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Analytic structures in the maximal ideal space of a uniform algebra

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Introduction

Let A be a uniform algebra with its maximal ideal space M_A and its Shilov boundary S_A. We say that M_A has a (one-dimensional) analytic structure at a point $x_0 \in M_A \setminus S_A$ if the following condition holds. There is an open neighborhood W of x_0 in M_A and some $f \in A$ such that $W \setminus \{x_0\} = V_1 \cup \ldots \cup V_n$, where V_i are disjoint open subsets of M_A each mapped homeomorphically by f onto the set $D \setminus \{0\}$ in C^1. Here D is the open unit disc and $f(x_0)=0$. The positive integer n above is called the branch-order of x_0.

If M_A has an analytic structure at the point x_0 as above, then J. Wermer has proved that if $g \in Z$ and if we define $g_i(z)=g(x_i(z))$ on $D \setminus \{0\}$, where $x_i(z)$ is the point in V_i for which $f(x_i(z))=z$ while $g_i(0)=g(x_0)$, then $g_1 \ldots g_n$ are analytic functions in D.

Conditions which guarantee that subsets of $M_A \setminus S_A$ have an analytic structure have originally been studied by J. Wermer in [5–6]. In Section 1 of this paper we prove some results which originally were obtained by Wermer under certain regularity conditions. The core of this section is the proof of Theorem 1.7. and in the final part we discuss some consequences of this result.

Section 1

Firstly we introduce some notations and collect some wellknown facts about uniform algebras. If X is a compact space and if $f \in C(X)$ we put $|f|_X = \sup\{|f(x)| : x \in X\}$. If W is a subset of X then ∂W denotes its topological boundary. If A is a uniform algebra and if F is closed subset of M_A we put $\operatorname{Hull}_A(F) = \{x \in M_A : |f(x)| \leqslant |f|_F$ for all f in $A\}$. We also introduce the uniform algebra $A(F) = \{g \in C(F) : \exists (f_n)$ in A with $\lim |f_n - g|_F = 0\}$. Here we know that $M_{A(F)}$ can be identified with $\operatorname{Hull}_A(F)$.

If A is a uniform algebra and if $f \in A$ we define the fibers $\pi_f^{-1}(z) = \{x \in M_A : f(x) = z\}$ for each $z \in C^1$. We recall the wellknown result below.

Lemma 1.1. *Let A be a uniform algebra and let $f \in A$. Suppose that U is an open component of the set $C^1 \setminus f(S_A)$. Then two cases are possible, either $U \cap f(M_A)$ is empty or else $U \subset f(M_A)$.*

Next we introduce a concept which appears in [2, p. 525].

Definition 1.2. *Let A be a uniform algebra and let $f \in A$. Let U be an open component of the set $C^1 \setminus f(S_A)$ such that the fibers $\pi_f^{-1}(z)$ contain at most n points for each $z \in U$, while equality holds for some point. Here $n \geq 0$ and we say that U is an f-regular component of multiplicity n.*

In [2] E. Bishop proves that if U is an f-regular component of multiplicity n, with $n > 0$, then there exists n subsets $W_1 \dots W_n$ in M_A each mapped homeomorphically by f onto U. If then $g \in A$ and if we put $g_i(z) = g(\pi_f^{-1}(z) \cap W_i)$ for all $z \in U$, then $g_1 \dots g_n$ are analytic in U. There also exists some g in A such that $g_1 \dots g_n$ are distinct. So if we put $D_{ij} = \{z \in U : g_i(z) = g_j(z)\}$, then $W_i \cap W_j$ is contained in the discrete set $\pi_f^{-1}(D_{ij})$. This result shows that the open set $\pi_f^{-1}(U)$ has analytic structure at all points and the branch-order is one except for a discrete subset where the branch-order varies between 2 and n.

Now we wish to state some criteria which guarantee that an open component of the set $C^1 \setminus f(S_A)$ is f-regular when f belongs to some A. Firstly we have the so called Local Principle which appears in [2, Lemma 17].

Lemma 1.3. *Let U be an open component of the set $C^1 \setminus f(S_A)$. Suppose that U contains an open subset V such that $\# \pi_f^{-1}(z) = n$ for all $z \in V$. Then U is f-regular of multiplicity n.*

Notice that Lemma 1.1. is the case when $n = 0$ in Lemma 1.3. The next result is the so-called Analytic Disc Lemma.

Lemma 1.4. *Let $f \in A$ and let Γ be a closed Jordan curve with its interior Δ. Suppose that Γ contains an open subarc J such that $\# \pi_f^{-1}(z) \leq n$ for all $z \in J$ and that the (possibly empty) set $\pi_f^{-1}(\Delta)$ is contained in $M_A \setminus S_A$. Then $\# \pi_f^{-1}(z) \leq n$ for all z in Δ.*

Proof. The case when $n = 1$ is contained in [6, Lemma 8]. If $n \geq 2$ we can choose $z_0 \in J$ so that $\# \pi_f^{-1}(z_0) = n$ say, while $\# \pi_f^{-1}(z) \leq n$ for all z in $J \cap D(z_0)$, where $D(z_0)$ is an open disc centered at z_0. Let $x_1 \dots x_n$ be the points in $\pi_f^{-1}(z_0)$ and choose disjoint open neighborhoods $W_1 \dots W_n$ of these points so that $f(W_i) \cap J$ is contained in $D(z_0)$ for each i.

Now we can easily find a closed Jordan curve Γ_0 with its interior Δ_0 such that $\Gamma_0 \cap \Gamma$ is a closed subarc of the set $J \cap D(z_0)$ while Δ_0 is contained in Δ. In addition we can arrange this so that $\pi_f^{-1}(\Delta_0 \cup \Gamma_0)$ is contained in $W_1 \cup \dots \cup W_n$, since this set is an open neighborhood of the whole fiber $\pi_f^{-1}(z_0)$.

From the case $n = 1$ we can conclude that $\pi_f^{-1}(z) \cap W_i$ contains at most one point for each i, and hence $\# \pi_f^{-1}(z) \leq n$ for all $z \in \Delta_0$. Here Δ_0 is an open subset of the connected set Δ, so an application of Lemma 1.3. completes the proof.

Using Lemma 1.4. and Theorem 1.7. below we can easily deduce the following result.

Proposition 1.5. *Let $f \in A$ and suppose that $f(S_A)$ contains a relatively open subset J which is homeomorphic with an open Jordan arc. Suppose also that the sets $S_z = \{x \in S_A : f(x) = z\}$ contain at most n points for each z in J. Suppose that U and V are the two (distinct) components of the set $C^1 \setminus f(S_A)$ which borders J, and that V is f-regular of multiplicity m. Here $m = 0$ may occur. Then it follows that U is f-regular of some multiplicity $\leq m + n$.*

Let us remark that Proposition 1.5. above was proved by J. Wermer in the case when J is real analytic and it also appears in [2, Lemma 21] when $n = 1$ under a less restrictive regularity assumption about J. Finally we wish to point out that Proposition 1.5. is contained in the work by H. Alexander in [1].

Next we wish to state a result, again due to J. Wermer which will be used in the proof of Theorem 1.7. below. Firstly we recall that a curve Γ in C^1 is real-analytic if there is a non-constant real-analytic function φ defined on T such that $\varphi(T)=\Gamma$. If we assume that the origin 0 belongs to $C^1\setminus\Gamma$ we can introduce the winding number of Γ with respect to 0 in the usual way. If Γ_1 and Γ_2 are two real-analytic curves we say that $\Gamma_1\subset\Gamma_2$ if Γ_1 is contained in the open component of the set $C^1\setminus\Gamma_2$ which contains 0.

Proposition 1.6. *Let A be a uniform algebra and let $S_A=K\cup K_1\cup\ldots\cup K_n$, where $K, K_1\ldots K_n$ are disjoint closed sets. Suppose that $f\in A$ is such that $|f|_K<\epsilon<\frac{1}{2}$ while each K_i is homeomorphic to the unit circle and $f\,|\,K_i$ determines a non-constant real-analytic function on K_i which maps K_i onto the real-analytic curve Γ_i. Here $\Gamma_1\subset\ldots\subset\Gamma_n$ holds and in addition $\Gamma_1\subset\{z\in C^1\colon |z|>1+\varepsilon\}$. Let now N be the maximum of the winding number of 0 for each Γ_i. If U is the open component of the set $C^1\setminus f(S_A)$ which contains the point 1, then U is f-regular of some multiplicity $\leqslant nN$.*

In Theorem 1.7. below we apply Proposition 1.6. to obtain a general result dealing with so called crossing over edges. The reader will see that we have taken great advantage of [2] in the proof. In particular Lemma 1.8. was already proved there.

Theorem 1.7. *Let A be a uniform algebra and let $f\in A$. Let U be an f-regular component of multiplicity n. Let $z_0\in\partial U$ be such that $\pi_f^{-1}(z_0)\cap S_A$ is a non-empty A-convex set $\longrightarrow$ in M_A. Then the set $\pi_f^{-1}(z_0)\setminus S_A$ contains at most n points and in addition M_A has analytic structure at each point in this (possibly empty) set.*

Proof. Let us assume that $n>0$ (the case when $n=0$ is easy and contained in the following proof using Lemma 1.8.) and choose a point λ_0 in U such that $\pi_f^{-1}(\lambda_0)$ contains n points $w_1\ldots w_n$. Denote by D_0 a small open disc centered at λ_0 so that $\pi_f^{-1}(D_0)=W_1\cup\ldots\cup W_n$, where W_i are disjoint open subsets of M_A each mapped homeomorphically by f onto D_0.

Next $\varepsilon>0$ is so small that if $D(\varepsilon)$ is the open disc of radius ε centered at λ_0, then $D(\varepsilon)\cup T(\varepsilon)$ is contained in D_0. Here $T(\varepsilon)$ is the circle of radius ε centered at λ_0. At this stage we do not fix ε, the following lemmas will be true for any ε as above, and in the final part of the proof we choose ε to be sufficiently small.

Let us put $X=M_A\setminus\pi_f^{-1}(D(\varepsilon))$, so that X is a closed subset of M_A. Here $\pi_f^{-1}(T(\varepsilon))=\partial X$ and we know that $\partial X=K_1\cup\ldots\cup K_n$, where K_i are disjoint closed sets each mapped homeomorphically by f onto $T(\varepsilon)$. Next we introduce the uniform algebra B on X which is generated by the restriction algebra $A\,|\,X$ and the function $(f-\lambda_0)^{-1}=f_0$ in $C(X)$. In the series of lemmas which follows we derive some properties of B which finally will enable us to prove Theorem 1.7.

Lemma 1.8. *Let B and X be as above. Then $M_B=X$ and S_B is contained in the set $(S_A\cup K_1\cup\ldots\cup K_n)$.*

Proof. Let $x_0\in M_B$ be given. Since B contains $A\,|\,X$ we get a point y_0 in M_A such that $g(x_0)=g(y_0)$ for all $g\in A\,|\,X$. Since $(f-\lambda_0)$ is invertible in B it follows that $f(M_B)$ does not contain λ_0. In addition $f(S_B)\subset f(X)$ which means that $D(\varepsilon)\subset C^1\setminus f(S_B)$. Then Lemma 1.1. shows that $f(M_B)\cap D(\varepsilon)$ is empty which implies that $y_0\in X$. But now it is easily seen that $f_0(x_0)=f_0(y_0)$ which implies that $x_0=y_0$ in M_B so that $M_B=X$ follows.

Suppose now that S_B is not contained in the set $R=S_A\cup K_1\cup\ ...\ \cup K_n$. Then $X\setminus R$ contains a point y which is a peak point for B. Here $y\in X\setminus\partial X$ so we can choose a closed A-convex neighborhood W of y in M_A while $W\subset X$. Here $f-\lambda_0\neq 0$ on W which means that $f_0|\ W$ belongs to $A(W)$. Hence $A(W)=B(W)$ and if we let $g\in B$ peak at y, then A contains a sequence (g_n) such that $\lim\ |\ g_n-g|_W=0$. But then we see that if n is sufficiently large the function g_n will determine a local peak set in the interior of W. Since $W\subset M_A\setminus S_A$ this contradicts the Local Maximum Principle. This proves that $S_B\subset R$ must hold.

Lemma 1.9. *$Hull_B(S_A)$ does not intersect $\pi_f^{-1}(U)$.*

Proof. Let $B_1=B(\mathrm{Hull}_B(S_A))$. Then B_1 is a uniform algebra and $M_{B_1}=\mathrm{Hull}_B(S_A)$ while $S_{B_1}=S_A$. Consider f as an element of B_1. Since $M_{B_1}\subset X$ we see that λ_0 belongs to $C^1\setminus f(M_{B_1})$ and in addition $U\subset C^1\setminus f(S_{B_1})$. Hence an application of Lemma 1.1. proves that $f(M_{B_1})\cap U$ is empty which gives the desired result.

Lemma 1.10. *Let B_1 be as in Lemma 1.9. Then $\pi_f^{-1}(z_0)\cap M_{B_1}$ is a peak set for B_1.*

Proof. Let us put $Y=f(M_{B_1})$ so that $Y\cap U$ is empty. Since U is connected and $z_0\in\partial U$ there exists a function $P\in C(Y)$ such that $P(z_0)=1$ while $|P|<1$ on $Y\setminus\{z_0\}$. In addition P can be uniformly approximated on Y by rational functions R_n which have poles in U only. Since $Y\cap U$ is empty the functions $R_n\circ f\in B_1$ and hence it follows that $P_1=P\circ f$ belongs to B_1. Clearly P_1 determines the peak set $\pi_f^{-1}(z_0)\cap M_{B_1}$.

Lemma 1.11. *Let $y\in\pi_f^{-1}(z_0)\setminus S_A$. Then y does not belong to $Hull_B(S_A)$.*

Proof. Suppose that $y\in\mathrm{Hull}_B(S_A)=M_{B_1}$. Since $\pi_f^{-1}(z_0)\cap M_{B_1}$ is a peak set for B_1 and since $S_{B_1}=S_A$ we can conclude that y even belongs to $\mathrm{Hull}_{B_1}(S_A\cap\pi_f^{-1}(z_0)$. If $K=\mathrm{Hull}_{B_1}(S_A\cap\pi_f^{-1}(z_0))$ it is obvious that K is contained in $M_{B_1}\cap\pi_f^{-1}(z_0)$, so the function f_0 is constant on K. This implies that $B_1(K)=B(K)=A(K)$ and hence $K=\mathrm{Hull}_A(S_A\cap\pi_f^{-1}(z_0))$, so by assumption $K=S_A\cap\pi_f^{-1}(z_0)$. But here $y\in M_A\setminus S_A$ so that $y\in M_A\setminus K$, a contradiction.

Continuation of the proof of Theorem 1.7. Let $y_0\in\pi_f^{-1}(z_0)\setminus S_A$ be a given point. Using Lemma 1.11. we can choose g in B so that $|g|S_A<\varepsilon_0<\frac{1}{2}$ while $g(y_0)=1$. Clearly we may change g slightly so we may assume that $g=hf_0^{-N}$ where $h\in A|X$ and $N\geqslant 0$. Here $N>0$ since g clearly cannot belong to $A|X$. In addition we may assume that the function h is such that $0<|h(w_1)|<\ ...\ <|h(w_n)|$ and that $dh_i/dz\neq 0$ at λ_0 for each i, where h_i are the analytic functions determined by h on D_0.

At this stage we make a good choice of ε. For we can choose ε so small that if $\Gamma_i=g(K_i)$, then Γ_i are analytic curves where $\Gamma_1\subset\ ...\ \subset\Gamma_n$ and each Γ_i has the winding number N with respect to 0. In addition we may arrange this so that $\Gamma_1\subset\{z\in C^1: |z|>1+\varepsilon_0\}$. So now an application of Proposition 1.6. shows that M_B has an analytic structure at the point y_0.

Here $y_0\in X\setminus\partial X$ so we can choose a small closed neighborhood W of y_0 in M_A as in Lemma 1.8. for which $A(W)=B(W)$. But then it follows that M_A also has an analytic structure at y_0.

Finally we show why $\#\ (\pi_f^{-1}(z_0)\setminus S_A)\leqslant n$. For if $y\in\pi_f^{-1}(z_0)\setminus S_A$ then the fact that M_A has an analytic structure at y implies that if f is not locally constant in a neighborhood of y, then $f(W)$ is a neighborhood of z_0 in C^1 if W is a neighborhood of y in M_A. So if $\pi_f^{-1}(z_0)\setminus S_A$ contains $n+1$ points $y_1 \dots y_{n+1}$ where f is not locally constant, then we choose disjoint neighborhoods $W_1 \dots W_{n+1}$ of these points and now we can find an open disc $D(z_0)$ which is contained in $f(W_1) \cap \dots \cap f(W_{n+1})$. Since $z_0 \in \partial U$ we can choose $z_1 \in D(z_0) \cap U$ and then $\pi_f^{-1}(z_1) \cap W_i$ are non empty for each i, so that $\# \pi_f^{-1}(z_1) \geq n+1$, a contradiction.

Hence the set $Q=\{y \in \pi_f^{-1}(z_0)\setminus S_A\colon f$ is not locally constant at $y\}$, contains at most n points. If we put $Z=Q \cup (\pi_f^{-1}(z_0) \cap S_A)$, then it is clear that $\pi_f^{-1}(z_0)\setminus Z$ is an open subset of M_A. Let us introduce the uniform algebra $A_1=A(\pi_f^{-1}(z_0))$. Using the Local Maximum Principle it follows that S_{A_1} is contained in Z. Since Q is a finite set it follows that $M_{A_1}=\pi_f^{-1}(z_0)$ is contained in $Q \cup \mathrm{Hull}_{A_1}(\pi_f^{-1}(z_0) \cap S_A)= Q \cup \mathrm{Hull}_A(\pi_f^{-1}(z_0) \cap S_A)=Q \cup (\pi_f^{-1}(z_0) \cap S_A)$. This completes the proof of Theorem 1.7.

The next result is fairly direct consequence of the preceeding proof.

Theorem 1.12. *Let A be a uniform algebra and let f be a function in A such that each open component of the set $C^1\setminus f(S_A)$ is f-regular of some multiplicity $n \geq 0$. In addition we assume that $R(f(S_A))=C(f(S_A))$ and that the sets $\pi_f^{-1}(z) \cap S_A$ are A-convex in M_A for all $z \in f(S_A)$. Then it follows that $M_A\setminus S_A$ has an analytic structure.*

Proof. Let $z_0 \in f(S_A)$ and assume that x is a point in $\pi_f^{-1}(z_0)\setminus S_A$. Using the assumption that $R(f(S_A))=C(f(S_A))$ the arguments used in Lemma 1.9–11 show that there exist finitely many components $U_1 \dots U_n$ of the set $C^1\setminus f(S_A)$ satisfying the following condition. If we take a point λ_i in each U_i and choose small open discs $D(\lambda_i)$ and let B be the uniform algebra on the set $X-M_A/(\pi_f^{-1}(D(\lambda_1) \cup \dots \cup D(\lambda_n))$ which is generated by $A\,|\,X$ and the functions $(f-\lambda_i)^{-1}$, then x does not belong to $\mathrm{Hull}_B(S_A)$. Using this fact the same argument as in the proof of Theorem 1.7. shows that M_A has an analytic structure at the point x_0.

In this final part we derive some consequences of Theorem 1.7. and 1.12. above. Firstly we notice that Theorem 1.7. together with Proposition 1.5. immediately shows that Proposition 1.6. holds in the case when Γ_i are curves with finitely many intersection points, i.e. $\Gamma_i=f(K_i)$ where K_i contains a finite set S_i such that $f(x) \neq f(y)$ for all pairs $x \neq y$ in $K_i\setminus S_i$.

Using the fact above we can for example derive the following result which was independently obtained by H. Alexander in [1].

Theorem 1.13. *Let J be a closed curve in C^1 with finitely many intersection points and let A be a uniform algebra on J which contains the coordinate function z. Then the (possibly empty) set $M_A\setminus S_A$ has an analytic structure.*

Finally we show how the result of G. Stolzenberg in [6] can be derived from Theorem 1.12.

Theorem 1.14. *Let A be a uniform algebra on the unit circle T such that A is generated by continuously differentiable functions. Then the (possibly empty) set $M_A\setminus S_A$ has an analytic structure.*

Before we begin the proof we insert some preliminary results. Suppose that $f \in C^1(T)$ and let E be a closed totally disconnected subset of T. Then $C^1\setminus f(E)$ is connected so if A is a uniform algebra on T which contains f, then an application of Mergelyan's Theorem to the simply connected set $f(E)$ shows that if $\varphi \in C(E)$ then $\varphi \circ f$ actually belongs to the closed restriction algebra $A(E)$. It follows that if A is generated by continuously differentiable functions then $A(E)=C(E)$.

The next result is essentially contained in [Theorem, p. 509 in 2], though we also need Proposition 1.5. and 1.6. with a smooth Jordan arc.

Proposition 1.15. *Let A be a uniform algebra on T and let $f \in C^1(T)$ be an element of A. Assume that T contains a closed totally disconnected subset E such that if $x \in T \setminus E$, then the set $S_x = \{y \in T: f(y) = f(x)\}$ is a finite subset of $T \setminus E$ and in addition $f'(x) \neq 0$. Then it follows that each open component of the set $C^1 \setminus f(T)$ is f-regular.*

Proof of Theorem 1.14. Firstly it is easily verified that A contains some $f \in C^1(T)$ satisfying the conditions in Proposition 1.15. with respect to a closed totally disconnected subset E of T. Clearly $S_A \subset T$ (in fact $S_A = T$ by Corollary, p. 85 in 6), so using Theorem 1.12. it is sufficient to show that $R(f(T)) = C(f(T))$ and that $\pi_f^{-1}(z_0) \cap T$ is A-convex for all $z_0 \in f(T)$. Now $f \in C^1(T)$ so the planar Lebesgue measure of $f(T)$ is zero and then it is wellknown that $R(f(T)) = C(f(T))$. Next $E(z_0) = \pi_f^{-1}(z_0) \cap T$ is always a closed disconnected subset of T when $z_0 \in f(T)$, so by the preceeding discussion $A(E(z_0)) = C(E(z_0))$ which means that $E(z_0) = M_{A(E(z0))} = \mathrm{Hull}_A(\pi_f^{-1}(z_0) \cap T)$. This completes the proof.

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