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Differentiability properties of solutions of systems of differential equations

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Introduction

Various algebraic characterizations of the differential equations with constant coefficients which only possess infinitely differentiable solutions have been given by Hörmander [4]. One of them is the following. The differential equation can be written

$$P(D)u=0, \tag{1}$$

where $P(\xi)=P(\xi_1, \ldots, \xi_\nu)$ is a polynomial and $P(D)$ is obtained by replacing ξ_j by $-i\,\partial/\partial x^j$, and u is a function of $x=(x^1, \ldots, x^\nu)$. Then, according to Theorem 3.7 in Hörmander [4], all (square integrable) solutions of (1) in a bounded domain Ω are infinitely differentiable functions (after correction on a null set) if and only if the set $V=\{\zeta; P(\zeta)=0\}$ satisfies the condition

$$\operatorname{Im}\zeta\to\infty \quad \text{when} \quad V\ni\zeta\to\infty. \tag{2}$$

An equivalent form of this condition is evidently that the distance from a real point ξ to V tends to infinity when $\xi\to\infty$. If (2) is fulfilled, it follows that every distribution u satisfying a differential equation

$$P(D)u=f \quad \text{in} \quad \Omega, \tag{3}$$

where Ω is an open set and $f\in C^\infty(\Omega)$, is itself in $C^\infty(\Omega)$. This theorem is not explicitly stated in the quoted paper except when $f=0$ (cf. the end of section 3.5) but is an immediate consequence of formula (3.5.3) and the well-known properties of convolutions.

With a terminology, which has recently become generally accepted, the differential operator $P(D)$ and the polynomial $P(\zeta)$ are called *hypoelliptic* if the solutions of (3) are infinitely differentiable where this is true for f, or, which is equivalent, if (2) is fulfilled.

We shall here extend these results to systems of differential equations with constant coefficients. This extension is straightforward unless the system is *overdetermined*, that is, contains more equations than unknowns. In this case our proof is a simple consequence of the following theorem of Lech [5]:

Let I be an ideal of polynomials in $\zeta\in C_\nu$ with complex coefficients and V_I the alge-

braic set defined by I, that is, the set where all $Q \in I$ vanish. Then there is a polynomial $R \in I$ such that

$$d(\xi, V_I)/d(\xi, V_R) \leqslant C, \quad \xi \in R_\nu, \tag{4}$$

where C is a constant, $d(\xi, V_I)$ is the distance from ξ to V_I and $d(\xi, V_R)$ the distance from ξ to the set $V_R = \{\zeta;\ R(\zeta) = 0\}$.

For the solutions of a hypoelliptic differential equation (1), estimates of the derivatives of high order were also given by Hörmander [4, Theorem 3.8]. We shall prove here that these estimates cannot be improved. Using the theorem of Lech we also extend them to arbitrary hypoelliptic systems of differential equations at the same time.

Conditions for hypoellipticity

Any system of differential equations with constant coefficients can be written

$$\mathcal{P}(D)u = f, \tag{5}$$

where $u = (u_1, \ldots, u_n)$ and $f = (f_1, \ldots, f_m)$ are vectors whose components are functions or distributions of $x \in \Omega$, an open set in R^ν, and $\mathcal{P}(D)$ is a matrix with m rows and n columns, whose elements $P_{jk}(D)$ are differential operators with constant coefficients. Both m and n are finite, but it would cause no difficulty to permit m to be infinite.

Definition. *If the distribution vector u is in $C^\infty(\Omega)$ in every open set Ω where $f = \mathcal{P}(D)u$ is in $C^\infty(\Omega)$, the system $\mathcal{P}(D)$ is called hypoelliptic.*

A necessary condition for hypoellipticity is contained in the following theorem.

Theorem 1. *Assume that there exists a domain Ω and an integer N such that all solutions of the homogeneous system of differential equations*

$$\mathcal{P}(D)u = 0 \tag{6}$$

in $C^N(\Omega)$ are in fact in $C^{N+1}(\Omega)$. Then, given any number A, there is a number B such that the rank of the matrix $\mathcal{P}(\zeta)$ is n if $|\operatorname{Im}\zeta| \leqslant A$ and $|\operatorname{Re}\zeta| \geqslant B$.

Proof. The theorem may be proved by a slight modification of the proof of Theorem 3.7 in Hörmander [4]. Thus we define a Banach space of N times continuously differentiable solutions of (6) in Ω which is large enough to contain all exponential solutions of (6). More precisely, we let U be the set of all solutions of (6) in $C^N(\Omega)$ such that

$$\|u\| = \sup_{x \in \Omega} e^{-|x|^2} \sum_{|\alpha| \leq N} |D_\alpha u(x)| < \infty$$

(if Ω is bounded, the exponential factor could be omitted). With $\|u\|$ as norm, U is obviously complete and hence a Banach space. Next let Ω' be a domain with compact closure in Ω. Let B be the space of functions in $C^{N+1}(\Omega')$ with the norm

$$|||u||| = \sup_{x \in \Omega'} \sum_{|\alpha| \leq N+1} |D_\alpha u(x)| < \infty.$$

Clearly, B is also a Banach space. Now note that by assumption every function in U is in $C^{N+1}(\Omega)$ and since Ω' is relatively compact in Ω, the derivatives of order up to $N+1$ must be bounded in Ω'. Therefore, the restriction of u to Ω' is in B, so that the restriction defines a linear mapping T of U into B, whose domain is the whole of U. Since T is obviously closed, it follows from the theorem on the closed graph (cf. Bourbaki [2]) that T is continuous, hence if $u \in U$ we have

$$|||u||| \leqslant C\|u\|, \tag{7}$$

where C is a constant.

Now let V be the set of all complex vectors ζ such that the rank of $\mathcal{P}(\zeta)$ is $< n$. If $\zeta \in V$, one can find a non trivial solution of (6) of the form

$$u(x) = e^{i\langle x, \zeta\rangle} a,$$

where a is a constant vector. Indeed, this function satisfies (6) if (and only if)

$$\mathcal{P}(\zeta) a = 0,$$

and this system of equations has a solution $a \neq 0$ since the rank of $\mathcal{P}(\zeta)$ is $<n$. Introducing this function in (7), we obtain after calculating the two sides

$$(1 + |\zeta_1| + \cdots + |\zeta_\nu|) \sup_{\Omega'} e^{-\langle x, \operatorname{Im}\zeta\rangle} \leqq C \sup_{\Omega} e^{-|x|^2 - \langle x, \operatorname{Im}\zeta\rangle}, \zeta \in V.$$

Assuming as we may that $0 \in \Omega'$, the supremum on the left is $\geqslant 1$. Since

$$-|x|^2 - \langle x, \operatorname{Im}\zeta\rangle \leqslant -|x|^2 + |x||\operatorname{Im}\zeta| \leqslant |\operatorname{Im}\zeta|^2/4,$$

we obtain

$$|\zeta_1| + \cdots + |\zeta_\nu| \leqslant C e^{|\operatorname{Im}\zeta|^2/4}, \zeta \in V.$$

This proves Theorem 1.

The algebraic condition of Theorem 1 may also be formulated as follows: The distance from the real point ξ to V tends to infinity with ξ. (If V is empty, we interpret the distance as ∞.) That this condition is also sufficient for hypoellipticity is proved by the following theorem.

Theorem 2. *Assume that the distance from the real point ξ to the set V where the rank of $\mathcal{P}(\zeta)$ is $<n$ tends to infinity with ξ. Then, if the distribution u satisfies the equation* (5) *in an open set Ω where $f \in C^\infty(\Omega)$, it follows that $u \in C^\infty(\Omega)$.*

Proof. Let $Q(\xi)$ be a determinant formed with n rows in $\mathcal{P}(\xi)$. (The assumption means in particular that there are at least n rows.) Then we have

$$Q(D) u_j \in C^\infty(\Omega), \quad 1 \leqslant j \leqslant n.$$

Indeed, assuming for simplicity in notations that Q is formed with the first n rows in $\mathcal{P}$ and denoting by $C_{jk}(\xi)$ the algebraic complement of the element $P_{jk}(\xi)$ in $Q(\xi)$, we obtain

$$Q(D) u_j = \sum_{i=1}^{n} \sum_{k=1}^{n} (C_{kj}(D) P_{ki}(D)) u_i = \sum_{k=1}^{n} C_{kj}(D) f_k \in C^\infty(\Omega).$$

Let I be the ideal generated by the determinants $Q(\xi)$ formed with n rows in $\mathcal{P}(\xi)$, that is, I is the set of all finite sums of such determinants multiplied with polynomials of ξ. It is obvious that

$$R(D)u_j \in C^\infty(\Omega), \quad R \in I. \tag{8}$$

(All differentiations are to be understood in the sense of the theory of distributions, so that they are certainly possible to perform.)

Now V is precisely the set where all polynomials in the ideal I vanish, for the rank of $\mathcal{P}(\zeta)$ is $< n$ if and only if all determinants formed with n rows in $\mathcal{P}(\zeta)$ vanish. We can thus apply the theorem of Lech [5] stated in the introduction. If we choose a polynomial $R \in I$ so that (4) holds, it follows that $d(\xi, V_R) \to \infty$. As recalled in the introduction, this means that the differential operator $R(D)$ is hypoelliptic, hence it follows from (8) that $u_j \in C^\infty(\Omega)$. This completes the proof.

Combining Theorems 1 and 2 we have proved:

Corollary. *A necessary and sufficient condition for the system* (5) *of differential equations with constant coefficients to be hypoelliptic is that the distance from the real point* ξ *to the set where the rank of* $\mathcal{P}(\zeta)$ *is* $< n$ *tends to infinity with* ξ.

Remark. If V is empty, the proof of Theorem 2 given above is not formally correct but should be read as follows. The theorem of Lech degenerates into Hilbert's Nullstellensatz, that is, the polynomial $R = 1$ is in I. The rest of the proof is as before and even shows that one can find polynomials R_{jk} so that $\mathcal{P}(D)u = f$ implies $u_j = \sum_1^m R_{jk}(D) f_k$. Thus there is no solution $\neq 0$ of the homogeneous system (6).

Estimates of high order derivatives of solutions of hypoelliptic systems of differential equations

We shall here extend Theorem 3.8 of Hörmander [4] to arbitrary systems and also show that it is the best possible result. We start by introducing some notations similar to those employed in [4]. $\mathcal{P}$ *will always denote a hypoelliptic system of differential operators.*

As before, we denote by V the algebraic set where the rank of $\mathcal{P}(\zeta)$ is $< n$. Since we shall study the solutions of (6) only, *we may assume that* V *is not empty,* for the only solution of (6) is otherwise $u = 0$ (cf. the remark above). Let y be a fixed real vector $\neq 0$ and set

$$M(\tau) = \inf |\zeta - \xi| \tag{9}$$

with the infimum taken over all $\zeta \in V$ and real ξ satisfying $|\langle y, \xi \rangle| = \tau$. Then there is a number $b > 0$ such that

$$M(\tau) = a\tau^b (1 + o(1)) \text{ when } \tau \to \infty, \tag{10}$$

where $a \neq 0$. To prove this we first note that if the matrix $\mathcal{P}(\zeta)$ is scalar, that is, $m = n = 1$, the definition (9) of $M(\tau)$ is identical to (3.4.9) in Hörmander [4], so that our assertion is precisely that of Lemma 3.9 in the quoted paper. In the general case the result follows if we note as in the proof of Theorem 2 that V consists of the zeros of all polynomials in the ideal I. If $R_1, \ldots, R_k$ is a basis for I, for instance all determinants formed with n rows of $\mathcal{P}$, the set V is thus defined by the equations

$$R_1(\zeta) = 0, \ldots, R_k(\zeta) = 0. \tag{11}$$

But then one can repeat the proof of Lemma 3.9 in Hörmander [4] with the only modification that in formula (3.4.10) the equation $P(\zeta) = 0$ has to be replaced by the equations (11).

The inverse of the number b in (10) will be denoted by ϱ, sometimes also by $\varrho(y)$ or $\varrho_{\mathcal{P}}(y)$ when we want to emphasize the dependance on P and y.

Theorem 3. *Let Ω be an open set and K a compact subset of Ω. Then, if u is a solution of the hypoelliptic system* (6) *in Ω, there is a constant C so that*

$$|\langle y, D\rangle^{\mu} u(x)| \leqslant C^{\mu+1}\Gamma(\varrho_{\mathcal{P}}(y)\mu), \quad x \in K, \quad \mu = 0, 1, \ldots, \tag{12}$$

where Γ is the gamma function.

Proof. The theorem is a trivial consequence of the theorem of Lech [5] stated in the introduction and Theorem 3.8 in Hörmander [4]. Indeed, (9) can be written

$$M(\tau) = \inf d(\xi, V_I), \qquad |\langle y, \xi\rangle| = \tau, \tag{9$'$}$$

and if we choose $R \in I$ so that

$$1 \leqslant d(\xi, V_I)/d(\xi, V_R) \leqslant C,$$

it follows that $\varrho_R(y) = \varrho_{\mathcal{P}}(y)$, with a natural notation. Now each component u_j of u satisfies the differential equation $R(D)u_j = 0$ (cf. (8)). Hence in virtue of Theorem 3.8 in Hörmander [4] we have with suitable constants C_j

$$|\langle y, D\rangle^{\mu} u_j(x)| \leqslant C_j^{\mu+1}\Gamma(\varrho_R(y)\mu), \quad x \in K, \ \mu = 0, 1, \ldots$$

Since $\varrho_R = \varrho_{\mathcal{P}}$, inequality (12) now follows with $C = C_1 + \cdots + C_n$.

Remarks. 1. The validity of this theorem is not restricted to hypoelliptic systems only but can easily be extended with slight modifications to any case where the number $\varrho_{\mathcal{P}}(y)$ exists. The present case seems, however, to be the most interesting one.

2. A method given by Schwartz [6] can be applied to give results of the same kind for the solutions of the inhomogeneous system (4). Thus, if we have an estimate similar to (12) for the derivatives of f, we still get the same result for u. However, we shall not repeat the argument of Schwartz here.

Our next task is to show that the estimate (12) cannot be improved.

Theorem 4. *Let Ω be a bounded open set and x_0 a point in Ω. If M_μ is a sequence such that every solution of* (6) *satisfies the inequality*

$$|\langle y, D\rangle^{\mu} u(x_0)| \leqslant K^{\mu+1} M_\mu, \ \mu = 0, 1, \ldots, \tag{13}$$

for some constant K, then there is a constant c such that

$$\Gamma(\varrho_{\mathcal{P}}(y)\mu) \leqslant c^{\mu} M_\mu. \tag{14}$$

Thus (13) is a consequence of (12) if we disregard the size of the constants C and K.

The proof requires the following lemma, which gives a definition of $\varrho(y)$ where the real vector ξ in (9) has been eliminated and therefore is easier to handle.

Lemma 1. *Let*
$$M'(\tau) = \inf |\operatorname{Im}\zeta| \tag{15}$$
with the infimum taken over all $\zeta \in V$ *such that* $|\langle y, \operatorname{Re}\zeta\rangle| = \tau$. *Then we have, if* $b = 1/\varrho(y)$, *with a positive* a'
$$M'(\tau)\tau^{-b} \to a' \quad \text{when } \tau \to \infty. \tag{16}$$

Proof. Arguing as in the proof of Lemma 3.9 in Hörmander [4], with the modifications indicated above, it is easy to show that there is a number b' such that
$$M'(\tau)\,\tau^{-b'} \to a' \neq 0.$$
We do not repeat this argument but shall prove that $b = b'$.

First note that we always have $M(\tau) \leqslant M'(\tau)$, if $M(\tau)$ is defined by (9). For if $\zeta \in V$ satisfies the equation $|\langle y, \operatorname{Re}\zeta\rangle| = \tau$ as required in (15), we can set $\xi = \operatorname{Re}\zeta$ in (9). Hence we have $b \leqslant b'$. Since we always have $0 < b \leqslant 1$, $0 < b' \leqslant 1$, it thus follows that $b' = 1$ if $b = 1$, so that the lemma is true in this case. Now assume that $b < 1$, hence $M(\tau) = o(\tau)$. The definition (9) of $M(\tau)$ shows that when $\tau > 0$ there are vectors ζ and ξ such that $\zeta \in V$ and
$$|\langle y, \xi\rangle| = \tau, \qquad |\xi - \operatorname{Re}\zeta| \leqq M(\tau) + 1, \qquad |\operatorname{Im}\zeta| \leqslant M(\tau) + 1.$$
If we set $\tau' = |\langle y, \operatorname{Re}\zeta\rangle|$, it thus follows that $M'(\tau') \leqslant M(\tau) + 1$ and $|\tau' - \tau| \leqslant |y|(M(\tau) + 1)$. When $\tau \to \infty$ we get
$$\tau'/\tau \to 1, \quad M'(\tau')\,\tau'^{-b'} \to a' \text{ when } \tau \to \infty,$$
hence
$$M'(\tau')\,\tau^{-b'} \to a' \neq 0.$$
Since $M'(\tau') \leqslant M(\tau) + 1$ and $M(\tau)\tau^{-b} \to a$, we can conclude that $b' \leqslant b$. This completes the proof of the lemma.

Proof of Theorem 4. The proof is parallel to that of Theorem 1. Thus we start with proving by means of functional analysis that certain inequalities must be fulfilled, and then we apply these inequalities to the exponential solutions.

Let U be the set of all solutions of (6) in Ω such that
$$\|u\| = \sup_{x \in \Omega} |u(x)| < \infty.$$
It is clear that U is a Banach space with this norm. Denote by F_r the set of those $u \in U$ such that
$$|\langle y, D\rangle^j u(x_0)| \leqslant r^{j+1} M_j, \quad j = 0, 1, 2, \ldots \tag{17}$$
The sets F_r are closed. For if we have a sequence of elements u_n in F_r such that $\|u_n - u\| \to 0$, it follows that u_n and all its derivatives are uniformly convergent on

every compact subset K of Ω.[1] Indeed, applying the theorem of the closed graph as in the proof of Theorem 1, using the hypoellipticity of $\mathcal{D}$, we can for arbitrary α prove an inequality

$$\sup_{x \in K} |D_\alpha u(x)| \leqslant C(\alpha, K) \|u\|, \quad u \in U.$$

But then it follows that (17) must be valid for the limit u since it holds for each u_n.

By assumption we have $\bigcup_1^\infty F_r = U$. Since all the sets F_r are closed, it follows from the category theorem (Bourbaki [1]) that some of them must contain an interior point. Let F_s have an interior point. Since F_s is convex and symmetric, the origin 0 must be an interior point. (This is also part of the argument in the proof of the theorem on the closed graph.) Hence there is a number $\delta > 0$ such that F_s contains all u with $\|u\| \leqslant \delta$. But then we have for all $u \in U$

$$|\langle y, D\rangle^j u(x_0)| \leqslant s^{j+1} M_j \|u\|/\delta, \tag{18}$$

for this inequality is homogeneous with respect to u and holds when $\|u\| = \delta$.

If $\zeta \in V$ we can, as in the proof of Theorem 1, find an exponential solution $u = e^{i\langle x, \zeta\rangle} a$ of (6), where a is a constant vector $\neq 0$. Then (18) yields

$$|\langle y, \zeta\rangle|^j e^{-\langle x_0, \operatorname{Im}\zeta\rangle} \leqslant s^{j+1} M_j \delta^{-1} \sup_{x \in \Omega} e^{-\langle x, \operatorname{Im}\zeta\rangle}. \tag{19}$$

If A is the diameter of Ω, (19) gives the following estimate of M_j

$$M_j \geqslant s^{-j-1} \delta |\langle y, \zeta\rangle|^j e^{-A|\operatorname{Im}\zeta|}, \quad \zeta \in V. \tag{20}$$

Replacing here $|\langle y, \zeta\rangle|$ by $|\langle y, \operatorname{Re}\zeta\rangle| = \tau$ and using the definition of $M'(\tau)$, we get

$$M_j \geqslant s^{-j-1} \delta \tau^j e^{-AM'(\tau)}. \tag{21}$$

According to Lemma 1 we can find to any number $\alpha > a'$ a number τ_0 so that $M'(\tau) \leqslant \alpha\tau^b$ when $\tau > \tau_0$. Hence

$$M_j \geqslant s^{-j-1} \delta \tau^j e^{-\alpha A \tau^b}, \quad \tau > \tau_0. \tag{22}$$

The right hand side of this inequality has a maximum when $\tau^b = j/\alpha A b$. This value of τ is $> \tau_0$ provided that $j > j_0$. Thus

$$M_j \geqslant c^j j^{j/b}, \quad j > j_0, \tag{23}$$

and taking if necessary a smaller value of c we can make (23) hold also for $j = 1, \ldots, j_0$. But Stirling's formula shows that this is equivalent to (14). The proof is complete.

Remark. If Ω is not bounded, it is still easy to prove that (14) must be valid if $\varrho_{\mathcal{D}}(y)$ is replaced by any larger number. This can be done by modifying the norm in

[1] In fact, it is true that convergence in the topology of distributions of solutions u_n of a hypoelliptic homogeneous system of differential equations implies uniform convergence of $D_\alpha u_n$ on every compact set for all α. This follows easily from formula (3.5.3) in Hörmander [4].

U as in the proof of Theorem 1 with an exponential factor $e^{-|x|^c}$, where c can be chosen arbitrarily large.

We shall finally give some remarks on $\varrho_P(y) = \varrho(y)$ as a function of y. We first prove

$$\varrho(t_1 y_1 + t_2 y_2) \leqslant \max(\varrho(y_1), \varrho(y_2)). \tag{24}$$

Since we always have $\varrho(y) \geqslant 1$, and we define $\varrho(0) = 1$, it is sufficient to prove (24) when the three vectors y_1, y_2 and $t_1 y_1 + t_2 y_2$ are all different from 0. Write $\varrho_i = \varrho(y_i)$ and $b_i = 1/\varrho_i$.

The definition of ϱ_i means that there is a constant A_i such that

$$|\langle y_i, \mathrm{Re}\,\zeta\rangle| \leqslant A_i(1 + |\mathrm{Im}\,\zeta|)^{\varrho_i}, \; \zeta \in V. \tag{25}$$

For writing $|\langle y_i, \mathrm{Re}\,\zeta\rangle| = \tau$, we have $|\mathrm{Im}\,\zeta| \geqslant M_i'(\tau) = a_i' \tau^{b_i}(1 + 0(1))$ with a modification of the notations of Lemma 1. Furthermore, (25) *does not hold if the exponent ϱ_i is replaced by a smaller number*. Now we get, if $\varrho = \max(\varrho_1, \varrho_2)$

$$|\langle t_1 y_1 + t_2 y_2, \mathrm{Re}\,\zeta\rangle| \leqslant \left(\sum_1^2 A_i |t_i|\right)(1 + |\mathrm{Im}\,\zeta|)^{\varrho}, \; \zeta \in V,$$

and since $\varrho(t_1 y_1 + t_2 y_2)$ is the smallest exponent on the right for which such an inequality can hold, (24) follows.

Our result may also be formulated as follows: The set

$$G_\sigma = \{y; \varrho(y) < \sigma\} \tag{26}$$

is a linear subspace of R^ν for every fixed σ. If we denote its dimension by $\nu(\sigma)$, the function $\nu(\sigma)$ is increasing and only assumes integer values between 0 and ν. Hence there are only a finite number of points $\sigma_1 < \sigma_2 < \cdots < \sigma_k$, where the dimension of G_σ increases. We have

$$\{0\} \subset G_{\sigma_1} \subset G_{\sigma_2} \subset \cdots \subset G_{\sigma_k} = R^\nu,$$

and $\varrho(y) = \sigma_j$ if G_{σ_j} is the first of these spaces containing y. Changing the notation slightly, we have thus proved the following result.

Theorem 5. *There exists a strictly increasing family of subspaces of R^ν,*

$$\{0\} = G_0 \subset G_1 \subset \cdots \subset G_k = R^\nu,$$

and a strictly increasing sequence of rational numbers $\sigma_1, \sigma_2, \ldots, \sigma_k$ all $\geqslant 1$ so that $\varrho(y) = \sigma_j$ if j is the smallest integer such that $y \in G_j$.

Theorem 5 generalizes the classical results for the equation of heat (cf. Gevrey [3]). It is very easy to get estimates also for the mixed derivatives such as are known for the equation of heat, but we are not going to do so here.

A system (5) of differential equations is called *elliptic* if all solutions are analytic when f is analytic. As a special case of our previous results we note that the condition for ellipticity is that $\varrho_P(y) = 1$ for all y. In this case the ideal I contains a polynomial R with the same property, $\varrho_R(y) = 1$; such a polynomial is elliptic in the usual sense that the principal part does not vanish for any real $\xi \neq 0$.

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