Communicated October 27, 1948 by F. Carlson

On a class of orthogonal series

By H. Bohman

1. Let $\{\varphi_n(x)\}$ denote a normalized orthogonal system, defined in the interval (a, b) and belonging to the class L^2. The famous theorem of Rademacher-Menchoff [I: 162] tells us that

$$\sum_{n=1}^{\infty} a_n \varphi_n(x)$$

is convergent almost everywhere in (a, b) if

$$\sum_{n=1}^{\infty} a_n^2 (\log n)^2 < \infty.$$

Conversely, if $\Sigma a_n^2 (\log n)^2 = \infty$ then it is possible to find a system $\{\varphi_n\}$ for which $\Sigma a_n \varphi_n$ is divergent almost everywhere.

In order that

$$\sum_{n=1}^{\infty} a_n^2 < \infty$$

should be a sufficient condition for the convergence almost everywhere of the series it will thus be necessary to specialize the orthogonal system. Kolmogoroff has found that if $\{\varphi_n\}$ is a system of independent random variables, then $\Sigma a_n^2 < \infty$ is a necessary and sufficient condition for the convergence almost everywhere of the series. The object of the following paper is to generalize slightly this result of Kolmogoroff.

2. In dealing with questions of this kind, the theory of the torus space seems to be very useful. Following Jessen, who made the first systematic study of this space, we denote it by Q_ω.

Q_ω is an ω-dimensional vector space, consisting of all infinite sequences of real numbers $\xi = (x_1, x_2, \ldots, x_n, \ldots)$ where $0 \leq x_n < 1$ for $n = 1, 2, \ldots$. The subspace Q_n consists of the points $\xi_n = (x_1, x_2, \ldots, x_n)$. In the same way $Q_{n,\omega}$ consists of the points $\xi_{n,\omega} = (x_{n+1}, x_{n+2}, \ldots)$. We may then consider Q_ω as the product space $Q_\omega = (Q_n, Q_{n,\omega})$. In accordance with this notation we may write $\xi = (\xi_n, \xi_{n,\omega})$.

The following two theorems of Jessen are fundamental for the theory.

If $f(\xi)$ is an integrable function in Q_ω then

$$(2.1)\qquad \int_{Q_\omega} f(\xi)\,d\xi = \lim_{n\to\infty}\int_{Q_n} f(\xi)\,d\xi_n = \lim_{n\to\infty}\int_0^1 d x_n \int_0^1 d x_{n-1}\cdots\int_0^1 f(\xi)\,d x_1$$

and

$$(2.2)\qquad \lim_{n\to\infty}\int_{Q_{n,\,\omega}} f(\xi)\,d\xi_{n,\,\omega} = f(\xi) \text{ almost everywhere.}$$

3. It is easily seen that an enumerable set of independent random variables may be represented in Q_ω as a system of real functions with the following property.

If $f_n(\xi)$ for $n = 1, 2, \ldots$ denotes a random variable then $f_n(\xi) = f_n(x_n)$, i. e. for each n $f_n(\xi)$ depends only on x_n.

Having noticed this fact we can state KOLMOGOROFF's theorem in the following form.

Let $\{\varphi_n(\xi)\}$ denote a normalized orthogonal system in Q_ω and suppose that $\varphi_n(\xi) = \varphi_n(x_n)$ for each n [III: 141]. Then the series

$$\sum_{n=1}^{\infty} a_n\,\varphi_n(x_n)$$

converges almost everywhere in Q_ω if $\Sigma a_n^2 < \infty$ but diverges almost everywhere if $\Sigma a_n^2 = \infty$.

The first part of this theorem may be proved as follows.

By the RIESZ-FISCHER theorem the partial sums

$$\sum_{n=1}^{N} a_n\,\varphi_n(x_n)$$

converge in mean to a function $\varphi(\xi)$.

By theorem (2.1) [II: 286]

$$\int_{Q_{N,\,\omega}} \varphi(\xi)\,d\xi_{N,\,\omega} = \sum_{n=1}^{N} a_n\,\varphi_n(x_n)$$

and hence by theorem (2.2)

$$\lim_{N\to\infty}\sum_{n=1}^{N} a_n\,\varphi_n(x_n) = \varphi(\xi) \text{ almost everywhere.}$$

Well-known examples of orthogonal systems of this type are

a) RADEMACHER's system $\{r_n(\xi)\}$, [I: 42], where $r_n(\xi) = \begin{cases} +1 & \text{if } 0 < x_n < \frac{1}{2} \\ 0 & \text{if } x_n = 0 \text{ or } \frac{1}{2} \\ -1 & \text{if } \frac{1}{2} < x_n < 1 \end{cases}$

b) The system $\{\theta_n(\xi)\}$, [I: 134], where $\theta_n(\xi) = e^{2\pi i x_n}$.

4. Let $\{\varphi_n(x_n)\}$ be a normalized orthogonal system of the type just mentioned. We will now define a new system of functions $\{\psi_n(\xi)\}$ in the following way. Let

$$n = \varepsilon_1 + \varepsilon_2 2 + \varepsilon_3 2^2 + \cdots$$

be the expression for n in the dyad scale. If ε_ν is 1 for $\nu = \nu_1, \nu_2, \ldots \nu_k$ and 0 otherwise, we denote by $\psi_n(\xi)$ the product

$$\varphi_{\nu_1} \cdot \varphi_{\nu_2} \cdot \varphi_{\nu_3} \cdots \varphi_{\nu_k}.$$

By virtue of FUBINI's theorem

$$\int\limits_{Q_\omega} \psi_n^2(\xi)\, d\xi = \int\limits_0^1 \varphi_{\nu_1}^2\, dx_{\nu_1} \int\limits_0^1 \varphi_{\nu_2}^2\, dx_{\nu_2} \cdots \int\limits_0^1 \varphi_{\nu_k}^2\, dx_{\nu_k} = 1$$

and hence $\{\psi_n(\xi)\}$ is normalized. It is also orthogonal. For

$$\int\limits_{Q_\omega} \psi_n(\xi) \cdot \psi_m(\xi)\, d\xi$$

may be expressed similarly as a product of integrals, one of which, at least, is of the form

$$\int\limits_0^1 \varphi_k\, dx_k = 0.$$

Starting for example with RADEMACHER's system $\{r_n\}$, [I: 132], we obtain a system $\{\psi_n\}$, which is easily identified with WALSH's system.

5. In this and the following sections we will deduce some theorems concerning real orthogonal systems of the type $\{\psi_n\}$.

5.1. We define the distribution function for a measurable function $\varphi(x)$ as

$$F(t) = m E(\varphi(x) \leq t).$$

Suppose that the distribution function of each φ_n is continuous. Then $\Sigma a_n \psi_n$ is either convergent almost everywhere or divergent almost everywhere.

To prove this theorem, we make the following purely formal decomposition of the series

$$S = \sum_{n=1}^{\infty} a_n \psi_n = \sum_{n=0}^{\infty} a_{2n+1} \psi_{2n+1} + \sum_{n=1}^{\infty} a_{2n} \psi_{2n} =$$

$$= \varphi_1(x_1) \sum_{n=0}^{\infty} a_{2n+1} \psi_{2n} + \sum_{n=1}^{\infty} a_{2n} \psi_{2n} = \varphi_1 \cdot S_1 + S_2$$

where S_1 and S_2 are independent of x_1.

If S and S_2 are convergent, then S_1 is convergent. If S is convergent but S_2 divergent, then S_1 is divergent. If α and β were two different values of φ_1 for which the latter case would occur, then $(\alpha - \beta) S_1$ would be convergent, contrary to the hypothesis. Consequently, for each $\xi_{1,\omega}$ there is at most one such value α. According to our assumption the distribution function of φ_1 is continuous; hence $m E(\varphi_1 = \alpha) = 0$.

We can therefore state: If S is convergent in a set E, both S_1 and S_2 are convergent almost everywhere in E. This may also be expressed in the following way: The measure of E is independent of x_1. S_1 and S_2 are, however, orthogonal series of the same type as S. The argument may be applied once again; $m E$ is independent of x_2. We proceed in that way and obtain:

For each n, $m E$ is independent of $x_1, x_2, \ldots x_n$. By an important lemma of JESSEN $m E$ is then either 0 or 1 [II: 270].

5.2. Consider again a system of the type $\{\varphi_n\}$.

As coefficients for the series we choose a system of functions $\{f_n\}$, belonging to L^2 and with the property

$$f_n = f_n(x_1, x_2, \ldots, x_{n-1}) \text{ for every } n.$$

Let S_ν for $\nu = 1, 2, \ldots$ denote the partial sums

$$f_1 \varphi_1 + f_2 \varphi_2 + \cdots + f_\nu \varphi_\nu$$

and E_N the set where

$$\overline{\text{bound}} \, \{|S_1|, |S_2|, \ldots, |S_N|\} > \varepsilon$$

then

$$m E_N < \frac{\sum_{\nu=1}^{N} \int_{Q_\omega} f_\nu^2 \, d\xi}{\varepsilon^2}.$$

To prove this, let B_n be the set where $|S_n| > \varepsilon$ and B_n^* the complementary set of B_n. We may evidently write [II: 275, III: 141]

$$E_N = B_1 + B_2 B_1^* + B_3 B_2^* B_1^* + \cdots + B_N B_{N-1}^* B_{N-2}^* \cdots B_1^* = A_1 + A_2 + A_3 + \cdots + A_N.$$

According to our hypothesis A_n is for every n a cylinderset in Q_ω with its base in Q_n.

$$\varepsilon^2 \cdot m A_n < \int_{A_n} S_n^2 \, d\xi \leq \int_{A_n} S_n^2 \, d\xi + \int_{A_n} (S_N - S_n)^2 \, d\xi = \int_{A_n} S_N^2 \, d\xi.$$

Adding these relations for $n = 1, 2, \ldots, N$, we get

$$\varepsilon^2 \cdot m E_N < \int_{E_N} S_N^2 \, d\xi \leq \int_{Q_\omega} S_N^2 \, d\xi = \sum_{\nu=1}^{N} \int_{Q_\omega} f_\nu^2 \, d\xi.$$

The inequality is thus proved. The method of proof also gives the following extension.

Let g be a function of the variables

$$x_{N+1},\ x_{N+2},\ \ldots \text{ only, i. e. } g = g(x_{N+1},\ x_{N+2},\ \ldots)$$

and let E_N be the set where

$$\overline{\text{bound}}\ \{|g S_1|,\ |g S_2|,\ \ldots,\ |g S_N|\} > \varepsilon$$

then

$$m E_N < \int\limits_{Q_\omega} g^2\, d\xi \, \frac{\sum\limits_{\nu=1}^{N} \int\limits_{Q_\omega} f_\nu^2\, d\xi}{\varepsilon^2}.$$

5.3. We return to series of the type $\Sigma a_n \psi_n$ and denote their partial sums by σ_N, i. e.

$$\sigma_N = \sum_{\nu=0}^{N} a_\nu \psi_\nu.$$

It is easily seen that the subsequence σ_{2n} is convergent almost everywhere, if $\Sigma a_\nu^2 < \infty$.

Since σ_N converges in mean to a function σ and

$$\int\limits_{Q_{N,\omega}} \sigma\, d\xi = \sigma_{2N}$$

we have by JESSEN's theorem (2.2)

$$\lim \sigma_{2N} = \sigma \text{ almost everywhere.}$$

6. We will now denote a function ψ by $\psi^{(k)}$ if it has k factors. We then divide the system $\{\psi_n\}$ into partial systems

$$\{\psi_n^{(k)}\} \qquad k = 1, 2, 3, \ldots$$

In each partial system the indices of the functions are changed, while the mutual order is kept unaltered.

We can now deduce the following theorem.

If $\Sigma a_n^2 < \infty$, then the series

$$\sum_{n=1}^{\infty} a_n \psi_n^{(k)}$$

is convergent almost everywhere for every k.

This can be proved by induction. For $k = 1$ the theorem holds true, and moreover we have the following inequality (cf. 5.2)

$$m E\left\{\overline{\text{bound}}_{1 \le \nu \le N}\ \{|S_\nu^{(1)}|\} > \varepsilon\right\} < \frac{1}{\varepsilon^2} \sum_{\nu=1}^{N} a_\nu^2.$$

Suppose that the theorem holds for k and that

$$m\,E\left\{\overline{\underset{1\le\nu\le N}{\text{bound}}}\,\{|S_\nu^{(k)}|\}>k\,\varepsilon\right\}<\frac{k}{\varepsilon^2}\sum_{\nu=1}^{N}a_\nu^2.$$

Under these assumptions we can prove the theorem for $k+1$.

By using the same method of proof as in 5.3 it is easily seen that the sequence $S^{(k+1)}_{\binom{k+1+\nu}{\nu}}$ is convergent almost everywhere, when ν runs through $0, 1, 2, \ldots$ and $\binom{k+1+\nu}{\nu}=\frac{(k+1+\nu)!}{(k+1)!\,\nu!}$.

For $\binom{k+1+\nu}{\nu}<n\le\binom{k+2+\nu}{\nu+1}$ the functions $S_n^{(k+1)}-S^{(k+1)}_{\binom{k+1+\nu}{\nu}}$ are of the form

$$\varphi_{k+2+\nu}\sum_{\mu=1}^{n-\binom{k+1+\nu}{\nu}}a_{\binom{k+1+\nu}{\nu}+\mu}\cdot\psi_\mu^{(k)}.$$

The upper bound for these $\binom{k+1+\nu}{\nu+1}$ functions is greater than $k\,\varepsilon$ in a set E_ν. The characteristic function of E_ν is denoted by $f(E_\nu)$

$$m\,E_\nu=\int_{Q_\omega}f(E_\nu)\,d\,\xi=\int_0^1 d\,x_{k+2+\nu}\int_{(Q_{k+1+\nu},\,Q_{k+2+\nu,\,\omega})}f(E_\nu)\,d\,x_1\,d\,x_2\cdots d\,x_{k+1+\nu}\,d\,x_{k+3+\nu}\cdots.$$

According to our assumptions the second integral is less than

$$\varphi^2_{k+\nu+2}\,\frac{k}{\varepsilon^2}\sum_{\mu=1}^{\binom{k+1+\nu}{\nu+1}}a^2_{\binom{k+1+\nu}{\nu}+\mu}$$

and hence

$$m\,E_\nu<\frac{k}{\varepsilon^2}\sum_{\mu=1}^{\binom{k+1+\nu}{\nu+1}}a^2_{\binom{k+1+\nu}{\nu}+\mu}\int_0^1\varphi^2_{k+2+\nu}\,d\,x_{k+2+\nu}=\frac{k}{\varepsilon^2}\sum_{\mu=1}^{\binom{k+1+\nu}{\nu+1}}a^2_{\binom{k+1+\nu}{\nu}+\mu}$$

$\sum_\nu m\,E_\nu$ is thus convergent; hence $m\,\bar{E}=m\ \limsup E_\nu=0$.

Choosing $\nu(n)$ so that

$$\binom{k+1+\nu(n)}{\nu(n)}<n\le\binom{k+2+\nu(n)}{\nu(n)+1}$$

we get

$$\limsup_{n\to\infty}\left|S_n^{(k+1)}-S^{(k+1)}_{\binom{k+1+\nu(n)}{\nu(n)}}\right|<k\,\varepsilon\ \text{ almost everywhere.}$$

ε is arbitrarily small, and hence

$$\lim_{n\to\infty}S_n^{(k+1)}=\lim_{\nu\to\infty}S^{(k+1)}_{\binom{k+1+\nu}{\nu}}\ \text{ almost everywhere.}$$

To prove the inequality for $k+1$, observe that the above proof gives the following relation.

Let E_1 be the set where

$$\overline{\text{bound}}_{1 \leq n \leq N} \left\{ \left| S_n^{(k+1)} - S^{(k+1)}_{\binom{k+1+\nu(n)}{\nu(n)}} \right| \right\} > \varepsilon$$

then

$$m E_1 < \frac{k}{\varepsilon^2} \sum_{\nu=1}^{N} a_\nu^2.$$

Next we have

$$S^{(k+1)}_{\binom{k+2+\nu}{\nu+1}} - S^{(k+1)}_{\binom{k+1+\nu}{\nu}} = \varphi_{k+2+\nu} \sum_{\mu=1}^{\binom{k+1+\nu}{\nu+1}} a_{\binom{k+1+\nu}{\nu}+\mu}\, \psi_\mu^{(k)}$$

i. e. the sequence

$$S^{(k+1)}_{\binom{k+1+\nu}{\nu}}, \qquad \nu = 0, 1, 2, \ldots,$$

is of the same type as the sequence S_ν, studied in 5. 2.

Let ν_0 be the smallest integer for which

$$\binom{k+2+\nu_0}{\nu_0+1} > N$$

and E_2 the set where

$$\overline{\text{bound}}_{0 \leq \nu \leq \nu_0} \left\{ \left| S^{(k+1)}_{\binom{k+1+\nu}{\nu}} \right| \right\} > \varepsilon$$

then, using theorem 5. 2, we get

$$m E_2 < \frac{1}{\varepsilon^2} \sum_{\nu=1}^{\binom{k+1+\nu_0}{\nu_0}} a_\nu^2 \leq \frac{1}{\varepsilon^2} \sum_{\nu=1}^{N} a_\nu^2.$$

From these relations we obtain the desired result

$$m E \left\{ \overline{\text{bound}}_{1 \leq n \leq N} \{ | S_n^{(k+1)} | \} > (k+1)\, \varepsilon \right\} < \frac{k+1}{\varepsilon^2} \sum_{\nu=1}^{N} a_\nu^2.$$

REFERENCES. [I] **Kaczmarz & Steinhaus:** Theorie der Orthogonalreihen. Warszawa 1935. — [II] **Jessen:** The theory of integration in a space of an infinite number of dimensions. Acta mathematica 63 (1934). — [III] **Lévy:** Théorie de l'addition des variables aléatoires. Paris 1937.

Tryckt den 10 februari 1949.

Uppsala 1949. Almqvist & Wiksells Boktryckeri AB