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The Stone-Weierstrass theorem in Lipschitz algebras

By LARS INGE HEDBERG

1. Introduction

A normed function algebra is said to have the Stone-Weierstrass property if every subalgebra which separates points and contains constant functions is dense in the algebra. The purpose of this paper is to investigate this property in certain algebras of real-valued functions with norm greater than the sup norm.

Let X be a compact metric space, connected or not, with metric $d(x,y)$. Let Lip (X,d^α), $0<\alpha\leqslant 1$, be the Banach algebra of all real-valued functions f on X such that

$$\|f\| = \operatorname{Max}\{\sup_{x\in X}|f(x)|, \sup_{x,y\in X}|f(x)-f(y)|/d(x,y)^\alpha\} < \infty$$

and let lip (X,d^α) be the subset of all f in Lip (X,d^α) with the property that

$$\sup\{|f(x)-f(y)|/d(x,y)^\alpha;\ x,y\in X, d(x,y)\leqslant\delta\}\to 0 \quad \text{as} \quad \delta\to 0.$$

If $0<\alpha<1$, lip (X,d^α) always contains plenty of functions, and it is a point-separating closed proper subalgebra of Lip (X,d^α). See [3] where these algebras are studied in detail.

It is natural to ask if lip (X,d^α) has the Stone-Weierstrass property (which, obviously, Lip (X,d^α) does not have). However, in [3], p. 249, reference is made to an unpublished example by Katznelson of a point-separating subalgebra of lip (X,d^α) which is not dense in lip (X,d^α). In the first part of this paper we give a necessary and sufficient condition, in terms of local properties of the functions, for a point-separating subalgebra of lip (X,d^α) to be dense (Theorem 1, Corollaries 1 and 2).

In the second part we consider algebras of periodic functions on the real line. For $0<\alpha<1$ we denote by Λ_α the algebra of all continuous real-valued functions with period 2π such that

$$\|f\| = \operatorname{Max}\left\{\sup_x|f(x)|, \sup_{t>0} t^{-\alpha}\int_{-\pi}^{\pi}|f(x+t)-f(x)|\,dx\right\} < \infty$$

and by λ_α the closed subalgebra of functions such that

$$\lim_{t\to 0} t^{-\alpha}\int_{-\pi}^{\pi}|f(x+t)-f(x)|\,dx = 0.$$

For $0<\alpha<2$ we denote by B_α the algebra of all continuous real-valued functions with period 2π such that

$$\|f\| = \text{Max}\left\{\sup_x |f(x)|, \left(\int_0^\pi t^{-1-\alpha}dt \int_{-\pi}^{\pi} (f(x+t)-f(x))^2 dx\right)^{\frac{1}{2}}\right\} < \infty$$

or equivalently

$$\|f\| = \text{Max}\left\{\sup_x |f(x)|, \left(\sum_{-\infty}^{\infty} |n|^\alpha |f_n|^2\right)^{\frac{1}{2}}\right\} < \infty,$$

where f_n are the Fourier coefficients of f.

We show that λ_α and B_α have the Stone-Weierstrass property for $0<\alpha<1$ (Theorem 2). On the other hand, B_α does not have this property for $\alpha \geqslant 1$ (see [1]).

2. Subalgebras of lip (X, d^α), $0<\alpha<1$

We use the following notation:

$$S_a(\delta) = \{x \in X;\ d(x,a) = \delta\},$$

$$B_a(\delta) = \{x \in X;\ d(x,a) \leqslant \delta\}.$$

The closure of a set E is denoted by $\bar{E}$.

We shall prove the following theorem.

Theorem 1. *Let X be a compact space with metric d, and let A be a point-separating subalgebra of* lip (X,d^α), $0<\alpha<1$, *which contains constant functions. Then A is dense in* lip (X,d^α) *if and only if for every $a \in X$ there are numbers $M_a>0$ and $\delta_a>0$ such that for every $x \in S_a(\delta)$, $\delta \leqslant \delta_a$, there is an $f \in A$ with $f(a)=1$, $f(x)=0$, and*

$$\sup\left\{\frac{|f(y)-f(z)|}{d(y,z)^\alpha};\ y,z \in B_a(\delta)\right\} < \frac{M_a}{\delta^\alpha}.$$

The condition is clearly necessary, since for all a and $\alpha<\beta<1$ the function $g(x) = 1-d(x,a)^\beta/\delta^\beta$ belongs to lip (X,d^α) and satisfies $g(a)=1$, $g(x)=0$ for $x \in S_a(\delta)$, and $|g(y)-g(z)|/d(y,z)^\alpha \leqslant \delta^{-\alpha}$ for $y,z \in B_a(\delta)$.

To prove the sufficiency we need the following lemmas.

Lemma 1. *Let $f_1,\dots,f_n \in A$, and let ϕ be a real-valued function which is defined and Lipschitz continuous with respect to the Euclidean metric in a neighborhood of the set $\Gamma = \{(f_1(x),\dots,f_n(x));\ x \in X\}$ in R^n. Then $\phi(f_1,\dots,f_n) \in \bar{A}$.*

Proof. If ϕ is continuously differentiable the assertion follows from the Weierstrass approximation theorem (Whitney [4], p. 74). Now suppose that ϕ is only Lipschitz continuous. It is then enough to show that there exists a sequence, $\{\phi_\nu\}_1^\infty$, of C^1 functions such that

$$\|(\phi-\phi_\nu)\ (f_1,\dots,f_n)\| \to 0,\ \nu \to \infty.$$

Let $g \in C^\infty(R^n)$ have support in $|u| \leqslant 1$, let $g \geqslant 0$, and $\int_{R^n} g\,du = 1$. Put $g_\nu(u) = \nu^n g(\nu u)$ for $\nu = 1, 2, \dots$ We can assume that ϕ is defined and Lipschitz continuous in all of R^n (see e.g. [3], p. 244), and then we put

$$\phi_\nu(v) = \int_{R^n} \phi(v-u) g_\nu(u)\, du = \int_{|v-u|\leqslant 1/\nu} \phi(u) g_\nu(v-u)\, du.$$

It follows that $|\phi(v) - \phi_\nu(v)| \leqslant \text{const. } 1/\nu$. Moreover

$$\frac{\partial \phi_\nu(v)}{\partial v_i} = \int_{|v-u|\leqslant 1/\nu} \phi(u) \frac{\partial g_\nu(v-u)}{\partial v_i}\, du = \int_{|v-u|\leqslant 1/\nu} (\phi(u) - \phi(v)) \frac{\partial g_\nu(v-u)}{\partial v_i}\, du.$$

This implies that $\phi_\nu \in C^1$ and that

$$\left|\frac{\partial \phi_\nu(v)}{\partial v_i}\right| \leqslant \text{Const.} \int_{|v-u|\leqslant 1/\nu} |u-v|\, \nu^{n+1} \left|\frac{\partial g(\nu(v-u))}{\partial v_i}\right| du \leqslant \text{Const.}$$

We now choose a sequence $\{\delta_\nu\}_1^\infty$ so that $\delta_\nu \to 0$ and $\nu\delta_\nu^\alpha \to \infty$, as $\nu \to \infty$. We then obtain for arbitrary ν

$$\|(\phi - \phi_\nu)(f_1, \ldots, f_n)\| \leqslant \text{Const. Max} \left\{ \sup_{d(x,y)\leqslant \delta_\nu} \frac{\left(\sum_{i=1}^n |f_i(x) - f_i(y)|^2\right)^{\frac{1}{2}}}{d(x,y)^\alpha} \frac{1}{\nu\delta_\nu^\alpha}, \frac{1}{\nu} \right\},$$

which tends to zero as $\nu \to \infty$. This proves the lemma.

We single out the following consequence as a separate lemma.

Lemma 2. *If $f_i \in A$, $i = 1, 2, \ldots, n$, then the function F defined by*

$$F(x) = \text{Max}_i \{f_1(x), f_2(x), \ldots, f_n(x)\}$$

belongs to $\bar{A}$, and $\|F\| \leqslant \text{Max}_i \|f_i\|$.

Proof. The first assertion follows from Lemma 1, and the second is trivial.

Lemma 3. *Let $a \in X$ be a point where the condition of Theorem 1 is satisfied. Then, for every positive $\delta \leqslant \delta_a$, there is a function $\psi \in \bar{A}$ with the following properties.*

$$0 \leqslant \psi(x) \leqslant 1, \quad x \in X, \tag{1}$$

$$\psi(x) = 1 \text{ in a neighborhood } \omega \text{ of } a, \tag{2}$$

$$\psi(x) = 0, \quad x \notin B_a(\delta), \tag{3}$$

$$\|\psi\| \leqslant M_a/\delta^\alpha. \tag{4}$$

Proof. Let $a \in X$ and suppose that M_a and δ_a are constants with the properties in Theorem 1.

Then, if $0 < \delta \leqslant \delta_a$, there is for every $x_0 \in S_a(\delta)$ an $f \in A$ such that $f(a) < 0$, $f(x) > 1$ in a neighborhood of x_0, and $\sup \{|f(y) - f(z)|/d(y,z)^\alpha;\ y, z \in B_a(\delta)\} < M_a/\delta^\alpha$.

Since $S_a(\delta)$ is compact it can be covered by a finite number of such open neighborhoods. We denote the corresponding functions f by f_i, $i = 1, 2, \ldots, n$. Then there is an $\eta > 0$ so that $F_n(x) = \text{Max}_{1\leqslant i\leqslant n} \{f_i(x)\} \geqslant 1 + 4\eta$, $x \in S_a(\delta)$. We put

$$K_1 = \{x;\ F_n(x) \leqslant 1 + 2\eta\} \cap B_a(\delta),\ K_2 = \{x;\ F_n(x) \geqslant 1 + 2\eta\} \cap B_a(\delta),\ K_3 = S_a(\delta) \cup \complement B_a(\delta).$$

These sets are compact, and $K_1 \cap K_3 = \phi$.

Since A separates points on X it is readily seen by means of a compactness argument in $X \times X$ that there are functions in A, $f_{n+1}, \dots, f_N$, say, such that the vector-valued function $\Phi(x) = \{f_1(x), \dots, f_N(x)\}$ maps K_1 and K_3 onto disjoint compact sets, $\Phi(K_1)$ and $\Phi(K_3)$, in R^N. K_2 is mapped onto $\Phi(K_2)$,

$$\Phi(K_2) \subset \{u \in R^N;\ \mathrm{Max}_{1 \leqslant i \leqslant n}\ u_i \geqslant 1 + 2\eta\},\ \text{and}\ \Phi(K_1) \subset \{u \in R^N;\ \mathrm{Max}_{1 \leqslant i \leqslant n}\ u_i \leqslant 1 + 2\eta\}.$$

Then there are compact neighborhoods N_i of the $\Phi(K_i)$ so that

$$N_1 \subset \{u \in R^N, \operatorname*{Max}_{1 \leqslant i \leqslant n} u_i \leqslant 1 + 3\eta\},$$

$$N_2 \subset \{u \in R^N, \operatorname*{Max}_{1 \leqslant i \leqslant n} u_i \geqslant 1 + \eta\}$$

and

$$N_1 \cap N_3 = \phi.$$

We can now define a function ϕ on $N_1 \cup N_2 \cup N_3$ by putting

$$\phi(u) = \mathrm{Min}\,\{1, \mathrm{Max}\,(0, 1 - \operatorname*{Max}_{1 \leqslant i \leqslant n} u_i)\}$$

for $u \in N_1 \cup N_2$ and $\phi(u) = 0$ for $u \in N_2 \cup N_3$. This is no contradiction, for both definitions give zero for $u \in N_2$. Since there is a positive distance between N_1 and N_3 it is clear that ϕ is Lipschitz continuous. Then, by Lemma 2, $\psi = \phi(f_1, \dots, f_N) \in \bar{A}$, and it is easy to see that ψ has the required properties.

Proof of Theorem 1. We assume that A satisfies the conditions in the theorem, and we shall show that a given function $g \in \mathrm{lip}\,(X, d^\alpha)$ belongs to $\bar{A}$. We assume that for all x and y with $d(x,y) \leqslant 2\delta$ we have

$$|g(x) - g(y)|/d(x,y)^\alpha \leqslant \eta(\delta)/(2\delta)^\alpha, \tag{5}$$

where $\eta(\delta)/\delta^\alpha \to 0$, as $\delta \to 0$.

We choose $\varepsilon > 0$, to be kept fixed, and then for every $a \in X$ we choose $\delta < \delta_a$ so that

$$\eta(\delta)/\delta^\alpha \leqslant \varepsilon\ \mathrm{Min}\ \{1, 1/M_a\}. \tag{6}$$

Corresponding to this δ, there is by Lemma 3 a function ψ satisfying the conditions in that lemma.

Since X is compact it can be covered by a finite number of the neighborhoods in (2), $\{\omega_i\}_1^p$, say. If the corresponding δ and ψ are denoted by $\{\delta_i\}_1^p$ and $\{\psi_i\}_1^p$, and the support of ψ_i is Ω_i, (5) gives that

$$\operatorname*{osc}_{x \in \Omega_i} g(x) \leqslant \eta(\delta_i) \tag{7}$$

and hence, by (4) and (6)

$$\operatorname*{osc}_{x \in \Omega_i} g(x)\, \|\psi_i\| \leqslant \varepsilon. \tag{8}$$

We shall now construct explicitly a function in $\bar{A}$ which approximates g. We choose numbers $l_0 < l_1 < \dots < l_q$ so that

$$l_0 \leqslant g(x) \leqslant l_q, \quad x \in X$$

and

$$l_j - l_{j-1} = \operatorname*{Min}_{1 \leqslant i \leqslant p} \eta(\delta_i), \quad j = 1, 2, \dots, q. \tag{9}$$

Let L_j be the compact set

$$L_j = \{x; g(x) \geqslant l_j\}, \quad j = 0, 1, 2, \ldots, q.$$

For each $i = 1, 2, \ldots, p$ there is an integer $\mu(i)$, such that $\Omega_i \subset L_{\mu(i)}$, but $\Omega_i \not\subset L_{\mu(i)+1}$, and there is an integer $\nu(i) \geqslant \mu(i)$ such that $\omega_i \cap L_{\nu(i)} \neq \phi$, but $\omega_i \cap L_{\nu(i)+1} = \phi$.

We put $\psi_i' = (l_{\nu(i)} - l_{\mu(i)})\psi_i$ if $\nu(i) > \mu(i)$, and $\psi_i' = (l_1 - l_0)\psi_i$ otherwise. It follows from (8) and (9) that

$$\|\psi_i'\| \leqslant (\operatorname*{osc}_{x \in \Omega_i} g(x) + l_{\mu(i)+1} - l_{\mu(i)}) \|\psi_i\| \leqslant 2\varepsilon.$$

Then we put

$$h_1(x) = l_0 + \operatorname*{Max}_{1 \leqslant i \leqslant p} \psi_i'(x).$$

Clearly $h_1(x) \geqslant l_0 + l_1 - l_0 = l_1$. Moreover, if $x \in L_j$, $j > 1$, and if x belongs to some ω_i such that Ω_i intersects $L_0 \setminus L_1$, then $\psi_i'(x) \geqslant l_j - l_0$, so $h_1(x) \geqslant l_j$.

We put $k_1(x) = \operatorname{Min}\{h_1(x), l_1\}$, and $h_2(x) = \operatorname{Max}\{h_1(x), k_1(x) + \operatorname{Max}\{\psi_i'(x); \Omega_i \subset L_1\}\}$. Then $h_2(x) \geqslant l_2$, $x \in L_2$, for either x belongs to some ω_i such that $\Omega_i \subset L_1$, and then $\psi_i'(x) \geqslant l_2 - l_1$, or else $h_1(x) \geqslant l_2$. If $x \in L_j$ for $j > 2$, and if x belongs to some ω_i such that Ω_i intersects $L_1 \setminus L_2$, then either $\Omega_i \subset L_1$, and $\psi_i'(x) \geqslant l_j - l_1$, so $h_2(x) \geqslant l_j$, or else Ω_i intersects $L_0 \setminus L_1$, so $h_1(x) \geqslant l_j$. For $x \notin L_1$, $h_2(x) = h_1(x)$.

Now assume that we have constructed h_r so that $h_r(x) \geqslant l_r$, $x \in L_r$, and so that $h_r(x) \geqslant l_j$, $j > r$, if $x \in L_j$ and belongs to some ω_i such that Ω_i is not contained in L_r.

Then we put

$$k_r(x) = \operatorname{Min}\{h_r(x), l_r\}$$

and

$$h_{r+1}(x) = \operatorname{Max}\{h_r(x), k_r(x) + \operatorname{Max}\{\psi_i'(x); \Omega_i \subset L_r\}\}.$$

If $x \notin L_r$, clearly $h_{r+1}(x) = h_r(x)$. If $x \in L_{r+1}$, we have either that x belongs to some $\omega_i \subset \Omega_i \subset L_r$, so $\psi_i'(x) \geqslant l_{r+1} - l_r$, or else $h_r(x) \geqslant l_{r+1}$, by the hypothesis. In both cases $h_{r+1}(x) \geqslant l_{r+1}$. If $x \in L_j$, $j > r+1$, and x belongs to some ω_i such that Ω_i is not contained in L_{r+1}, then either Ω_i is contained in L_r, and $\psi_i'(x) \geqslant l_j - l_r$, or else $h_r(x) \geqslant l_j$. In both cases $h_{r+1}(x) \geqslant l_j$.

The procedure breaks off for $r = q$, so we finally put $h_q(x) = h(x)$, and we shall show that $\|g - h\|$ is small. Since $h \in \bar{A}$ by Lemma 2, this will prove the theorem.

We already know that $h(x) \geqslant l_j$, $x \in L_j$. On the other hand we shall see that

$$h(x) \leqslant l_j + 2 \operatorname*{Max}_i \eta(\delta_i), \quad x \in L \setminus L_{j+1}.$$

This is certainly true for $h_1(x)$, for by (7) and (9)

$$h_1(x) \leqslant l_0 + \operatorname{Max} \operatorname*{osc}_{x \in \Omega_i} g(x) + l_1 - l_0 \leqslant l_0 + 2 \operatorname*{Max}_i \eta(\delta_i).$$

Assume that

$$h_r(x) \leqslant l_j + 2 \operatorname*{Max}_i \eta(\delta_i), \quad x \in L_j \setminus L_{j+1}, \quad j = 1, 2, \ldots$$

Then, for $x \notin L_r$, $h_{r+1}(x) = h_r(x)$, and for $x \in L_r$, either

$$h_{r+1}(x) = l_r + \psi_i'(x) \leqslant l_r + \operatorname*{Max}_i \operatorname*{osc}_{x \in \Omega_i} g(x) + l_{r+1} - l_r \leqslant l_r + 2 \operatorname*{Max}_i \eta(\delta_i),$$

or else $h_{r+1}(x) = h_r(x)$, which proves the assertion.

It follows that $|g(x)-h(x)| \leqslant 2 \operatorname{Max}_i \eta(\delta_i)$, and hence for $d(x,y) \geqslant 2 \operatorname{Max}_i \delta_i$ we have by (6)

$$\frac{|g(x)-h(x)-g(y)+h(y)|}{d(x,y)^\alpha} \leqslant \frac{4 \operatorname{Max}_i \eta(\delta_i)}{(2 \operatorname{Max}_i \delta_i)^\alpha} \leqslant 4\varepsilon.$$

For $d(x,y) \leqslant 2 \operatorname{Max}_i \delta_i$ we have by (5) and (6)

$$\frac{|g(x)-h(x)-g(y)+h(y)|}{d(x,y)^\alpha} \leqslant \frac{|g(x)-g(y)|}{d(x,y)^\alpha} + \frac{|h(x)-h(y)|}{d(x,y)^\alpha} \leqslant \varepsilon + \frac{|h(x)-h(y)|}{d(x,y)^\alpha}$$

and it is thus enough to show that $|h(x)-h(y)|/d(x,y)^\alpha$ is small if $d(x,y) \leqslant 2 \operatorname{Max}_i \delta_i$.

Let x and y be given with $d(x,y) \leqslant 2 \operatorname{Max}_i \delta_i$, and assume that $h(x) \geqslant h(y)$. Then $h(x) = l_j + \psi_i'(x) = h_{j+1}(x)$ for certain i and j so that $\Omega_i \subset L_j$, and either $y \in L_j$, $y \in L_{j-1} \setminus L_j$, or $y \notin L_{j-1}$.

In the first case $h(y) \geqslant l_j + \psi_i'(y)$, so $h(x)-h(y) \leqslant \psi_i'(x) - \psi_i'(y)$.

In the second case it is clear that $h(y) = h_j(y)$, so $h(x)-h(y) \leqslant h_{j+1}(x) - l_j + h_j(x) - h_j(y) \leqslant 2 \operatorname{Max}_\nu |\psi_\nu'(x) - \psi_\nu'(y)|$, since $|h_j(x)-h_j(y)| \leqslant \operatorname{Max}_\nu |\psi_\nu'(x)-\psi_\nu'(y)|$, and $\psi_i'(y) = 0$.

In the third case, finally, $h(x)-h(y) = h(x) - l_j + l_j - h(y) \leqslant \psi_i'(x) - \psi_i'(y) + g(x) - g(y) + g(y) - h(y) \leqslant \psi_i'(x) - \psi_i'(y) + g(x) - g(y) + \operatorname{Min}_i \eta(\delta_i)$. In this case $d(x,y) \geqslant \operatorname{Min}_i \delta_i$, so we find in all cases

$$\frac{h(x)-h(y)}{d(x,y)^\alpha} \leqslant \operatorname{Max}_i \|\psi_i'\| + 2\varepsilon < 3\varepsilon$$

which proves the theorem.

Now let $E \subset X$ be a compact, totally disconnected set, and let A_E be the subalgebra of lip (X, d^α) which consists of all functions in lip (X, d^α) which are constant in every component of some open neighborhood of E. Then A_E obviously separates points in X. The following corollary translates the condition in Theorem 1 into a metric condition on the set E.

Corollary 1. *A_E is dense in* lip (X, d^α), $0 < \alpha < 1$, *if and only if for every $a \in E$ there are numbers $M_a > 0$ and $\delta_a > 0$ such that for every $\delta < \delta_a$ there is an open set O containing $E \cap B_a(\delta)$ with the following property:*

O has finitely many components ω_i, $i = 1, 2, \ldots, p$, $\omega_0 = \complement B_a(\delta)$, with distances $d(\omega_i, \omega_j) = d_{ij}$, and

$$\operatorname{Min}\left\{\sum_{i=1}^{q} d^\alpha_{\nu_{i-1}\nu_i}\right\} \geqslant \delta^\alpha / M_a,$$

where the minimum is taken over all sums such that $\nu_0 = 0$, $a \in \omega_{\nu_q}$, and over all $q = 1, 2, \ldots, p$.

Proof. Suppose E satisfies the condition in the corollary. Choose $a \in E$, and a $\delta < \delta_a$, so that there is a set $O \supset E \cap B_a(\delta)$ with the required property.

We define a function f on $\bar{O}$ by $f(x) = 0$, for $x \in \bar{\omega}_0$, and for $x \in \bar{\omega}_j$, $j = 1, 2, \ldots, p$, by

$$f(x) = \operatorname{Min}\left\{\sum_{i=1}^{q} d^\alpha_{\nu_{i-1}\nu_i}\right\},$$

where the minimum is taken over all sums such that $\nu_0=0$, $\nu_q=j$, and over all $q=1, 2, ..., p$.

Suppose $x\in\bar{\omega}_j$ and $y\in\bar{\omega}_k$. Then $d(x,y)\geqslant d_{jk}$. If $f(x)\geqslant f(y)$ it follows from the definition that $f(x)\leqslant f(y)+d_{jk}^{\alpha}$, so $|f(x)-f(y)|/d(x,y)^{\alpha}\leqslant 1$. Also $f(a)\geqslant \delta^{\alpha}/M_a$, and $f(x)=0$ for $x\notin B_a(\delta)$.

Moreover, for $\alpha<\alpha'<1$,

$$\sup_{x,y\in \bar{O}} |f(x)-f(y)|/d(x,y)^{\alpha'}\leqslant \sup_{j\neq k} d_{jk}^{\alpha-\alpha'}\leqslant 2,$$

if $\alpha'-\alpha$ is small enough. But f can be extended to a function in Lip $(X,d^{\alpha'})$ with the same Lipschitz constant ([3], p. 244). Thus

$$|f(x)-f(y)|/d(x,y)^{\alpha}\leqslant 2d(x,y)^{\alpha'-\alpha}, \quad x,y\in X,$$

so $f\in$ lip (X,d^{α}). This proves the sufficiency part of the corollary, since it is obvious that functions f with the required properties exist for $a\notin E$.

The necessity is obvious.

In the case when X is an interval $I\subset R^1$ and $d(x,y)=|x-y|$, the corollary gets a particularly simple form.

If $O=\bigcup_1^{\infty}\omega_i$ is a union of disjoint intervals, we put $M_{\alpha}(O)=\Sigma_1^{\infty}|\omega_i|^{\alpha}$.

Corollary 2. *Let $E\subset I$ be compact and totally disconnected. Then A_E is dense in* lip (I,d^{α}) *if and only if for every $a\in E$*

$$\varliminf_{\delta\to 0} M_{\alpha}((I\backslash E)\cap B_a(\delta))/\delta^{\alpha}>0.$$

The proof is obvious.

3. The Stone-Weierstrass theorem in λ_α and B_α, $0<\alpha<1$

We refer to the introduction for the notation. It is easy to prove the following fact.

Lemma 4. *λ_α is a closed subalgebra of Λ_α, $0<\alpha<1$.*

The purpose of this section is to prove the following theorem.

Theorem 2. *The algebras λ_α and B_α, $0<\alpha<1$, have the Stone-Weierstrass property.*

Remark. The algebras B_α, $\alpha>1$, were studied in [1]. It was shown there that they do not have the Stone-Weierstrass property (p. 85 f., p. 94). It is easy to see that this extends to $\alpha=1$.

Lemma 5. *The C^1 functions are dense in λ_α and B_α.*

The proofs are straightforward, and are omitted.

In what follows, A is a subalgebra of λ_α or B_α which separates points and contains constant functions.

Lemma 6. *Let $\delta > 0$ and $-\pi \leqslant a-\delta < a \leqslant b < b+\delta \leqslant \pi$. Then there is a function $f \in \bar{A}$ such that*

$$0 \leqslant f(x) \leqslant 1,$$

$$f(x) = 1, \quad a - \frac{\delta}{2} \leqslant x \leqslant b,$$

$$f(x) = 0, \quad -\pi \leqslant x \leqslant a-\delta \quad \text{and} \quad b + \frac{\delta}{2} \leqslant x \leqslant \pi,$$

and

$$t^{-\alpha} \int_{-\pi}^{\pi} |f(x+t) - f(x)|\, dx \leqslant 4\delta^{1-\alpha}, \quad 0 < t < \frac{\delta}{2},$$

or

$$\int_0^{\delta/2} t^{-1-\alpha} dt \int_{-\pi}^{\pi} |f(x+t) - f(x)|^2 dx \leqslant \frac{4}{1-\alpha} \delta^{1-\alpha},$$

respectively.

Proof. For a moment we consider the functions in A as functions on the circle. We then make the following observation.

If I_1 and I_2 are two open intervals which together cover the circle, and if $f_1 \in A$ and $f_2 \in A$ are equal on $I_1 \cap I_2$, then the function f_3, defined as f_1 on I_1 and as f_2 on I_2, belongs to $\bar{A}$.

This is proved as in the proof of Lemma 3.

Now let a, b, and δ be given as stated. By assumption there is a function $h_1 \in A$ such that $h_1(a-(\delta/2)) > 1$, and $h_1(a-\delta) < 0$. Let h_2 be the truncation of h_1 by 0 and 1. Then $h_2 \in \bar{A}$ by Lemma 2, which clearly applies. Similarly there is a function $h_3 \in A$ such that $h_3(b) > 1$, and $h_3(b+(\delta/2)) < 0$, with a corresponding truncation h_4. By the above observation the function h_5, defined by

$$h_5(x) = \begin{cases} h_2(x), & a-\delta \leqslant x \leqslant a - \dfrac{\delta}{2}, \\ 1, & a - \dfrac{\delta}{2} \leqslant x \leqslant b, \\ h_4(x), & b \leqslant x \leqslant b + \dfrac{\delta}{2}, \\ 0, & \text{elsewhere in } (-\pi, \pi), \end{cases}$$

also belongs to $\bar{A}$.

Now denote h_5 by h and consider the interval $(a-\delta, a) = I$. By a well-known lemma of Riesz (see [2], p. 6) the set

$$O = \{x \in I;\ \exists y \in I,\ y > x, \quad \text{such that} \quad h(y) < h(x)\}$$

is open, $O = \bigcup_1^\infty \omega_\nu$, $\omega_\nu = (\alpha_\nu, \beta_\nu)$, and $h(\alpha_\nu) = h(\beta_\nu)$. Clearly, $h(x) > h(\alpha_\nu)$ for $\alpha_\nu < x < \beta_\nu$, and $h(x) \geqslant h(\beta_\nu)$ for $\beta_\nu \leqslant x \leqslant a$.

For any $\varepsilon > 0$ we now define a function h_ε by

$$h_\varepsilon(x) = \begin{cases} h(x), & x \in I \setminus O \\ \operatorname{Min}\{h(x), h(\alpha_\nu) + \varepsilon\}, & x \in \omega_\nu, \ \nu = 1, 2, \ldots \end{cases}$$

We shall show that if h_ε is defined in a similar way in $(b, b+\delta)$, and as h elsewhere, then h_ε is the required function, if ε is chosen small enough. It is immediate from Lemma 2 and the earlier observation that $h_\varepsilon \in \bar{A}$, because $h(x) < h(\alpha_\nu) + \varepsilon$ in all but a finite number of the ω_ν.

Let $x \in I$, $0 < t < \delta/2$, and assume that $h_\varepsilon(x-t) > h_\varepsilon(x)$. We claim that then

$$h_\varepsilon(x-t) - h_\varepsilon(x) \leqslant \operatorname{Min}\{\varepsilon, h(x-t) - h(x)\}.$$

In fact, it is clear from the definition of h_ε that $h_\varepsilon(x-t) - h_\varepsilon(x) \leqslant \varepsilon$. On the other hand, $h_\varepsilon(x-t) > h_\varepsilon(x)$ implies that $h_\varepsilon(x) = h(x)$, for otherwise we would have $h(x) > h(\alpha_\mu) + \varepsilon$ for some μ, $\alpha_\mu < x < \beta_\mu$, and then $h_\varepsilon(x) = h(\alpha_\mu) + \varepsilon \geqslant h_\varepsilon(x-t)$. Thus

$$h_\varepsilon(x-t) - h_\varepsilon(x) = h_\varepsilon(x-t) - h(x) \leqslant h(x-t) - h(x).$$

For any f it is true that $|f| = f + 2f^-$, where $f^- = \operatorname{Max}\{-f, 0\}$. We find for λ_α,

$$\int_I |h_\varepsilon(x) - h_\varepsilon(x-t)|\,dx = \int_I (h_\varepsilon(x) - h_\varepsilon(x-t))\,dx + 2\int_I (h_\varepsilon(x) - h_\varepsilon(x-t))^-\,dx.$$

Here

$$t^{-\alpha}\int_I (h_\varepsilon(x) - h_\varepsilon(x-t))\,dx = t^{-\alpha}\int_{a-\delta}^{a} h_\varepsilon(x)\,dx - t^{-\alpha}\int_{a-\delta-t}^{a-t} h_\varepsilon(x)\,dx \leqslant t^{1-\alpha} \leqslant \left(\frac{\delta}{2}\right)^{1-\alpha}$$

and

$$2t^{-\alpha}\int_I (h_\varepsilon(x) - h_\varepsilon(x-t))^-\,dx \leqslant 2t^{-\alpha}\int_I |h(x) - h(x-t)|\,dx \leqslant 2t^{-\alpha}\int_{-\pi}^{\pi} |h(x) - h(x-t)|\,dx$$

or

$$2t^{-\alpha}\int_I (h_\varepsilon(x) - h_\varepsilon(x-t))^-\,dx \leqslant 2t^{-\alpha}\varepsilon.$$

Now choose $t_0 > 0$ so that $2t^{-\alpha}\int_{-\pi}^{\pi}|h(x) - h(x-t)|\,dx < \delta^{1-\alpha}$ for $0 < t < t_0$, and then choose ε so small that $2t_0^{-\alpha}\varepsilon < \delta^{1-\alpha}$. It follows that

$$t^{-\alpha}\int_{a-\delta}^{a} |h_\varepsilon(x) - h_\varepsilon(x-t)|\,dx < 2\delta^{1-\alpha}, \quad 0 < t < \frac{\delta}{2}.$$

This follows in the same way for $(b, b+\delta)$, and for all other x the integrand is 0, so the lemma is proved for λ_α.

For B_α the proof is similar, because $|h_\varepsilon(x) - h_\varepsilon(x-t)|^2 \leqslant |h_\varepsilon^2(x) - h_\varepsilon^2(x-t)|$.

Proof of Theorem 2. Let g be a C^1 function which is non-decreasing in an interval (a, c), non-increasing in an interval (c, b), and 0 elsewhere. It suffices to show that $g \in \bar{A}$. We can also assume that $|g'(x)| \leqslant 1$.

Choose $\delta > 0$ and let $l_1 < \ldots < l_{p+1}$ be such that $l_1 \leqslant g(x) \leqslant l_{p+1}$ and $l_{\nu+1} - l_\nu = \delta$ for all $\nu = 1, 2, \ldots p$. Then, for every ν, there is an interval $I_\nu = (a_\nu, b_\nu)$ such that $g(x) > l_\nu$ for $x \in I_\nu$, and $g(x) \leqslant l_\nu$ for $x \notin I_\nu$. Clearly $b_{\nu-1} - b_\nu \geqslant \delta$ and $a_\nu - a_{\nu-1} \geqslant \delta$.

By Lemma 6, for every $\nu = 1, \ldots, p$. there is a function $f_\nu \in \bar{A}$ such that $f_\nu(x) = \delta$ on I_ν, $f_\nu(x) = 0$ outside $I_{\nu-1}$, and (if $f(x+t) - f(x) = \Delta_t f(x)$)

$$t^{-\alpha} \int_{-\pi}^{\pi} |\Delta_t f_\nu(x)|\, dx \leqslant K\delta^{2-\alpha}, \quad 0 < t < \delta/2$$

or

$$\int_0^{\delta/2} t^{-1-\alpha} dt \int_{-\pi}^{\pi} |\Delta_t f_\nu(x)|^2 dx \leqslant K\delta^{3-\alpha},$$

respectively.

Then $|g(x) - \sum_1^p f_\nu(x)| \leqslant \delta$ for all x. Moreover, for λ_α,

$$t^{-\alpha} \int_{-\pi}^{\pi} \left| \Delta_t \left(g - \sum_1^p f_\nu \right) \right| dx \leqslant 2\pi t^{-\alpha} 2\delta \leqslant 8\pi\delta^{1-\alpha}, \quad t \geqslant \delta/2,$$

and

$$t^{-\alpha} \int_{-\pi}^{\pi} \left| \Delta_t \left(g - \sum_1^p f_\nu \right) \right| dx \leqslant t^{-\alpha} \int_{-\pi}^{\pi} |\Delta_t g|\, dx + \sum_{\nu=1}^{p} t^{-\alpha} \int_{-\pi}^{\pi} |\Delta_t f_\nu|\, dx$$

$$\leqslant 2\pi t^{1-\alpha} + pK\delta^{2-\alpha} \leqslant (2+K)\pi\delta^{1-\alpha}, \; 0 < t < \delta/2,$$

because $p\delta \leqslant \operatorname{osc} g \leqslant \pi$. But δ is arbitrary, which proves the theorem for λ_α. For B_α the proof is similar, and is omitted.

University of Uppsala and Massachusetts Institute of Technology

REFERENCES

1. Hedberg, L. I., The Stone-Weierstrass theorem in certain Banach algebras of Fourier type. Ark. Mat. *6*, 77–102 (1965).
2. Riesz, F., and Sz.-Nagy, B., Leçons d'analyse fonctionnelle. Budapest and Paris, 1955.
3. Sherbert, D. R., The structure of ideals and point derivations in Banach algebras of Lipschitz functions. Trans. Amer. Math. Soc. *111*, 240–272 (1964).
4. Whitney, H., Analytic extensions of differentiable functions defined in closed sets. Trans. Amer. Math. Soc. *36*, 63–89 (1934).

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