ARKIV FÖR MATEMATIK Band 3 nr 45

Communicated 9 April 1958 by OTTO FROSTMAN and LENNART CARLESON

A generalization of Picard's theorem[1]

By OLLI LEHTO

1. Introduction

1. Let $f(z)$ be meromorphic outside a closed point set E in the complex plane, and let $f(z)$ possess at least one singularity in E. If $f(z)$ cannot omit more than two values in the complement of E, we call E a *Picard set*.

By Picard's theorem, sets with only a finite number of points are Picard sets. In this paper, we shall generalize this result and show that all sufficiently thin countable sets with one limit point are also Picard sets.

Let E: $a_1, a_2, \ldots$ be a point set whose points converge to infinity. If a function $f(z)$, meromorphic outside E, is singular at some point a_ν, then of course $f(z)$ cannot omit more than two values. Hence, on studying whether E is a Picard set or not, we may restrict ourselves to functions $f(z)$ possessing their only singularity at infinity. In other words, we consider functions $f(z)$ meromorphic for $z \neq \infty$.

2. If $f(z)$ omits two values w_1 and w_2 in the whole finite plane, it is clear that $f(z)$ takes all other values outside E if the points of E tend to infinity with sufficient rapidity. For $f(z)$ is then at least of order 1, and this implies an upper bound for the velocity with which for any $w \neq w_1, w_2$, the w-points converge towards infinity.

If however, $f(z)$ omits only one value or none at all for $z \neq \infty$, no similar conclusions can be drawn. For it is possible to construct entire or meromorphic functions for which all w-points tend to infinity as rapidly as we please. Removing from the plane the w-points for three different values w (for two values for entire functions), we obtain sets E which are certainly not Picard sets.

The following example shows that in such a case, even the distance of any two points of E can be made arbitrarily large. Put

$$f(z) = \prod_{\nu=1}^{\infty} \frac{1 - z/b_\nu}{1 + z/b_\nu}, \tag{1}$$

where $b_\nu > 0$, $b_1 < b_2 < \ldots$, and b_ν tends rapidly to infinity. Clearly, $f(z)$ has its zeros at $z = b_\nu$, poles at $z = -b_\nu$, and 1-points on the imaginary axis. If the 1-points are denoted by $\pm i c_\nu$, $c_0 = 0, c_\nu > 0, \nu = 1, 2, \ldots$, we have the identity

[1] This research was done at the University of Uppsala under a grant from the Mittag-Leffler Foundation. The author takes pleasure in thanking Professor Lennart Carleson for helpful suggestions.

$$\prod_{\nu=1}^{\infty}(1-i\,c_k/b_\nu)=\prod_{\nu=1}^{\infty}(1+i\,c_k/b_\nu),\quad k=0,1,2,\ldots$$

This can also be written in the form

$$\sum_{\nu=1}^{\infty}\overline{\text{arc}}\ \text{tg}\ (c_k/b_\nu)=n\pi,$$

where n is a positive integer. If the points b_ν tend to infinity with sufficient rapidity, this implies that, except for c_0, there is just one c_k in every interval (b_{2k}, b_{2k+1}). Hence, $c_k - c_{k-1} > b_{k+1} - b_k$. Puncturing the plane at the zeros, 1-points and poles of the function (1), we thus obtain sets E the points of which tend to infinity as rapidly as we please and in which the distance of any two points can be made arbitrarily large. In spite of this, these sets are not Picard sets.

3. Hence, neither the rapidity of convergence of the points of E towards infinity nor large distances between individual points of E are sufficient to ensure E to be a Picard set. There is, however, still another way to make the set E thin, namely, to impose the condition that for large values of ν, $|a_{\nu+1}|$ must be much larger than $|a_\nu|$. If this requirement is strong enough, we arrive at sets which are always Picard sets.

With this result, which will be established below, the study on Picard sets is by no means completed. The given density condition, guaranteeing E to be always a Picard set, is scarcely necessary. Moreover, besides these thin Picard sets, there may also exist much denser Picard sets. Even the existence of arbitrarily dense Picard sets is not unlikely.

II. Existence of Picard sets

4. The result to be established is as follows:

Theorem 1. *Let $f(z)$ be meromorphic outside a set E: $a_1, a_2, \ldots, a_\nu \to \infty$, and possess at least one singularity in E. If the points of E satisfy the condition*

$$(\log |a_\nu|)^{2+\delta} = O\ (\log |a_{\nu+1}|),\quad \delta > 0, \tag{2}$$

then E is a Picard set, i.e. $f(z)$ can omit at most two values outside E.

Proof. It is obviously sufficient if we can prove that the assumption of the existence of a function $f(z)$, meromorphic and non-rational for $z \neq \infty$ and different from $0, 1, \infty$ outside E, leads to a contradiction.

We introduce the standard notations: $T(r)$ is the characteristic function of $f(z)$, $N(r,a)$ the counting function, and $\bar{N}(r,a)$ the counting function which counts all a-points only once, irrespective of multiplicity.

Nevanlinna's second main theorem, applied to $f(z)$ for the values $w=0, 1, \infty$, yields

$$T(r) < \bar{N}(r,0) + \bar{N}(r,1) + \bar{N}(r,\infty) + O\ (\log r).$$

Hence, considering the density condition (2), it follows by an easy computation that for any $\eta > 0$,

$$T(r) = o\,(\log^{1+\eta} r). \tag{3}$$

Let b_i, c_j, d_k denote the zeros, 1-points and poles, respectively, of $f(z)$. By (3), at least two of the sequences b_i, c_j, d_k contain an infinite number of points. In view of the transformations $1/f$ and $1-f$, permuting the zeros, 1-points and poles, there is no restriction to suppose that the number of zeros and 1-points is infinite.

From (3) it also follows that $f(z)$ admits a representation

$$f(z) = f(0)\frac{\Pi(1-z/b_i)^{\lambda_i}}{\Pi(1-z/d_k)^{\mu_k}} \quad (f(0)\neq 0, \infty),$$

where λ_i and μ_k denote the multiplicity of the zero or pole in question. The product over the poles may be finite or even reduce to the constant 1.

Setting $z=c_j$, we get the relations

$$f(0)\prod_i(1-c_j/b_i)^{\lambda_i} = \prod_k(1-c_j/d_k)^{\mu_k}, \quad j=1,2,\ldots \tag{4}$$

between the zeros, 1-points and poles.

5. We shall now prove that, under the condition (2), the equations (4) cannot be true for all values of j. To begin with, we point out that the numbers λ_i and μ_k cannot be very large.

In fact, we conclude from (3) that

$$N(r,0) = o(\log^{1+\eta} r).$$

Hence,

$$N(2|b_i|, 0) = o(\log^{1+\eta}|b_i|),$$

while on the other hand,

$$N(2|b_i|,0) = \sum_{|b_h|<2|b_i|}\lambda_h \log(2|b_i/b_h|) \geq \lambda_i \log 2.$$

Consequently,

$$\lambda_i = o(\log^{1+\eta}|b_i|), \tag{5}$$

and similarly,

$$\mu_k = o(\log^{1+\eta}|d_k|). \tag{5'}$$

6. Reverting to the relation (4), we start from the identity

$$\prod_i(1-c_j/b_i)^{\lambda_i} = \prod_{|b_i|<|c_j|}(c_j/b_i)^{\lambda_i}\prod_{|b_i|<|c_j|}(b_i/c_j-1)^{\lambda_i}\prod_{|b_i|>|c_j|}(1-c_j/b_i)^{\lambda_i} \tag{6}$$

and show that for $j\to\infty$, the moduli of the two last products tend to 1.

For the first of these products we get, considering (5),

$$|\log\prod_{|b_i|<|c_j|}|b_i/c_j-1|^{\lambda_i}| = O\Big(\sum_{|b_i|<|c_j|}\lambda_i|b_i/c_j|\Big) = o\Big(|c_j|^{-1}\log^{1+\eta}|c_j|\sum_{|b_i|<|c_j|}|b_i|\Big).$$

If b_N is the zero with largest modulus less than $|c_j|$, it follows immediately from (2) that

$$\sum_{i=1}^{N}|b_i| = O(|b_N|).$$

Hence,
$$\log \prod_{|b_i|<|c_j|} |b_i/c_j - 1|^{\lambda_i} = o(|b_N/c_j| \log^{1+\eta}|c_j|),$$

and by (2), the left-hand side tends to zero as $j \to \infty$.

In a similar manner we obtain for the last product in (6),

$$|\log \prod_{|b_i|>|c_j|} |1 - c_j/b_i|^{\lambda_i}| = O\Big(\sum_{|b_i|>|c_j|} \lambda_i |c_j/b_i|\Big) = o(|c_j/b_{N+1}| \log^{1+\eta}|b_{N+1}|) = o(1).$$

Hence, it follows that the relation (4) can be written in the form

$$c_j^{\Sigma\mu_k - \Sigma\lambda_i} \prod_{|b_i|<|c_j|} b_i^{\lambda_i} = A_j \prod_{|d_k|<|c_j|} d_k^{\mu_k}, \tag{7}$$

where A_j is bounded with respect to j.

It is the handling of the equation (7) that has determined the density condition (2). By this condition, $|c_j|$ is so much larger than $\Pi |d_k|^{\mu_k}$ and $\Pi |b_i|^{\lambda_i}$ for large values of j that a contradiction follows immediately, unless

$$\Sigma \lambda_i = \Sigma \mu_k.$$

If this condition is satisfied,

$$A_j \Pi d_k^{\mu_k} = \Pi b_i^{\lambda_i}. \tag{8}$$

But again, as $j \to \infty$ the term corresponding to the largest of the numbers $|b_i|$, $|d_k|$ becomes so large compared with all other terms that (8) cannot remain true. A contradiction has thus been found, and the proof is completed.

7. The condition (2), guaranteeing E to be a Picard set, yields quite thin sets. On the other hand, E must not be too dense in order to always be a Picard set, as we shall now show by means of an example.

We shall construct a set E which is not a Picard set, although

$$|a_{\nu+1}/a_\nu| \geq q > 1 \tag{9}$$

from a certain ν on.

Let $F(w)$ be a double-periodic function with periods $\omega > 0$ and $2\pi i$. Let further $F(w)$ be even and take every value twice in its periodicity rectangle.

The function $F(\log z)$ is then single-valued and meromorphic for $z \neq 0, \infty$. Obviously, it admits a representation

$$F(\log z) = f(z) + g(z),$$

where $f(z)$ and $g(z)$ are meromorphic for $z \neq \infty$ and $z \neq 0$, respectively, and $g(\infty) = 0$.

Let now $-\omega/2 < \alpha_1 < \alpha_2 < \alpha_3 < \omega/2$ such that $F(\alpha_i) = w_i$, $i = 1, 2, 3$, are different from each other. The function $F(\log z)$ takes the three values w_i at the points $z = e^{\pm\alpha_i \pm \nu\omega}$, $\nu = 0, 1, 2, \ldots$ Let $\zeta = e^{\pm\alpha_i + \nu\omega}$ denote an arbitrary w_i-point of $F(\log z)$ in the vicinity of $z = \infty$. From $|F(\log z) - f(z)| = O(1/|z|)$ it follows that there is just one w_i-point of $f(z)$ in the disc $|z - \zeta| < \varrho$, where $\varrho = O(1/|\zeta|)$. Moreover, outside such discs, $f(z) \neq w_i$. Hence, the w_i-points of $f(z)$ satisfy (9), thus constituting a desired example.

III. Entire functions

8. Let $f(z)$ be a non-rational entire function for $z \neq \infty$. We call E a *Picard set for entire functions* if $f(z)$ cannot omit more than one finite value outside E. Of course, every Picard set in the above more general sense is a Picard set for entire functions, while the converse need not always be true.

In the special case of entire functions, the density condition (2) can be considerably relaxed:

Theorem 2. *A set E: $a_1, a_2, \ldots, a_\nu \to \infty$, is a Picard set for entire functions if*

$$|a_\nu/a_{\nu+1}| = O(\nu^{-2}). \tag{10}$$

Proof. Let $f(z)$ be non-rational and entire for $z \neq \infty$, and let $f(z) \neq 0,1$ outside E. Adopting the same notations as above, we conclude first that

$$T(r) = o(\log^2 r).$$

Hence, $f(z)$ must possess an infinite number of both zeros and 1-points and

$$\lambda_i = o(\log^2 |b_i|).$$

We write as above,

$$\prod_i (1 - c_j/b_i)^{\lambda_i} = \prod_{|b_i|<|c_j|} (c_j/b_i)^{\lambda_i} \prod_{|b_i|<|c_j|} (b_i/c_j - 1)^{\lambda_i} \prod_{|b_i|>|c_j|} (1 - c_j/b_i)^{\lambda_i}. \tag{11}$$

For the last product we obtain

$$|\log \prod_i |1 - c_j/b_i|^{\lambda_i}| = O\left(|c_j| \sum_{|b_i|>|c_j|} \frac{\log^2 |b_i|}{|b_i|}\right)$$

and by the condition (10), this is $O(\log |c_j|)$. The same holds for the second product in (11), and it follows that the relation corresponding to (7) now becomes

$$\log \prod_{|b_i|<|c_j|} |c_j/b_i|^{\lambda_i} = O(\log |c_j|).$$

This, however, contains a contradiction. For the left-hand side is the counting function $N(|c_j|, 0)$, and since $f(z)$ cannot have any finite values with positive deficiency, the relation $N(|c_j|, 0) = O(\log |c_j|)$ is impossible.

9. If the points of E lie on a ray, the condition (10) can still be weakened. In this case, the condition

$$|a_{\nu+1}/a_\nu| \geq q > 1 \tag{12}$$

already ensures E to be a Picard set.[1]

For if $f(z)$ is entire and $\neq 0,1$ outside a set E lying on a ray Γ, it follows from the Weierstrassian product representation that $f(z)$ is bounded on certain segments $(a_{\nu_i}, a_{\nu_{i+1}})$ of Γ clustering to infinity. Since $f(z)$ omits three values outside E, an upper bound is obtained for the spherical derivative of $f(z)$ in terms of the hyperbolic metric $d\sigma$ of the complement of E:

[1] This is a joint result with Dr. K. I. Virtanen.

$$\frac{|f'(z)|}{1+|f(z)|^2}=O\left(\frac{d\sigma}{|dz|}\right).$$

By the condition (12), this implies that on all rays sufficiently near Γ, the lower limit at infinity of $f(z)$ is finite. This, however, contradicts the fact that $f(z)$ must tend to infinity on every ray except for Γ.

The sufficient condition (12) cannot be very far from a necessary condition. In fact, the entire function $\frac{1}{2}(1+\cos\sqrt{z})$ takes the values 0 and 1 at the points $z=\nu^2\pi^2$, $\nu=0, 1, 2, \ldots$ Hence,

$$|a_{\nu+1}/a_\nu|>1+\frac{2}{\nu}$$

does not imply E to be a Picard set.

Let it be recalled that, as we showed above, (12) is not a sufficient condition in the general case of meromorphic functions.

10. Certain well-known modifications of Picard's theorem concerning non-zero entire functions admit obvious generalizations for functions with zeros in a Picard set. For instance, we obtain immediately.

Theorem 3. *If f_1 and f_2 are entire functions with zeros in a Picard set for entire functions, and if identically $f_1+f_2=1$, then f_1 and f_2 are rational functions.*

If $f_1\, f_2, f_3$ are entire functions with zeros lying in a Picard set for meromorphic functions, and if identically

$$c_1 f_1 + c_2 f_2 + c_3 f_3 = 0$$

with non-zero coefficients, then the quotient of any two of these functions is a rational function.

University of Helsinki.

Tryckt den 19 september 1958

Uppsala 1958. Almqvist & Wiksells Boktryckeri AB