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Hardy-fields

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Introduction

In his work [1] Hardy discusses the problem of describing how rapidly a function $f: R_1 \to R_1$, where R_1 is the system of real numbers, tends to infinity. This is a special case of the more general problem to describe how such a function behaves in the vicinity of $+\infty$. For simplicity we write ∞ instead of $+\infty$ in the sequel. We will only be interested in functions that are infinitely differentiable in a neighborbood of ∞. Thus let

$C^\infty = \{f \mid f$ is a real-valued function such that its domain lies in R_1 and contains a neighborhood of ∞ in which f is infinitely differentiable$\}$.

To discuss the above problem we are naturally led to considering two functions f, g belonging to C^∞ as essentially equal, written $f \sim g$, if they are equal in a neighborhood of ∞, i.e. if there is a number N such that $f(x) = g(x)$ for $x \geqslant N$. It is obvious that $\sim$ is an equivalence relation and that the equivalence classes are the same as the residue classes of the ideal $I = \{f \mid f \sim 0\}$ in the ring C^∞. Let $R = C^\infty / I$ and let $I(f)$ denote the residue class of f with respect to I. Then R is a ring with differentiation where $(I(f))' = I(f')$ and it can be said to represent all the ways a function in C^∞ can behave in a neighborhood of ∞.

The concept, Hardy-field, which will be studied in this paper, was essentially introduced in [1] but was, as far as I know, first formally defined in [2]. The definition given in [2] is equivalent to the following.

Definition. *A field H contained in R, such that $y \in H$ implies that $y' \in H$, is said to be a Hardy-field.*

An example of a subset of R, which constitutes a Hardy-field, is the set of residue classes of the rational functions.

An intersection of an arbitrary non-empty family of Hardy-fields is a Hardy-field. Thus if $A \subset R$ and if there is a Hardy-field containing both a given Hardy-field H and the set A then there is also a smallest Hardy-field, denoted by $H\{A\}$, containing both H and A. If $H\{A\}$ exists then A is said to be adjoinable to H. If $A = \{a_1, \ldots, a_n\}$ then we write $H\{A\} = H\{a_1, \ldots, a_n\}$ instead of $H\{\{a_1, \ldots, a_n\}\}$ and we say that $a_1, \ldots, a_n$ are adjoinable to H.

It will be one of the main concerns in this work to extend a given Hardy-field in such a way that the extension has nice algebraic and analytic properties. A natural and important means to extend a Hardy-field is the method of adjunction as described above. Thus it will be important to have conditions on A and H that guarantee the existence of $H\{A\}$. The following result is of that nature. It is proved in [2].

Theorem. *Let H be a Hardy-field and suppose that $y' = ay + b$ where $a, b \in H$, $y \in R$. Then y is adjoinable to H.*

Several partial orderings can be defined in R. The one which will be most frequently used in this paper is defined by letting $I(f) < I(g)$ mean that $f(x) < g(x)$ in a neighborhood of ∞. This definition is obviously independent of the choice of the representatives f and g. It is easy to see that if H is a Hardy-field then H with the restriction of $<$ to H is a totally ordered field.

Hardy and Bourbaki use Hardy-fields as "scales" of the order of increase of a function in C^∞. To decide how far-reaching such scales are one faces a number of problems. Given a Hardy-field H there might for instance exist functions in C^∞ that increase more rapidly than any of the representatives of any element in H. More precisely, is there an element $a \in R$ such that $a > h$ for all $h \in H$? If this is the case H is said to be bounded and a is said to be an upper bound for H which will be written $a > H$. On the other hand if no such element exists, then H is said to be unbounded. Thus a natural and important problem would be to decide whether or not there are unbounded Hardy-fields. A related problem is to decide whether there is a Hardy-field H such that for each $a \in R$ there is an $h \in H$ such that $a < h$. Such a Hardy-field will be said to be cofinal with R or simply cofinal. It follows immediately from the definitions that a cofinal Hardy-field is unbounded. These two problems will be of major interest in the sequel and we shall prove that there are in fact unbounded Hardy-fields and that the continuum hypothesis implies the existence of cofinal Hardy-fields.

From Zorn's lemma it follows in the usual way that each Hardy-field is contained in a maximal Hardy-field. The theorem above and some theorems which will be proved later on shows that a maximal Hardy-field is unbounded, real-closed and closed under the solution of differential equations of the type above. However we shall prove that there is no unique maximal Hardy-field, i.e. there is no greatest Hardy-field. When we have obtained this result we conclude the paper with an investigation of the properties of the intersection of all maximal Hardy-fields. We show that this intersection is a Hardy-field which is closed under composition and bounded, in fact we can give a countable subset of it such that each element in the intersection is less than some element in the countable subset.

For later convenience we now make a few observations and introduce some terminology.

The symbol $F(A)$ will denote field-extension and the symbol $F[A]$ ring-extension of the field F with respect to A. For example it is easy to see that in Bourbaki's theorem above $H\{y\} = H(y)$. The transcendental elements will be denoted by X or X_i. An element $y \in R$ is said to be comparable to 0 if $y > 0$, $y = 0$ or $y < 0$. Suppose that H is a Hardy-field and that $a \in R$ is such that $p(a, a', \ldots, a^{(n)})$ is comparable to 0 for each n and each $p \in H[X_0, \ldots, X_n]$. Then it follows from the rules of differentiation that $H\{a\}$ exists and consists of all elements $r(a, a', \ldots, a^{(n)})$ where $n \geqslant 0$ and $r \in H(X_0, \ldots, X_n)$ is such that it can be written p/q, with $p, q \in H[X_0, \ldots, X_n]$ and $q(a, a', \ldots, a^{(n)}) \neq 0$. If y is comparable to 0 then $|y|$ will denote y if $y > 0$ and $-y$ if $y < 0$.

By abuse of language the residue class of the function $f(t) = c$ for all t will also be denoted by c and the residue class of the function $f(t)=t$ for all t will be denoted by x. Similarly if $a=I(f)$ then $\exp a$ will denote $I(\exp f)$ and $\log a$ is analogously defined.

§ 1. Real-closed Hardy-fields

The object of this part is to give a result on extensions of Hardy-fields related to algebraic properties. The partial ordering $<$ plays a minor role in this section but the results given here will be of help later on when a more explicit study of the ordering $<$ is made.

First we give an extension-theorem of algebraic nature.

Theorem 1. *Let H be a Hardy-field and let $H^* = \{a \mid a \in R$ and a is algebraic over $H\}$. Then H^* is a Hardy-field. Furthermore, H^* is real-closed, i.e. each polynomial of odd degree over H^* has a zero in H^* and each positive element in H^* has a square-root in H^* and consequently $H^*(\sqrt{-1})$ is algebraically closed.*

The proof of this theorem is contained in the following lemmas.

Lemma 1. *H^* is a Hardy-field.*

Proof. Exactly as in the classical field-theory one proves that H^* is a field. Let $a \in H^*$ and let $q \in H[X]$ be of minimal degree such that $q(a)=0$. Thus $q'(a) \neq 0$ since the characteristic of H is 0. Let $q(X)=b_nX^n+\dots+b_0$. Differentiating, one obtains

$$a'q'(a)+q_1(a)=0$$

where $q_1(X)=b_n'X^n+\dots+b_0'$. Consequently $q_1 \in H[X]$

$$a' = \frac{-q_1(a)}{q'(a)} \in H(a) \subset H^*.$$

Thus it follows that H^* is a Hardy-field.

To prove that H^* is real-closed we shall need the following well-known result.

Lemma 2. *Suppose that $p(X)$ is a polynomial of odd degree with real coefficients. Then the real zeros of $p(X)$ are analytic functions of the coefficients of p at each point where the discriminant of p is different from 0.*

Furthermore, we shall need

Lemma 3. *Let H be a Hardy-field. Then each polynomial over H of odd degree has a zero in R and each positive element of H has a square-root in R.*

Proof. Let $p \in H[Y]$ be of odd degree. It is enough to assume that p is irreducible. Let

$$p(Y)=b_nY^n+\dots+b_0,\ b_n \neq 0.$$

Let D be the discriminant of p. Since p is irreducible we can conclude that $D \neq 0$. Let $b_i=I(f_i)$, $0 \leqslant i \leqslant n$, and let

$$P_x(Y) = f_n(x)\, Y^n + \dots + f_0(x).$$

Let $F(x)$ be the discriminant of P_x. Then $I(F) = D$. Thus there is an N such that $f_n(x) \neq 0$, $F(x) \neq 0$ for $x \geqslant N$ and $f_i \in C^\infty[N, \infty)$, $0 \leqslant i \leqslant n$. Let, for $x \geqslant N$, $d(x)$ be the greatest real zero of P_x. Then it follows from lemma 2 that $d \in C^\infty[N, \infty)$. Consequently $I(d) \in R$ is a zero of p.

Let $a \in H$ be such that $a > 0$. Let $a = I(f)$. Thus there is an N such that $f \in C^\infty[N, \infty)$ and $f(x) > 0$ when $x \geqslant N$. Therefore $\sqrt{f} \in C^\infty[N, \infty)$ and $a = I(f) = I((\sqrt{f})^2) = I(\sqrt{f})^2$ i.e. a has a square-root in R.

Proof of theorem 1. The only thing that remains to be proved is that H^* is real-closed. Suppose that $p \in H^*[X]$ is of odd degree. Then according to Lemma 3 there is an element $a \in R$ such that $p(a) = 0$. But a is then algebraic over H^* and therefore also over H. Consequently $a \in H^*$ i.e. p has a zero in H^*.

Suppose that $a \in H^*$ and that $a > 0$. Thus according to Lemma 3 there is an element $b \in R$ such that $b^2 = a$. But then b is algebraic over H^* and it follows as above that $b \in H^*$. Consequently H^* is real-closed.

§ 2. Linearly closed Hardy-fields

Theorem 1 gives a possibility to extend an arbitrary Hardy-field in such a way that the extension has nice algebraic properties. We will now give a result which enables us to extend a Hardy-field in such a way that the extension has certain pleasant properties related to the solution of linear differential equations. This result will be strongly connected with the above mentioned theorem in [2]. First we give a definition.

Definition. *A Hardy-field H is said to be linearly closed if for any $a, b \in H$ we have that $y' = ay + b$ implies that $y \in H$.*

Let H be a Hardy-field. Then $y \in R$ is said to be connected to H by the chain $y_1, \dots, y_n$, where $y_n = y$, if there are a_i, b_i, $1 \leqslant i \leqslant n$, such that $y_i' = a_i y_i + b_i$, where $a_1, b_1 \in H$, $a_i, b_i \in H\{y_1, \dots, y_{i-1}\}$. The extensions $H\{y_1, \dots, y_{i-1}\}$ exist according to the theorem in the introduction.

We can now state

Proposition 1. *Let H be a Hardy-field and let $L(H)$ be the set of all $y \in R$ such that y is connected to H by a chain. Then $L(H)$ is a linearly closed Hardy-field and is furthermore the smallest linearly closed Hardy-field containing H.*

Proof. It follows from the definition of $L(H)$ that it is contained in each linearly closed Hardy-field containing H. Thus it will be sufficient to prove that $L(H)$ is a linearly closed Hardy-field. Suppose that $y, z \in L(H)$, where $z \neq 0$. Let $y_1, \dots, y_m$ and $z_1, \dots, z_n$ be chains connecting y and z respectively to H. Consequently

$$y, z \in H\{y_1, \dots, y_m, z_1, \dots, z_n\}$$

so that $y - z$, y/z, $y' \in L(H)$. Thus $L(H)$ is a Hardy-field. Suppose that $u' = yu + z$, where z may now be 0. Then u is connected to H by the chain $y_1, \ldots, y_m, z_1, \ldots, z_n, u$. Hence $u \in L(H)$. Consequently $L(H)$ is linearly closed.

We will now combine Theorems 1 and 2 in order to give an extension of a Hardy-field that has both the algebraic property of Theorem 1 and the analytic property of Proposition 1.

Proposition 2. *Let H be a Hardy-field. Let $H_1 = L(H)$, $H_2 = H_1^*$ and let H_n be inductively defined by $H_{2n+1} = L(H_{2n})$, $H_{2n+2} = H_{2n+1}^*$. Let $\tilde{H} = \bigcup_1^\infty H_n$. Then $\tilde{H}$ is a linearly closed and real-closed Hardy-field containing H and is furthermore the smallest Hardy-field with these properties.*

The proof is obvious.

§ 3. Boundedness of Hardy-field extensions

In this and the following sections we will consider problems concerning the partial ordering $<$ in R. We wish to establish the existence of unbounded and, if possible, even cofinal Hardy-fields.

To get an idea of the properties of the ordering $<$ and for later use the following lemma will be helpful.

Lemma 4. *Any countable set in R is bounded.*

Proof. Let $A = \{a_n\}_1^\infty$, $a_n \in R$. Let $a_n = I(f_n)$, $f_n \in C^\infty$. Let N_n be a real number such that $f_n \in C^\infty[N_n, \infty)$. Let

$$g_n(x) = \begin{cases} f_n(x), & x \geq N_n \\ f_n(N_n), & x < N_n. \end{cases}$$

Thus $a_n = I(g_n)$. The functions g_n are continuous for all $x \in R_1$. It is then a well-known result that there is a function $f \in C^\infty[0, \infty)$ such that $I(f) > I(f_n)$ for all $n \geq 1$ (see e.g. [1] page 8 where a theorem by du Bois-Reymond is proved that immediately implies the above statement). Hence $I(f)$ is an upper bound for A. Similarly one can construct a lower bound for A.

In the following let $b > A$ where $b \in R$, $A \subset R$ denote that $b > a$ for all $a \in A$ and let $b \geq A$, $b < A$, $b \leq A$ be defined similarly. Let $A < B$, where $A, B \subset R$, denote that there is an element $b \in B$ such that $A < b$. Let $A \leq B$ denote that for each $a \in A$ there is an element $b \in B$ such that $a \leq b$. Let $A \sim B$ denote that $A \leq B$ and $B \leq A$ are both valid. Let $a \ll b$, where $a, b \in R$, mean that $na < b$ for all natural numbers n. Generally a symbol $ means that S is not valid.

First we will examine whether the extension-theorems developed so far will yield an unbounded Hardy-field. The answer is in the negative which the following theorem expresses.

Theorem 2. *Suppose that H is a bounded Hardy-field. Then $\tilde{H}$ is bounded.*

The proof of this theorem is contained in the following lemmas.

Lemma 5. *Let H be a Hardy-field. Then $H \sim H^*$.*

Proof. We have $H \subset H^*$. Thus $H \leqslant H^*$. Suppose that $y \in H^*$. We may suppose that $y \neq 0$. Let $p(X) = a_n X^n + \ldots + a_0$, $a_n \neq 0$, $a_0, \ldots, a_n \in H$ be such that $p(y) = 0$. Let

$$a = \max\left(\left|\frac{a_1}{a_n}\right|, \ldots, \left|\frac{a_{n-1}}{a_n}\right|, 1\right).$$

Then $a \in H$ and $a_i/a \leqslant |a_n|$. Suppose that $y > (n+1)a$. Now

$$|p(y)| = |y^n| \left| a_n + \frac{a_{n-1}}{y} + \ldots + \frac{a_0}{y^n} \right|$$

and

$$\left|\frac{a_{n-i}}{y^i}\right| \leqslant \frac{|a_{n-i}|}{(n+1)a} \leqslant \frac{|a_n|}{n+1} \text{ for } 1 \leqslant i \leqslant n$$

so that

$$0 = |p(y)| \geqslant |y^n| \left| |a_n| - \left| \frac{a_{n-1}}{y} + \ldots + \frac{a_0}{y^n} \right| \right|.$$

But

$$\left| \frac{a_{n-1}}{y} + \ldots + \frac{a_0}{y^n} \right| \leqslant n \frac{|a_n|}{n+1} < |a_n|.$$

Hence $|p(y)| > 0$ which is a contradiction. Thus $y \leqslant (n+1)a \in H$. It follows that $H^* \leqslant H$. Consequently $H \sim H^*$.

In the sequel an element $a \in R$ will be called hardian if there is a Hardy-field H such that $a \in H$.

Lemma 6. *Let $a \in R$, where $a \neq 0$, and suppose that a' is hardian. Then*

$$\min\left(\frac{1}{x}, a'^2\right) < |a| < a'^2 + x^2$$

Proof. Let $a = I(f)$. Thus $a' = I(f')$ and since a' and a'' are hardian we can choose a $t_0 \geqslant 0$ such that each of $f'(t)$ and $f''(t)$ has a constant sign or is identically equal to 0 for $t \geqslant t_0$. Thus $\lim_{t\to\infty} f(t) = c$, where $-\infty \leqslant c \leqslant \infty$, exists and there is a $t_1 \geqslant t_0$ such that $f(t)$ has a constant sign for $t \geqslant t_1$. Hence $|a| \in R$. Since $|a|' = \pm a'$ we may suppose that $a \geqslant 0$. Consequently $0 \leqslant c \leqslant \infty$. The proof is clear if $0 < c < \infty$. Suppose that $c = 0$. Then we only have to prove the left inequality. Now $f(T) = f(t) + \int_t^T f'(s)\,ds$. Thus $\int_t^\infty f'(s)\,ds$ converges and $f(t) = -\int_t^\infty f'(s)\,ds$ since both sides have the same derivative and tend to 0 when t tends to ∞. Since a' is hardian $f'(s)$ is monotonic for sufficiently large values of s and we can conclude that $\lim_{t\to\infty} f'(t)$ exists and thus must equal 0. In the same way we obtain $\lim_{t\to\infty} f''(t) = 0$. Since $f(t) > 0$ for $t \geqslant t_1$ we can conclude that $f'(t) < 0$ for $t \geqslant t_1$ and similarly we obtain a $t_2 \geqslant t_1$ such that $0 < f''(t) < \frac{1}{2}$ if $t \geqslant t_2$. Thus we get for $t \geqslant t_2$

$$f(t) = -\int_t^\infty f'(s)\,ds > -\int_t^\infty 2 f'(s) f''(s)\,ds = f'(t)^2.$$

Hence $a > a'^2$.

Now suppose that $c=\infty$. Then we only have to prove the right inequality. Now $0<f'(s)<f'(t_1)+f'(t)$ for $s\geq t_1$ since $f'(s)$ is monotonic for $s\geq t_1$. Consequently

$$f(t)=f(t_1)+\int_{t_1}^{t} f'(s)\,ds<$$

$$<f(t_1)+tf'(t_1)+tf'(t).$$

Thus
$$a<f(t_1)+xf'(t_1)+xa'<$$

$$<\frac{x^2}{2}+\frac{x^2+a'^2}{2}<x^2+a'2.$$

Lemma 7. *Let H be a Hardy-field such that $x\in H$. Let $y'=ay+b$ where $a, b\in H$, $y\neq 0$. Then there are c, $d\in H$ such that* $\exp c<|y|<\exp d$.

Proof. Let $A'=-a$. Thus
$$(y\exp A)'=b\exp A$$

Now $b\exp A$ is hardian since $\exp A$ is adjoinable to H by the theorem by Bourbaki mentioned in the introduction.
Hence by Lemma 6

$$\min\left(\frac{1}{x},\ b^2\exp 2A\right)<|y|\exp A<b^2\exp 2A+x^2$$

Thus
$$\min\left(\frac{1}{x}\exp(-A), b^2\exp A\right)<|y|<b^2\exp A+x^2\exp(-A).$$

If $b=0$ then $y=c_1\exp(-A)$ where $c_1\in R_1$ and the result follows from Lemma 6 since $x\in H$. Now suppose that $b\neq 0$. By Lemma 6 $|A|<a^2+x^2$. Thus using the fact that $\alpha>\exp(-1/\alpha)$ if $\alpha>0$ we obtain $|y|\geq\min(\exp(-x-A), \exp(A-[1/b^2]))>\min(\exp(-x-a^2-x^2), \exp(-a^2-x^2-[1/b^2]))=\exp c$ where $c\in H$ since $x\in H$.

Turning to the right inequality we obtain $|y|<\exp(b^2+A)+\exp(x^2-A)<\exp(b^2+a^2+x^2)+\exp(2x^2+a^2)<\exp(1+2x^2+a^2+b^2)=\exp d$ where $d\in H$.

Lemma 8. *Let H be a Hardy-field such that $x\in H$. Suppose that $r>H$, $y'=ay+b$ where $a, b\in H$. Then* $\exp r>H\{y\}$.

Proof. It is sufficient to prove that $\exp r>H^*\{y\}$. Now $r>H^*$ since $H\sim H^*$. Suppose that $z\in H^*\{y\}$. Then $z=a_0[p(y)/q(y)]$, $a_0\in H^*$, where p, q are products of factors of one of the two types $y-\alpha$ and $(y-\alpha)^2+\beta^2$ where $\alpha, \beta\in H^*$ and $y\neq\alpha$, $\beta\neq 0$. Now

$$(y-\alpha)'=y'-\alpha'=ay+b-\alpha'=a(y-\alpha)+a\alpha+b-\alpha'$$

where $a\alpha+b-\alpha'\in H^*$. By Lemma 7 this implies the existence of elements $\gamma, \delta\in H^*$ such that $\exp\gamma<|y-\alpha|<\exp\delta$ where $\delta\geq 0$. Thus $\exp(-[1/\beta^2])<(y-\alpha)^2+\beta^2<$

$\exp 2\delta+\exp \beta^2<\exp(2\delta+1+\beta^2)$. Using these inequalities we obtain an element $c\in H^*$ such that $|z|<\exp c$. Hence $|z|<\exp r$. Thus $\exp r>H\{y\}$.

Lemma 9. *Let H be a Hardy-field and let $r\in R$. Suppose that $r>H$. Then $re^x>H\{x\}$.*

Proof. It is sufficient to prove that $re^x>H^*\{x\}$. Let $y\in H^*\{x\}$. Then $y=a_0[p(x)/q(x)]$, $a_0\in H^*$, where p, q are products of factors of one of the two types $x-\alpha$ and $(x-\alpha)^2+\beta^2$ where $\alpha, \beta\in H^*$ and $x\neq\alpha, \beta\neq 0$. Now $(x-\alpha)'=1-\alpha'\in H^*$.

Let $|x-\alpha|=I(f)$. Then $\lim_{t\to\infty} f(t)=c, 0\leqslant c\leqslant\infty$ exists. If $c>0$ then $|x-\alpha|>1/m$ for some m. If $c=0$ it follows as in the proof of Lemma 6 that $|x-\alpha|>(1-\alpha')^2>0$. Thus in either case there exists an element $\gamma\in H^*$ such that $|x-\alpha|>\gamma>0$. On the other hand $|x-\alpha|\leqslant x+|\alpha|$ and $\beta^2\leqslant(x-\alpha)^2+\beta^2\leqslant(x+|\alpha|+|\beta|)^2$. Using these inequalities we obtain an element $a\in H^*$, where $a>1$, and a natural number n such that

$$|y|<a^n(x+a)^n<a^n(e^{x/n}a)^n=a^{2n}e^x<re^x. \text{ Thus } re^x>H\{x\}.$$

Lemma 10. *Let H be a bounded Hardy-field. Then $L(H)$ is bounded.*

Proof. According to Lemma 9 $K=H\{x\}$ is bounded. Since $x'=1\in H$ we can conclude that $L(K)=L(H)$. Let $r>K$. Let r_n be defined inductively by $r_0=r$, $r_n=\exp r_{n-1}$. Suppose that $y\in L(K)$. Let $y_1, \dots, y_n$ be a chain connecting y to K. Then using Lemma 8 and induction it follows that $r_n>H\{y_1, \dots, y_n\}$. Thus $y<r_n$. But according to Lemma 4 there is an element $s\in R$ such that $s>r_n$ for all $n\geqslant 0$. Thus $s>L(H)$ so that $L(H)$ is bounded.

Proof of theorem 2. Let H_n be as in the definition of $\tilde{H}$. Let $H_0=H$. Thus $\tilde{H}=\bigcup_0^\infty H_{2n}$. Now H_0 is bounded. Suppose that H_{2n} is bounded. Hence by Lemma 10 $H_{2n+1}=L(H_{2n})$ is bounded. But $H_{2n+2}=H^*_{2n+1}\sim H_{2n+1}$ and so H_{2n+2} is also bounded. Thus it follows by induction that each H_{2n} is bounded. Let $r_n>H_{2n}$ for each $n\geqslant 0$ and let $s>r_n$ for all $n\geqslant 0$. Then $s>\tilde{H}$ so that $\tilde{H}$ is bounded.

§ 4. Hardy-field extensions surpassing a given bound

Suppose that we have a Hardy-field H and an element $r\in R$ such that $r>H$. We will now solve the following problem.

Problem. Find a Hardy-field K such that $H\subset K$ and $r<a$ for some $a\in K$.

If we can solve this problem we can obviously also solve the problem of finding a Hardy-field K such that $H\subset K$ and r is not an upper bound for K. However the solution of each of these problems require the construction of a Hardy-field K such that $H\subset K$ and $H<K$.

According to Theorem 2 $\tilde{H}$ will also be bounded so it might happen that $r>\tilde{H}$ and in this case the application of the extension-theorems so far developed will not even suffice to eliminate the bounding property of r.

A natural first attempt to obtain a method, that would always do to construct, to a given bounded Hardy-field H, a Hardy-field K which is an extension of H and such that $H < K$, would be the following procedure. Find an element $y \in R$ such that $y > H$ and $y^{(i)}$ dominates all powers of $y^{(i+1)}$ for each $i \geq 0$ (or $y^{(i+1)}$ dominates all powers of $y^{(i)}$ for each $i \geq 0$). Then y would be adjoinable to H and $H\{y\} > H$. However, if $y = I(f)$ is such an element and $e^x \in H$ then $[\log f(t)/\log t] \to \infty$ when $t \to \infty$ which according to [2] page 112 implies that $[f''(t)f(t)/f'(t)^2] \to 1$ when $t \to \infty$. Consequently the procedure described above is, in general, impossible. Therefore we have to apply another method. It turns out that the first of the following definitions gives an adequate operation on y to make the counterpart of the above procedure possible. The second definition selects those elements in R which will be especially interesting when this new operation is considered.

Definition. *Let $y \in R$ and let $y_{(0)} = y$. The sentence "$y_{(n)}$ exists" and the value $y_{(n)}$ itself are inductively defined as follows. We say that $y_{(1)}$ exists if y is comparable to 0, in that case we put $y_{(1)} = y'/y$ if $y \neq 0$ and $y_{(1)} = 0$ if $y = 0$. We say that $y_{(k+1)}$ exists if $y_{(k)}$ is comparable to 0, in that case we put $y_{(k+1)} = (y_{(k)})_{(1)}$.*

Definition. *Suppose that $y \in R$ and that for all $k \geq 0$ $y_{(k)}$ exists, $y_{(k)} > 0$, and $y_{(k)} > y_{(k+1)}$. Then y is said to be overhardian.*

The name overhardian is motivated by the fact, proved in Section 9, that y cannot be adjoined to all Hardy-fields if it satisfies the above conditions.

Lemma 11. *Suppose that $y > 0$, y hardian, $y_{(1)} > 0$. Then $y_{(1)}^n < y$ for all $n \geq 1$.*

Proof. Let $\alpha > 0$ be a real number. By the earlier-mentioned theorem by Bourbaki $y^{1+\alpha}$ and y' lie in a common Hardy-field and are consequently comparable. Suppose that $y' \geq y^{1+\alpha}$. Let $y = I(z)$ where $z \in C^\infty[N, \infty)$. Thus we can choose $x_0 \geq N$ such that $z > 0$ and $z' \geq z^{1+\alpha}$ for $x \geq x_0$. Integrating the last inequality we obtain

$$\frac{-z(x)^{-\alpha}}{\alpha} + \frac{z(x_0)^{-\alpha}}{\alpha} \geq x - x_0 \quad \text{for} \quad x \geq x_0.$$

But the left-hand side of this inequality is less than $z(x_0)^{-\alpha}/\alpha$ for $x \geq x_0$ while the right-hand side tends to ∞ with x. This is a contradiction. Hence $y' < y^{1+\alpha}$ for all $\alpha > 0$. Putting $\alpha = 1/n$ we obtain $y' < y^{1+1/n}$ which implies $y_{(1)}^n < y$.

Lemma 12. *Suppose that $y_{(k)}$ exists for all $k \geq 0$. Then $y^{(n)} = p_n(y_{(0)}, \ldots, y_{(n)})$ where p_n is a polynomial with natural numbers as coefficients.*

Proof. Follows by induction using the relation $y'_{(i)} = y_{(i)} y_{(i+1)}$.

Lemma 13. *Suppose that y is overhardian. Then $y_{(k)} \gg 1$ for all $k \geq 0$. Furthermore $y_{(k)} > y_{(k+1)}^n$ for all $k \geq 0$ and $n \geq 1$.*

Proof. Let $y = I(f)$ and let $f_i = f_{(i)}$ be defined as for y. Then if x_0 is sufficiently large

$$f'_{k+1}(x) = f_{k+1}(x) f_{k+2}(x) > 0 \quad \text{for} \quad x \geq x_0$$

Thus

$$f_{k+1}(x) > f_{k+1}(x_0) > 0 \text{ for } x \geq x_0$$

Hence $\quad f_k(x)=f_k(x_0)\exp\int_{x_0}^{x} f_{k+1}(t)\,dt\to\infty\quad$ when $x\to\infty$, i.e. $y_{(k)}\gg 1$.

Now for sufficiently large x

$$\left(\log\frac{f_k(x)}{f_{k+1}^n(x)}\right)'=f_{k+1}(x)-nf_{k+2}(x)=f_{k+2}(x)\left(\frac{f_{k+1}(x)}{f_{k+2}(x)}-n\right).$$

Thus it is sufficient to prove that $[f_{k+1}(x)/[f_{k+2}(x)]$ tends to infinity with x. But

$$\left(\log\frac{f_{k+1}(x)}{f_{k+2}(x)}\right)'=f_{k+2}(x)-f_{k+3}(x)$$

and $f'_{k+2}(x)-f'_{k+3}(x)=f_{k+3}(f_{k+2}-f_{k+4})(x)>0$ for $x\geqslant x_0$. Consequently

$$\left(\log\frac{f_{k+1}(x)}{f_{k+2}(x)}\right)'\geqslant f_{k+2}(x_0)-f_{k+3}(x_0)>0$$

for $x\geqslant x_0$ so that $$\frac{f_{k+1}(x)}{f_{k+2}(x)}\to\infty\quad\text{when }x\to\infty$$

which completes the proof.

Theorem 3. *Suppose that y is overhardian. Then y is hardian.*

Proof. Let $z=p(y^{(0)},\ldots,y^{(n)})$, where p is an arbitrary polynomial with integral coefficients. It is sufficient to prove that z is comparable to 0. By Lemma 12 it follows that $z=q(y_{(0)},\ldots,y_{(n)})$ where q is a polynomial with integral coefficients. If the degree of q is zero then z is comparable to 0 since z then is an integer. Suppose that the degree of q is greater than zero.

Let the degree of $y_{(0)}^{\nu_0}\ldots y_{(n)}^{\nu_n}$ be the $(n+1)$-tuple $(\nu_0,\ldots,\nu_n)$ and let the degrees of the terms in q be lexicographically ordered i.e. $(\nu_0,\ldots,\nu_n)>(\eta_0,\ldots,\eta_n)$ if the first non-vanishing difference $\nu_i-\eta_i$ is positive. Then the term in q which has the greatest degree dominates the other terms.

For suppose that $(\eta_0,\ldots,\eta_n)<(\nu_0,\ldots,\nu_n)$ then

$$\left|\frac{y_{(0)}^{\eta_0}\cdots y_{(n)}^{\eta_n}}{y_{(0)}^{\nu_0}\cdots y_{(n)}^{\nu_n}}\right|=|y_{(0)}^{\eta_0-\nu_0}\cdots y_{(n)}^{\eta_n-\nu_n}|\ll 1$$

since the first non-vanishing exponent $\eta_i-\nu_i$ is negative and $y_{(i)}\gg y_{(j)}^m$ for all j, m such that $i<j\leqslant n$, $m\geqslant 1$ by Lemma 13 and the fact that $y_{(k)}\gg 1$ for $0\leqslant k\leqslant n$.

Consequently z is comparable to 0 and hence y is hardian.

In the sequel we will need

Definition. *Let $e_0=x$. Let e_n, $n\geqslant 0$, be defined inductively by $e_n=\exp e_{n-1}$.*

Thus $e_1=\exp x=I$ (exp). If H is a linearly closed Hardy-field then $e_n\in H$ for all n since $e'_n=e_{n-1}e_{n-2}\ldots e_1,\ldots,e'_1=e_1$, $x'=1\in H$ so that $e_n\in H$ follows by induction.

Lemma 14. *Suppose that $y \in R$, $y_{(k)}$ exists for $0 \leqslant k \leqslant n$ and $y_{(n)} \leqslant 0$. Then $y < e_n$.*

Proof. We may suppose that $y_{(k)} > 0$, $0 \leqslant k \leqslant n-1$, in the sequel. If $n=0$ then $y = y_{(0)} \leqslant 0 < x = e_0$. Suppose that $n=1$. Then $y_{(1)} \leqslant 0$ and since $y_{(0)} > 0$ we get $y' \leqslant 0$. Consequently $y \leqslant c \in R_1$ so that $y < e_1$. Suppose that $n=2$. Then $(\log y)' \leqslant c \in R_1$ by the just proved statement for $n=1$. Thus $\log y \leqslant cx + d$ where $d \in R_1$ so that $y \leqslant \exp(cx+d) < \exp(\exp x) = \exp e_1 = e_2$. Suppose that the theorem has been proved for $n-1$, where $n-1 \geqslant 2$. Then $(y_{(1)})_{(n-1)} = y_{(n)} \leqslant 0$. Hence $(\log y)' = y_{(1)} < e_{n-1}$. But $e'_{n-1} = e_{n-1} \dots e_1 > 2e_{n-1}$ so that $(\log y)' < (\frac{1}{2} e_{n-1})'$. Thus $\log y < \frac{1}{2} e_{n-1} + c$ where $c \in R_1$ and consequently $y < \exp(\frac{1}{2} e_{n-1} + c) < \exp e_{n-1} = e_n$.

§ 5. Simple Hardy-field extensions

Theorem 4. *Suppose that H is a Hardy-field. Then*

> *H linearly closed, $y > H$, y is adjoinable to $H \Rightarrow y$ is overhardian and $y_{(k)} > H$ for all $k \geqslant 0$*

and

> *y is overhardian and $y_{(k)} > H$ for all $k \geqslant 0 \Rightarrow y$ is adjoinable to H.*

Proof. Suppose that H is linearly closed, $y > H$ and that y is adjoinable to H. Thus y is hardian. Hence $y_{(k)}$ exists for all $k \geqslant 0$. Suppose that $y_{(k)} \leqslant 0$ for some k. Then $y < e_k \in H$ since H is linearly closed. But this is a contradiction. Consequently $y_{(k)} > 0$ for all $k \geqslant 0$ and since $y_{(k)}$ is hardian it follows from Lemma 11 that $y_{(k)} > y_{(k+1)}$ so that y is overhardian.

Now suppose that there is a k such that $y_{(k)} \not> H$ and let k be minimal with this property. Then $k \geqslant 1$ and there is a $z \in H$ such that $y_{(k)} \leqslant z$. Let $y_{(k-1)} = I(f)$, $z = I(g)$. Then if x_0 is large enough

$$f(x) = f(x_0) \exp \int_{x_0}^{x} f_{(1)}(t)\, dt \leqslant g(x_0) \exp \int_{x_0}^{x} g(t)\, dt \quad \text{for } x \geqslant x_0.$$

Thus
$$y_{(k-1)} = I(f) \leqslant I\left(g(x_0) \exp \int_{x_0}^{x} g(t)\, dt\right) = u.$$

But $u' = zu$. Hence $u \in H$ since H is linearly closed. This implies that $y_{(k-1)} \not> H$ which contradicts the minimality of k. It follows that $y_{(k)} > H$ for all $k \geqslant 0$.

Suppose on the other hand that y is overhardian and $y_{(k)} > H$ for all $k \geqslant 0$. Let $z = p(y^{(0)}, \dots, y^{(n)})$ where $p \in H[X_0, \dots, X_n]$. By Lemma 12 it follows that $z = q(y_{(0)}, \dots, y_{(n)})$ where $q \in H[X_0, \dots, X_n]$. If q is a constant then it follows that $z \in H$ and consequently that z is comparable to 0. Suppose that q is not a constant. Since $y_{(k)} > H$ for all $k \geqslant 0$ it follows exactly as in the proof of Theorem 3 that the term with the greatest degree in q dominates the other terms. Thus z is comparable to 0 whence y is adjoinable to H.

§ 6. Construction of overhardian elements

Now suppose that H is a Hardy-field and that $r > H$. We want to solve the problem in Section 4 i.e. we want to find an element $y \in R$ such that y can be adjoined to H and such that $y > r$. Theorem 4 gives information about the type of function required to achieve this and it suggests that we should try to find an overhardian element y such that $y_{(k)} > r$ for all $k \geq 0$. Anyway it shows that such an element y will fulfil all our requirements.

Suppose that y is such an element, put $y = I(G)$ and, for convenience, put $G_{(k)} = G_k$. Then there is a strictly increasing sequence $\{a_k\}_1^\infty$ of real numbers tending to infinity with k such that $G_k(x)$ is defined for $x \geq a_k$. Furthermore

$$G_{k-1}(x) = \begin{cases} G_{k-1}(x) & \text{for } a_{k-1} \leq x \leq a_k \\ G_{k-1}(a_k) \exp \int_{a_k}^{x} G_k(t)\,dt & \text{for } x \geq a_k \end{cases}$$

so that each G_n is determined by this downwards recursive formula once we know the values of each G_k in $[a_k, a_{k+1}]$.

On the other hand let $\{a_k\}_1^\infty$ be a strictly increasing sequence of real numbers tending to infinity with n. We want to use the formula above in order to construct, from functions given on $[a_i, a_{i+1}]$, functions G_k defined on $[a_k, \infty)$. This is so done that we first define, for each fixed n, G_k on $[a_k, a_n]$ for $0 \leq k < n$. Then we observe that the definition of $G_k(x)$ for a fixed value of $x \geq a_k$ is independent of n and we can therefore, now by letting k be fixed and varying n, define $G_k(x)$ for $x \geq a_k$.

Thus, let $h_{k,k+1} \in C^\infty[a_k, a_{k+1}]$ be given for $k \geq 0$. Let $n \geq 1$ be fixed and define $h_{k,n}$ on $[a_k, a_n]$ for $0 \leq k < n$ recursively downwards by the same formula as before, i.e.

$$h_{k-1,n}(x) = \begin{cases} h_{k-1,k}(x) & \text{for } a_{k-1} \leq x \leq a_k \\ h_{k-1,k}(a_k) \exp \int_{a_k}^{x} h_{k,n}(t)\,dt & \text{for } a_k \leq x \leq a_n. \end{cases}$$

From the definition it follows that $h_{k,n}(x) = h_{k,m}(x)$ where they are both defined. Thus we can define functions G_k on $[a_k, \infty)$ by putting $G_k(x) = h_{k,n}(x)$ for $a_k \leq x \leq a_n$. Let $G_0 = G$. Then, by construction, we get $G_{(k)}(x) = G_k(x)$ for $x \geq a_k$. It might therefore be expected that G would do for our purpose if the functions $h_{k-1,k}$ have suitable properties. We know from Theorem 4 that it is sufficient if we can achieve that $G \in C^\infty$ and $r < G_{k-1} < G_k$ for each $k \geq 1$.

Suppose that we have chosen $h_{k-1,k}$ such that $h_{k-1,k}(x) > r(x)$ for $a_{k-1} < x \leq a_k$ and that we know that $h_{k,n}(x) > r(x)$ for $a_k < x \leq a_n$. Then

$$h_{k-1,k}(a_k) \exp \int_{a_k}^{x} h_{k,n}(t)\,dt > r(a_k) \exp \int_{a_k}^{x} r(t)\,dt \quad \text{for } a_k \leq x \leq a_n$$

i.e. we may continue the induction if we know that

$$r(a_k) \exp \int_{a_k}^{x} r(t)\,dt > r(x) \quad \text{for } x > a_k.$$

We require the strict inequality for reasons that will soon appear. This inequality may be written

$$\int_{a_k}^{x} r(t)\,dt > \log r(x) - \log r(a_k) = \int_{a_k}^{x} \frac{r'(t)}{r(t)}\,dt$$

and is consequently satisfied for $k \geqslant 0$, $x > a_k$ if $r(x) > r'(x)/r(x)$ for $x > a_0$. This may be written $r'(x)/r(x)^2 < 1$ or

$$\left(\frac{1}{r(x)}\right)' > -1 \quad \text{for} \quad x > a_0.$$

The requirement $G_k > r$ for each $k \geqslant 0$ is consequently satisfied if we choose $h_{k-1,k}(x) > r(x)$ for $a_{k-1} < x \leqslant a_k$, provided that $(1/r(x))' > -1$. Of course $(1/r(x))' > -1$ may not be satisfied but if we can find a function $s > r$ such that $(1/s(x))' > -1$ for sufficiently large x then we may work with s instead of r and thus dispose of that question. This amounts to finding a positive function $p < 1/r$ such that $p'(x) > -1$ for sufficiently large x since then $s = 1/p$ would do. However, by a well-known construction we can obtain a positive, convex, decreasing polygon under $1/r$. Then we can smooth this polygon to a positive, convex, decreasing function p still below $1/r$ such that $p \in C^\infty$. Then it is easy to see that if a_0 is chosen sufficiently large then $p'(x) > -1$ for $x > a_0$.

From now on we work with the s so constructed and let a_0 be chosen as above. We also assume that $h_{k-1,k}(x) > s(x)$ for $a_{k-1} < x \leqslant a_k$ for $k \geqslant 1$. For convenience we denote $s(a) \exp \int_a^x s(t)\,dt$ by $g_a(x)$ so that $s(x) < g_a(x)$ if $x > a \geqslant a_0$.

We now want to show that we can choose the functions $h_{k-1,k}$ such that $G_k > G_{k+1}$ for $k \geqslant 0$. These inequalities will all be satisfied if for $0 \leqslant k < n-1$

$$h_{k,n}(x) > h_{k+1,n}(x) \quad \text{for} \quad a_{k+1} \leqslant x \leqslant a_n.$$

First let $k = n-2$. Then

$$h_{n-2,n}(x) = h_{n-2,n}(a_{n-1}) \exp \int_{a_{n-1}}^{x} h_{n-1,n}(t)\,dt > s(a_{n-1}) \exp \int_{a_{n-1}}^{x} s(t)\,dt = g_{a_{n-1}}(x).$$

Consequently the inequality will be satisfied if $h_{n-1,n}(x) < g_{a_{n-1}}(x)$ for $a_{n-1} < x \leqslant a_n$. Since $s(x) < g_{a_{n-1}}(x)$ for $x > a_{n-1}$ this is compatible with $h_{n-1,n}(x) > s(x)$ for $a_{n-1} < x \leqslant a_n$.

Now suppose that for each $n \geqslant 1$ $s(x) < h_{n-1,n}(x) < g_{a_{n-1}}(x)$ for $a_{n-1} < x \leqslant a_n$ and that we have shown that $h_{i,n}(x) > h_{i+1,n}(x)$ for $a_{i+1} \leqslant x \leqslant a_n$. Then it follows from the above for $a_i \leqslant x \leqslant a_{i+1}$ that

$$h_{i-1,n}(x) = h_{i-1,i+1}(x) > h_{i,i+1}(x) = h_{i,n}(x)$$

For $a_{i+1} \leqslant x \leqslant a_n$ we get by the induction hypothesis

$$h_{i,n}(x) = h_{i-1,n}(a_{i+1}) \exp \int_{a_{i+1}}^{x} h_{i,n}(t)\,dt > h_{i,n}(a_{i+1}) \exp \int_{a_{i+1}}^{x} h_{i+1,n}(t)\,dt = h_{i,n}(x).$$

Thus the result follows by induction downwards for all i, n such that $0 \leqslant i \leqslant n-2$ and consequently $G_0 > G_1 > \dots$

From the investigation above it follows that in order to obtain the required function G it is sufficient to choose

$$h_{n-1,n}\in C^{\infty}[a_{n-1},a_n] \tag{1}$$

such that

$$s(x) < h_{n-1,n}(x) < g_{a_{n-1}}(x) \quad \text{for} \quad a_{n-1} < x \leqslant a_n \tag{2}$$

and such that $G\in C^{\infty}$, which will certainly be the case if

$$h_{n-1,n}(a_n)\exp\int_{a_n}^{x} h_{n,n+1}(t)\,dt$$

constitutes a continuation of $h_{n-1,n}$ belonging to $C^{\infty}[a_{n-1},a_{n+1}]$. This last condition is obviously satisfied if

$$h^{(k)}_{n-1,n}(a_{n-1}) = s^{(k)}(a_{n-1}) \quad \text{for } k \geqslant 0$$

and

$$h^{(k)}_{n-1,n}(a_n) = \left(h_{n-1,n}(a_n)\exp\int_{a_n}^{x} s(t)\,dt\right)^{(k}_{(x=a_n)} \quad \text{for } k\geqslant 1. \tag{3}$$

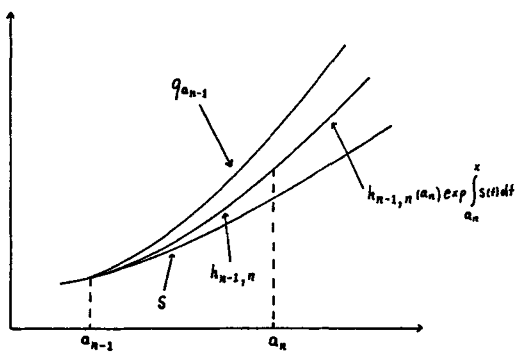

Fig. 1.

It is easy to see that (1), (2) and (3) can be simultaneously satisfied. But this means that we can construct a function $G\in C^{\infty}$ such that $G_{(0)} > G_{(1)} > \dots$ and $G > s > r$, i.e. by putting $y = I(G)$ and using Theorem 4 we have proved the following result.

Theorem 5. *Suppose that H is a Hardy-field and that $r > H$. Then there is an element $y\in R$ such that y is adjoinable to H and $y > r$.*

§ 7. Unbounded and cofinal Hardy-fields

Thus it is always possible to "get above" a given bound of H by adjoining a single element to H. One might ask if this procedure yields an unbounded Hardy-field. However, this is not so and we can say even more. To do that we first extend the notion of cofinality. Suppose that $A\subset B\subset R$. Then A is said to be cofinal in B if for each $b\in B$ there is an $a\in A$ such that $a\geqslant b$. It follows from Lemma 4 that if A is countable and cofinal in B then B is bounded.

Lemma 15. *Suppose that H is a Hardy-field, y is overhardian and that $y_{(k)} > H$ for all $k \geq 0$. Then $\{y^n\}_1^\infty$ is cofinal in $H\{y\}$.*

Proof. Let $z \in H\{y\}$. Then

$$z = \frac{p(y_{(0)}, \dots, y_{(n)})}{q(y_{(0)}, \dots, y_{(m)})}$$

where p and q are polynomials over H. Reasoning as in the proof of Theorem 4 we get

$$\frac{1}{q(y_{(0)}, \dots, y_{(m)})} < h \text{ for some } h \in H \text{ and}$$

$$|p(y_{(0)}, \dots, y_{(n)})| < |y|^{k+\frac{1}{2}}$$

where k is the greatest appearing exponent of $y = y_{(0)}$ in $p(y_{(0)}, \dots, y_{(n)})$. Thus

$$|z| < |y|^{k+1}$$

It is easy to prove the more general result that if H is a bounded Hardy-field and y is adjoinable to H then $H\ \{y\}$ is bounded. We do not need this result. However, the following theorem, which is an immediate consequence of Theorem 5, establishes the existence of unbounded Hardy-fields.

Theorem 6. *A maximal Hardy-field is unbounded.*

This theorem says nothing about the existence of cofinal Hardy-fields. We can give the following result.

Theorem 7. *Suppose that $2^{\aleph_0} = \aleph_1$ i.e., make the continuum hypothesis. Then there exists a cofinal Hardy-field.*

Proof. R_1 is imbedded in R. Hence $2^{\aleph_0} = \text{card } R_1 \leq \text{card } R$. For each $r \in R$ choose an $f \in C^\infty[n_f, \infty)$, where n_f is a natural number, such that $I(f) = r$ and let $\varphi(r) = f$. Then φ is $1-1$ into the set $A = \bigcup_{-\infty < n < \infty} C^\infty[n, \infty)$ and card $A = 2^{\aleph_0}$. Consequently card $R \leq 2^{\aleph_0}$ so that card $R = 2^{\aleph_0} = \aleph_1$. Thus the elements of R can be indexed with the ordinals $\alpha < w_1$ in such a way that $R = \{x_\alpha\}_{\alpha < w_1}$.

Let $\mathcal{A}$ be the family of bounded Hardy-fields and let $\mathcal{A}$ be partially ordered by inclusion. $H \in \mathcal{A}$ especially if H is a Hardy-field with a countable cofinal subset. Let $H_0 = R_1$. Let $\alpha < w_1$ and suppose that for each $\xi < \alpha$ $H_\xi \in \mathcal{A}$ has been defined in such a way that $\eta < \xi < \alpha$ implies $H_\eta \subset H_\xi$ and that there is an $h \in H_\xi$ such that $h > x_\xi$. Let $K = \bigcup_{\xi < \alpha} H_\xi$. Then K is a Hardy-field. Let y_ξ, $\xi < \alpha$ be an upper bound for H_ξ. Then $\{y_\xi\}_{\xi < \alpha}$ is countable. Thus there is a $z \in R$ such that $z > y_\xi$ for all $\xi < \alpha$ and $z > x_\alpha$. Hence $z > h$ for all $h \in K$ so that $K \in \mathcal{A}$. Thus by Theorem 5 there is an $H_\alpha \in \mathcal{A}$ such that $K \subset H_\alpha$ and such that there is an $h \in H_\alpha$ with $h > z$ and consequently also $h > x_\alpha$. Hence it follows from the principle of transfinite induction that it is possible to define for each $\xi < w_1$, a Hardy-field $H_\xi \in \mathcal{A}$ in such a way that $\eta < \xi < w_1$ implies $H_\eta \subset H_\xi$ and for each $\xi < w_1$ there is an $h \in H_\xi$ such that $h > x_\xi$. Let $H = \bigcup_{\xi < w_1} H_\xi$. Then H is a Hardy-field. Let $r \in R$. Now $r = x_\xi$ for some $\xi < w_1$ and there is an $h \in H_\xi \subset H$ such that $h > x_\xi = r$. Consequently H is a cofinal Hardy-field.

§ 8. General structure of overhardian elements

We will now consider the problem of deciding if there is only one maximal Hardy-field. We will prove that this is not so. To do this we adopt a method which is, in a way, opposite to the one which led to Theorem 5. More precisely we try to show that each overhardian element can in fact be obtained by the construction preceding that theorem.

Motivated by the reasoning leading to Theorem 5 and for later convenience we adopt the following terminology.

Definition. *A function $s \in C^\infty$ is said to be prehardian if there is a real number N_s such that $s(x) > 0$ and $s'(x)/s(x)^2 < 1$ for $x \geqslant N_s$. By $g_{s,a}$ or simply g_a, when no confusion is possible, we denote the function defined by*

$$g_{s,a} = s(a) \exp \int_a^x s(t)\, dt \quad \text{for} \quad x \geqslant a.$$

Consequently $s(x) < g_{s,a}(x)$ for $x > a$ if $a \geqslant N_s$.

Definition. *Suppose that $s \in C^\infty$ is prehardian. Then a sequence of functions $\mathcal{H} = \{h_{n-1,n}\}_{n=1}^{\infty}$ is said to be a supplement to s if for a strictly increasing sequence $\{a_n\}_1^\infty$, where $a_0 \geqslant N_s$, of real numbers tending to infinity, the functions $h_{n-1,n}$ satisfy the conditions* (1), (2) *and* (3).

When a supplement $\mathcal{H}$ to a prehardian function s is given then we can construct, exactly as in the proof of Theorem 5, the functions $h_{k,n}$, G_k and we will denote the function G_0 by $F(\mathcal{H})$ and say that s is support of $F(\mathcal{H})$.

Lemma 16. *Suppose that G is overhardian. Then G has a support.*

Proof. Let in the following $G_{(n)}$ be denoted by G_n. Let $a_0 = a_{-1}$ be such that

$$G_0(x) > G_1(x) > G_2(x) > 0 \quad \text{for} \quad x \geqslant a_0$$

and define $a_n > \max(n, a_{n-1})$ inductively in such a way that

$$G_n(x) > G_{n+1}(x) > G_{n+2}(x) > 0 \quad \text{for} \quad x \geqslant a_n$$

Let
$$g_n(x) = \frac{1}{G_n(x)} \quad \text{for} \quad x \geqslant a_{n-1}.$$

Then
$$g_n'(x) = -\frac{G_{n+1}(x)}{G_n(x)} > -1 \quad \text{for} \quad x \geqslant a_{n-1}$$

and $0 < g_n(x) < g_{n+1}(x)$ for $x \geqslant a_n$.
Let $\theta_n \in C^\infty[a_n, a_{n+1}]$ for each $n \geqslant 0$ be such that $\theta_n^{(k)}(a_n) = 0$ for $k \geqslant 0$, $\theta_n'(x) > 0$ for $a_n < x < a_{n+1}$, $\theta_n(a_{n+1}) = 1$ and $\theta_n^{(k)}(a_{n+1}) = 0$ for $k \geqslant 1$. Define $h(x)$ for $x \geqslant a_0$ by

$$h(x) = (1-\theta_n)g_n(x)+\theta_n g_{n+1}(x) \quad \text{for} \quad a_n \leqslant x \leqslant a_{n+1}.$$

Then it is easy to see that $h \in C^\infty[a_0, \infty)$ and that

$$g_n(x) < h(x) < g_{n+1}(x) \quad \text{for} \quad a_n < x < a_{n+1}.$$

Furthermore

$$h'(x) = (1-\theta_n)g_n'(x)+\theta_n g_{n+1}'(x)+\theta_n'(g_{n+1}-g_n)(x) > -1+\theta_n'(g_{n+1}-g_n)(x) \geqslant -1$$

for $a_n \leqslant x \leqslant a_{n+1}$ and thus for each $x \geqslant a_0$.
Consequently $s = 1/h$ is prehardian and

$$G_n(x) < s(x) < G_{n-1}(x) \quad \text{for} \quad a_{n-1} < x < a_n,\ s(a_n) = G_n(a_n).$$

Let $h_{n-1,n} = G_{n-1}/[a_{n-1}, a_n]$ for $n \geqslant 1$.
Now $h^{(k)}(a_{n-1}) = g_{n-1}^{(k)}(a_{n-1})$ for $k \geqslant 1$. Thus

$$h_{n-1,n}^{(k)}(a_{n-1}) = G_{n-1}^{(k)}(a_{n-1}) = s^{(k)}(a_{n-1}) \quad \text{for} \quad k \geqslant 0$$

and
$$h_{n-1,n}^{(k)}(a_n) = G_{n-1}^{(k)}(a_n) = \left(G_{n-1}(a_n) \exp \int_{a_n}^x G_n(t)\,dt\right)^{(k)}_{(x=a_n)}$$

$$= \left(h_{n-1,n}(a_n) \exp \int_{a_n}^x s(t)\,dt\right)^{(k)}_{(x=a_n)} \text{ since } G_n^{(k)}(a_n) = s^{(k)}(a_n) \text{ for } k \geqslant 0.$$

Furthermore

$$s(x) < G_{n-1}(x) = h_{n-1,n}(x) \quad \text{for} \quad a_{n-1} < x \leqslant a_n$$

and
$$h_{n-1,n}(x) = G_{n-1}(a_{n-1}) \exp \int_{a_{n-1}}^x G_n(t)\,dt$$

$$< s(a_{n-1}) \exp \int_{a_{n-1}}^x s(t)\,dt \quad \text{for} \quad a_{n-1} < x \leqslant a_n.$$

Consequently $\mathcal{H} = \{h_{n-1,n}\}_{n=1}^\infty$ is a supplement to s and from the definition of $h_{n-1,n}$ it follows that $F(\mathcal{H}) = G$. Thus s is a support of G.

Lemma 17. *Let h be prehardian and let $\mathcal{F} = \{f_{n-1,n}\}_1^\infty$, $f_{n-1,n} \in C^\infty[a_{n-1}, a_n]$ be a supplement to h. Suppose that $\mathcal{G} = \{g_{n-1,n}\}_1^\infty$, $g_{n-1,n} \in C^\infty[a_{n-1}, a_n]$ is another supplement to f such that for some $k \geqslant 1$.*

$$g_{k-1,k}(a_k) > f_{k-1,k}(a_k), \quad g_{n-1,n}(x) \geqslant f_{n-1,n}(x)$$

for all x such that $a_{n-1} \leqslant x \leqslant a_n$ if $n > k$. Then $F(\mathcal{G})(x)/F(\mathcal{F})(x) \to \infty$ when $x \to \infty$.

Proof. Let $F = F(\mathcal{F})$, $G = F(\mathcal{G})$, $F_{(n)} = F_n$, $G_{(n)} = G_n$. It follows from the assumptions that

$$F_k(x) \leqslant G_k(x) \quad \text{for} \quad x \geqslant a_k$$

Since $G_{k-1}(a_k) = g_{k-1,k}(a_k) > f_{k-1,k}(a_k) = F_{k-1}(a_k)$ we get

$$G_{k-1}(x) = G_{k-1}(a_k) \exp \int_{a_k}^{x} G_k(t)\,dt > F_{k-1}(a_k) \exp \int_{a_k}^{x} F_k(t)\,dt = F_{k-1}(x)$$

when $x \geqslant a_k$.

But
$$(G_{k-1} - F_{k-1})'(x) = G_{k-1} G_k(x) - F_{k-1} F_k(x) \geqslant F_{k-1} G_k(x) - F_{k-1} F_k(x)$$
$$= F_{k-1}(G_k - F_k)(x) \geqslant 0 \text{ for } x \geqslant a_k.$$

Thus
$$G_{k-1}(x) - F_{k-1}(x) \geqslant G_{k-1}(a_k) - F_{k-1}(a_k) = a > 0,\ x \geqslant a_k$$

$$\left(\log \frac{G_{k-2}}{F_{k-2}}\right)'(x) = G_{k-1}(x) - F_{k-1}(x) \geqslant a,\ x \geqslant a_k.$$

Hence
$$\frac{G_{k-2}(x)}{F_{k-2}(x)} \to \infty \text{ when } x \to \infty.$$

Suppose that it has been proved that $G_i(x)/F_i(x) \to \infty$ when $x \to \infty$ for some $i > 0$. But $F_i(x) \to \infty$ when $x \to \infty$ since F is overhardian. Consequently

$$\left(\log \frac{G_{i-1}}{F_{i-1}}\right)'(x) = G_i(x) - F_i(x) = F_i(x)\left(\frac{G_i(x)}{F_i(x)} - 1\right) \to \infty \text{ when}$$

$x \to \infty$ and hence
$$\frac{G_{i-1}(x)}{F_{i-1}(x)} \to \infty \text{ when } x \to \infty.$$

Thus it follows by induction downwards that $G(x)/F(x) \to \infty$ when $x \to \infty$.

Lemma 18. *Suppose that h is prehardian and that $\mathcal{F} = \{f_{n-1,n}\}_1^\infty$, is a supplement to h. Then there is a supplement $\mathcal{G} = \{g_{n-1,n}\}_1^\infty$ to h such that $I(F(\mathcal{F}))$ and $I(F(\mathcal{G}))$ are not comparable.*

Proof. Let $F = F(\mathcal{F})$ and let $h_{n-1,n} \in C^\infty[a_{n-1}, a_n]$. Let $h_{0,1} \in C^\infty[a_0, a_1]$ be such that it satisfies the conditions given in the definition of supplement and such that

$g_{0,1}(a_1) > f_{0,1}(a_1)$. Let $n_1 = 1$ and let $P_1 = \{p_{n-1,n,1}\}_1^\infty$ be defined by

$p_{0,1,1} = g_{0,1}$ and $p_{n-1,n,1} = f_{n-1,n}$ for $n \geqslant 2$. Thus $F(P_1)(a_{n_1}) > F(a_{n_1})$.

Now let P_k, n_k, $p_{n-1,n,k}$ be defined recursively by the following procedure. Suppose that P_k, n_k, $p_{n-1,n,k}$ have been defined for some $k \geqslant 1$ and that

$$F(P_k)(a_{n_k}) < F(a_{n_k}) \qquad (F(P_k)(a_{n_k}) > F(a_{n_k})).$$

Let $h_{n_k, n_k+1} \in C^\infty [a_{n_k}, a_{n_k+1}]$ be such that it satisfies the conditions given in the definition of supplement and such that

$$h_{n_k, n_k+1}(a_{n_k+1}) > f_{n_k, n_k+1}(a_{n_k+1})$$

$$(h_{n_k, n_k+1}(a_{n_k+1}) < f_{n_k, n_k+1}(a_{n_k+1})).$$

Let $\quad P_{k+1} = \{p_{n-1,n,k+1}\}_{n=1}^\infty$ where $p_{n-1,n,k+1} = p_{n-1,n,k}, \quad 1 \leqslant n \leqslant n_k$

$$p_{n_k, n_k+1, k+1} = h_{n_k, n_k+1}, \quad p_{n-1,n,k+1} = f_{n-1,n}, \quad n \geqslant n_k + 2.$$

By Lemma 21 it is possible to choose $n_{k+1} > n_k$ such that $F(P_{k+1})(a_{n_{k+1}}) > F(a_{n_{k+1}})$ $(F(P_{k+1})(a_{n_{k+1}}) < F(a_{n_{k+1}}))$. Let $\mathcal{G} = \{g_{n-1,n}\}_1^\infty$ where $g_{n-1,n} = p_{n-1,n,k}$ for $n \leqslant n_k$. Let $G = F(\mathcal{G})$. Then $G(a_{n_k}) = F(P_k)(a_{n_k})$. Hence $G(a_{n_k}) > F(a_{n_k})$ if k is odd and $G(a_{n_k}) < F(a_{n_k})$ if k is even. Consequently $I(F)$ and $I(G)$ are not comparable.

§ 9. The intersection of all maximal Hardy-fields

Definition. $H_0 = \cap \{H \mid H$ *is a maximal Hardy-field*$\}$

Theorem 8. $H_0 = \{y \mid y$ *is adjoinable to each Hardy-field*$\}$.

Proof. Suppose that $y \in H_0$ and let H be a Hardy-field. Let H_1 be a maximal Hardy-field containing H. Thus $H_0 \subset H_1$ so that $y \in H_1$. Hence y is adjoinable to H. Suppose that y is adjoinable to every Hardy-field. Let H be a maximal Hardy-field. Since y is adjoinable to H and H is maximal we get that $y \in H$, whence $y \in H_0$.

From the definition of H_0 (or from the possible equivalent definition given by Theorem 8 it follows directly that H_0 is a linearly closed and real-closed Hardy-field. The problem to decide whether there is a unique maximal Hardy-field is the same as to decide if H_0 is a maximal Hardy-field or not. However, this is not so which the following theorem shows.

Theorem 9. *Suppose that y is overhardian. Then* $y \notin H_0$.

Proof. Let $y = I(F)$ and let f be a support of F and $\mathcal{F}$ a supplement to f such that $F(x) = F(\mathcal{F})(x)$, $x \geqslant x_0$. Let $\mathcal{G}$ be a supplement to f such that $I(F(\mathcal{F}))$ and $I(F(\mathcal{G}))$ are not comparable. Let H be a Hardy-field such that $I(F(\mathcal{G})) \in H$. Then $y = I(F(\mathcal{F}))$ cannot be adjoined to H since y and $I(F(\mathcal{G}))$ are not comparable. Consequently $y \notin H_0$.

Theorem 10. *The sequence* $\{e_n\}_1^\infty$ *is cofinal in* H_0, *which consequently is bounded.*

Proof. Suppose that $y \in H_0$. Hence y is not overhardian. Thus, by Lemma 14, there is a $k \geqslant 0$ such that $y_{(k)} \leqslant 0$. But by Lemma 14 it then follows that $y < e_k$.

It turns out that H_0 has a lot of nice structural properties. One of these is concerned with the following notion of composition in R.

Definition. *Suppose that* $y \in R$, $z \in R$, $z \gg 1$, $y = I(f)$, $z = I(g)$. *Then let* $y \circ z$ *be defined by* $y \circ z = I(f \circ g)$.

It is obvious from the definition of I that $y \circ z$ does not depend on the choice of representatives for y and z in the above definition.

If $y \gg 1$, $y' > 0$ then there is a unique element y^{-1} such that $y \circ y^{-1} = y^{-1} \circ y = x$ and in point of fact we have $y^{-1} = I(f^{-1})$ where $y = I(f)$, $f^{-1} = (f \,|\, [a, \infty))^{-1}$, where a is chosen such that $f'(x) > 0$ for $x \geqslant a$, so that also $y^{-1} \gg 1$. Especially if y is hardian and $y \gg 1$ then $y' > 0$ so that y^{-1} exists.

Definition. *Suppose that* H *is a Hardy-field and that* $y \gg 1$. *Then let*

$$H \circ y = \{h \circ y \,|\, h \in H\}.$$

According to the differentiation rules

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

i.e.

$$(f^{-1})' = \frac{1}{f' \circ f^{-1}}.$$

Lemma 19. *Suppose that* H *is a Hardy-field,* $y \gg 1$, $y \in H$. *Then* $H \circ y^{-1}$ *is a Hardy-field.*

Proof. Suppose that $h_1, h_2 \in H$. Thus $h_1 \circ y^{-1} - h_2 \circ y^{-1} = (h_1 - h_2) \circ y^{-1} \in H \circ y^{-1}$. Suppose furthermore that $h_2 \neq 0$. Then

$$\frac{h_1 \circ y^{-1}}{h_2 \circ y^{-1}} = \left(\frac{h_1}{h_2}\right) \circ y^{-1} \in H \circ y^{-1} \text{ so that } H \circ y^{-1}$$

is a field. Let $h \in H$. Then

$$(h \circ y^{-1})' = (h' \circ y^{-1})\,(y^{-1})' = \left(\frac{h'}{y'}\right) \circ y^{-1} \in H \circ y^{-1}$$

consequently $H \circ y^{-1}$ is a Hardy-field.

Lemma 20. *Suppose that* H *is a maximal Hardy-field,* $y \in H$, $y \gg 1$. *Then* $H \circ y^{-1}$ *is a maximal Hardy-field.*

Proof. Suppose that $H \circ y^{-1} \subset K$ where K is a Hardy-field. Since $x \in H$ we get that $y^{-1} = x \circ y^{-1} \in H \circ y^{-1} \subset K$. Hence $K \circ (y^{-1})^{-1} = K \circ y$ is a Hardy-field and $H = H \circ y^{-1} \circ y \subset K \circ y$ so that $H = K \circ y$. Suppose that $h \in K$. Then $h \circ y = h_1 \in H$ so that $h = h_1 \circ y^{-1} \in H \circ y^{-1}$ whence $H \circ y^{-1} = K$. Thus $H \circ y^{-1}$ is a maximal Hardy-field.

Theorem 11. *Suppose* $z, y \in H_0$, $y \gg 1$. *Then* $z \circ y \in H_0$.

Proof. Suppose that K is a maximal Hardy-field. Then $K \circ y^{-1}$ is a maximal Hardy-field whence $z \in K \circ y^{-1}$. Thus $z = g \circ y^{-1}$ where $g \in K$ and $z \circ y = g \in K$. But K is an arbitrary maximal Hardy-field. Hence $z \circ y \in H_0$.

Proof without use of maximal Hardy-fields.

Let K be an arbitrary Hardy-field. Let $H = K\{y, x\}$. Thus $y \in H$ and $H \circ y^{-1}$ is a Hardy-field. Hence z is adjoinable to $H \circ y^{-1}$. Let $P = H \circ y^{-1}\{z\}$. Then $y^{-1} \in P$ so that $P \circ y = P \circ (y^{-1})^{-1}$ is a Hardy-field. Now $P \circ y \supset H \circ y^{-1} \circ y = H = K\{y, x\}$ and $z \in P$ whence $z \circ y \in P \circ y$ and $K \subset P \circ y$. Thus $z \circ y$ is adjoinable to K. Consequently $z \circ y \in H_0$.

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