3).$$

The equation

$$x^2 + x + 21 = y^3 \tag{17}$$

gives

$$(x - \varrho) = \mathfrak{a}^3. \tag{18}$$

$\mathfrak{a} \sim (1)$ gives no solution of (17).

If $\mathfrak{a} \sim (3, \varrho)$ we get from (18)

$$(\varrho' + 3)(x - \varrho) = (a + b\varrho)^3$$

with

$$3y = a^2 - ab + 21b^2.$$

$$2x - 21 + \varrho(-x - 3) = a^3 - 63ab^2 + 21b^3 + \varrho(3a^2b - 3ab^2 - 20b^3).$$

Hence

$$-27 = a^3 + 6a^2b - 69ab^2 - 19b^3. \tag{19}$$

The equation (19) has the solution $a = -3$, $b = 0$ which gives the solution $y = 3$ of (17).

At the end of this paper we give a table containing solutions of the equation

$$x^2 + x + \tfrac{1}{4}(D + 1) = y^3$$

when D is a prime < 100.

§ 7.

Let us examine the case $q = 5$, when $h(\sqrt{-D})$ is not divisible by 5. We get from (5)

$$2^4\left(\frac{D}{5}\right) = 5c^4 - 10c^2D + D^2. \tag{20}$$

Hence we find that the equation

$$x^2 + x + \tfrac{1}{4}(D + 1) = y^5$$

has at most one solution y. For if there were different values of c for a given D, we get from (20)

$$5c_1^4 - 10c_1^2D = 5c_2^4 - 10c_2^2D$$

and hence

$$c_1^2 + c_2^2 = 2D$$

i. e.

$$2D \equiv 2 \pmod{8}$$

which is impossible since $D \equiv 3 \pmod{4}$.

If we put with an odd z

$$D - 5c^2 = 2z \tag{21}$$

the equation (20) is transformed into

$$4\left(\frac{D}{5}\right) = z^2 - 5c^4. \tag{22}$$

I. If $\left(\frac{D}{5}\right) = 1$ we get

$$z^2 - 4 = 5c^4. \tag{22'}$$

(22′) has the only solution $z = 3$ and $c = 1$ in odd positive z and c. This can be proved in the following way. (22′) can be written

$$(z+2)(z-2) = 5c^4$$

with

$$(z+2,\ z-2) = 1.$$

Hence we get the system $(z > 0)$

$$z \pm 2 = 5a^4; \quad z \mp 2 = b^4 \tag{A}$$

where $(a, b) = 1$ and $ab = c$.
Hence

$$\pm 4 = 5a^4 - b^4$$

where the lower sign is impossible, since the right member is congruent 4 modulo 16. Hence

$$b^4 + 4 = 5a^4$$

which can be written

$$(b^2 + 2b + 2)(b^2 - 2b + 2) = 5a^4$$

where

$$(b^2 + 2b + 2,\ b^2 - 2b + 2) = (b^2 + 2b + 2,\ 4b) = 1.$$

Hence we get the system

$$b^2 \pm 2b + 2 = 5f^4; \quad b^2 \mp 2b + 2 = g^4$$

with $(f, g) = 1$ and $fg = a$.
The last equation can be written

$$(b \mp 1)^2 + 1 = g^4$$

but as is well known the diophantine equation

$$x^4 - y^4 = z^2$$

has the only solution $z = 0$. Hence $b = \pm 1$ is the only solution of the system (A) and hence $z = \pm 3$, $c^2 = 1$ the only solutions of (22′) in odd integers.

This gives by (21) $D = 11$ as the only case when the equation

$$x^2 + x + \tfrac{1}{4}(D + 1) = y^5$$

is solvable if $\left(\frac{D}{5}\right) = 1$.

$$15^2 + 15 + \tfrac{1}{4}(11 + 1) = 3^5.$$

II. If $\left(\frac{D}{5}\right) = -1$, (22) becomes

$$z^2 + 4 = 5\,c^4 \qquad (22'')$$

The equation (22″) has only the solutions $z = \pm 1$, $c^2 = 1$ according to an information from W. LJUNGGREN not yet published.

This gives by (21) $D = 7$ and $D = 3$ as the only cases of solvability if $\left(\frac{D}{5}\right) = -1$.

$$5^2 + 5 + \tfrac{1}{4}(7 + 1) = 2^5.$$

We get the following result:

Theorem 6. *If D is a positive integer $\equiv 3$ (mod. 4) having no squared factor > 1 and if $h(\sqrt{-D})$ is not divisible by 5, the equation*

$$x^2 + x + \tfrac{1}{4}(D + 1) = y^5$$

is solvable in integers x and y only when $D = 3$, 7 and 11. In these cases the equation has a single solution y.

§ 8.

Finally we consider the special case $q = 7$, when $h(\sqrt{-D})$ is not divisible by 7. We shall show that the equation

$$x^2 + x + \tfrac{1}{4}(D + 1) = y^7 \qquad (23)$$

has at most one solution y for a given D.

From (5) we get

$$7\,c^6 - 35\,c^4 D + 21\,c^2 D^2 - D^3 + 2^6 \left(\frac{D}{7}\right) = 0. \qquad (24)$$

Hence we get from (6) for the solutions y of the equation (23)

$$7\,y^3 - 14\,D\,y^2 + 7\,D^2 y - D^3 + \left(\frac{D}{7}\right) = 0. \qquad (25)$$

Let y_1, y_2 and y_3 be the roots of the equation (25). We have

$$y_1 + y_2 + y_3 = 2D \tag{I}$$

$$y_1 y_2 + y_2 y_3 + y_3 y_1 = D^2 \tag{II}$$

$$7 y_1 y_2 y_3 = D^3 - \left(\frac{D}{7}\right) \tag{III}$$

Hence we see that the equation (23) cannot have three solutions y. For if $D \equiv -1 \pmod{8}$ we would have by (23) $y_i \equiv 0 \pmod{2}$ for $i = 1, 2, 3$ against (II). If $D \equiv 3 \pmod{8}$ we would have by (23) $y_i \equiv 1 \pmod{2}$ against (I). Neither can we have two solutions for $D \equiv -1 \pmod{8}$ according to (II). That it is the same in the case $D \equiv 3 \pmod{8}$ can be seen in the following way.

Let (25) have three integral roots y_1, y_2, y_3. We put $y_1 + y_2 = u$; $y_1 y_2 = v$. Hence by (I) and (II)

$$u + y_3 = 2D; \quad v + y_3 u = D^2$$

and after elimination of y_3

$$y_1 y_2 = v = (D - u)^2 \tag{26}$$

The y_i are relatively prime two and two by (II), since $(D, y_i) = 1$ by (25). Hence all y_i are squares according to (26). If y_1 and y_2 are odd we get $y_3 \equiv 4 \pmod{8}$ by (I) and $\left(\frac{D}{7}\right) = -1$ in (III). Hence the relation (III) can be written

$$7A^2 = D^3 + 1$$

where A is an integer.

This equation has the only integral solutions $D = -1$ and $D = 3$.[7] With $D = 3$ we get $y_1 = y_2 = 1$ and $y_3 = 4$.

If we put

$$D = 4z + 7c^2$$

the equation (24) is transformed into

$$z^3 - 7c^4 z - 7c^6 = \left(\frac{D}{7}\right). \tag{24'}$$

For $c = 1$ we get, if $\left(\frac{D}{7}\right) = -1$, the solutions $z = 3$; $z = -1$ and $z = -2$ of (24'). The two first values give $D = 19$ and $D = 3$ respectively. Hence the equation

$$x^2 + x + 5 = y^7$$

has the only solution $y = 5$.

$$279^2 + 279 + \tfrac{1}{4}(19 + 1) = 5^7.$$

Table

containing solutions of the equation

$$x^2 + x + \tfrac{1}{4}(D + 1) = y^3$$

when D is a prime < 100.

D	$h(\sqrt{-D})$	$D =$ $3c^2 + 4$	$D =$ $3c^2 - 4$	Solution when $\alpha \sim (1)$		Other solutions	
		c	c	y	$x\ (> 0)$	y	$x\ (\geqq 0)$
7	1	1	—	2	2	—	—
11	1	—	—	—	—	—	—
19	1	—	—	—	—	—	—
23	3	—	3	8	22	2 6 12	1 14 41
31	3	3	—	10	31	2 4 140	0 7 1656
43	1	—	—	—	—	—	—
47	5	—	—	—	—	—	—
59	3	—	—	—	—	3 5	3 10
67	1	—	—	—	—	—	—
71	7	—	5	24	117	—	—
79	5	5	—	26	132	—	—
83	3	—	—	—	—	3	2

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