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The set of extreme points of a compact convex set

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0.1. Introduction

The purpose of this note is to establish necessary and sufficient conditions for a set S in a Fréchet space X to be the set of extreme points of some compact convex set.[1]

We shall first study the case where X is a Euclidean space R^n. Suppose that S is the set of extreme points of some compact convex set K. As is well known, S need not be closed. But it is also well known that K must be the convex hull of S and, consequently,

the closure of S is contained in the convex hull of S. (b)

Further obvious necessary properties of S are that

the closure of S is compact (a)

and that

no point of S is a barycenter of a finite set of other points of S. (c′)

The following is obviously an equivalent formulation of (c′): For all $A \subset S$, no point of S is in the convex hull of A without being in A, or, with obvious notation,

$$S \cap H(A) \subset A \quad \text{for all } A \subset S. \tag{c}$$

Thus, in a Euclidean space, conditions (a, b, c) are necessary. In Theorem 2.1 they are proved to be sufficient. We shall also prove that some apparently stronger conditions are necessary.

When X is a general Fréchet space (Theorem 3.1), we have to make some changes in conditions (b) and (c), for it develops that (b) is not necessary and (a, b, c) are not sufficient. However, if the convex hull occurring in (b) and (c) is replaced by a certain larger "hull", the resulting conditions will be necessary and sufficient. This new hull H_C of a set S was considered by Choquet [4, 5, 6]. It is defined as the set of barycentra of those positive Radon measures μ of total mass one on $\bar{S}$ which are contained in S in the sense that $(\bar{S}-S)$ has μ-measure zero. Choquet [5 or 6] proved that each compact convex set K in a Fréchet space is the H_C-hull of its set of extreme points. The Krein-

[1] For terminology, see [1] and [2].

Milman theorem only tells us that every point of K is the barycenter of a measure on $\bar{S}$. Choquet's result means that in a certain sense there need not be any masses outside of S. At first it might seem more natural to require the masses to be contained in S in the stronger sense that each x in K is the barycenter of a measure with its support (compact by definition) contained in S. However, it follows from examples given by Choquet (see Lemma 3.2), that this is not possible. Accordingly we shall find that the corresponding "hull operation" (the H_{st} of section 0.2) cannot be introduced in conditions (b) and (c).

0.2. Notation and definitions

Let X be a linear space. Let $S \subset X$ and let $K \subset X$ be convex. We shall consider the following sets:

$$H_m(S) = \{x \in X \mid x = \sum_{i=1}^{m} \lambda_i x_i,\ x_i \in S,\ \lambda_i \geq 0,\ \sum_{i=1}^{m} \lambda_i = 1\};$$

$$H(S) = \text{convex hull of } S = \bigcup_{m=1}^{\infty} H_m(S);$$

and

$$E(K) = \text{the set of extreme points of } K$$
$$= \{x \in K \mid x \in H_2(A) \Rightarrow x \in A \text{ for all } A \subset K\}.$$

When X is a topological linear space, we shall also consider:

$\bar{S}$ = closure of S;

$\bar{H}(S) = \overline{H(\bar{S})}$ = closed convex hull of S;

$H_C(S) = \{x \in X \mid \exists$ Radon measure μ on $\bar{S}$ with $x = \int_{\bar{S}} t\, d\mu(t),\ \mu \geq 0,\ \mu(\bar{S}) = 1,\ \mu(\bar{S} - S) = 0\}$;

and

$H_{st}(S)$, defined as $H_C(S)$ but with "$\mu(\bar{S} - S) = 0$" replaced by "support $(\mu) \subset S$". (The support is the smallest compact set outside which μ vanishes.)

Evidently,

$$S = H_1(S) \subset H_2(S) \subset \ldots \subset H(S) \subset H_{st}(S) \subset H_C(S) \subset \bar{H}(S).$$

If x and μ are related as in the definition of $H_C(S)$, we shall write $x = \int_{(S)} t\, d\mu$.

1. Non-topological preliminaries

In this section, K (convex) and S are sets in a linear space X.

We first formulate without proof two remarks which are simple consequences of the definition of an extreme point:

1.1. Remark. If $E(K_1) \subset K_2 \subset K_1$, then $E(K_1) \subset E(K_2)$.

1.2. Remark. $E(H(S)) \subset S$ for all S.

1.3. Lemma. *If $S = E(K)$ for some convex K, then $S = E(H(S))$.*

Proof: Take $K_1 = K$ and $K_2 = H(S)$ in Remark 1.1. We get $S \subset E(H(S))$, and the lemma follows from Remark 1.2.

1.4. Remark. In contrast with the case in Theorem 2.1, it does not follow from $S = E(K)$ that $K = H(S)$.

1.5. Lemma. *The following three properties of a point x and a set S are equivalent*:

(I) $x \in E(H(S))$.
(II) $x \in H(A)$ *implies* $x \in A$ *for all* $A \subset S$.
(III) $x \in H(B)$ *implies* $x \in B$ *for all* $B \subset H(S)$.

This was observed by Klee [7, Theorem 1]. (That (I) $\Rightarrow$ (III) is very near the definition of an extreme point, (III) $\Rightarrow$ (II) is obvious, and (II) $\Rightarrow$ (I) is proved by straightforward calculation.)

The following lemma states that condition (c) of section 0.1 is necessary and sufficient for S to be the set of extreme points of some convex set.

1.6. Lemma. *The following three properties of a set S are equivalent*:

(1) $S = E(K)$ *for some convex* K.
(2) $S \cap H(A) \subset A$ *for all* $A \subset S$.
(3) $S \cap H(B) \subset B$ *for all* $B \subset H(S)$.

Proof: Obviously, (3) implies (2). Next suppose (2) is true and let $x \in S$. Then x satisfies (II) of Lemma 1.5, and from (I) we get $S \subset E(H(S))$. Remark 1.2 then completes the proof of (1) with $K = H(S)$. Finally, suppose (1) is true and let $x \in S \cap H(B)$ with $B \subset H(S)$. From Lemma 1.3, x satisfies (I) of Lemma 1.5, and so from (III) we see that $x \in B$, which completes the proof of (3) and of Lemma 1.6.

2. Compact convex sets in a Euclidean space

2.1. Theorem. *The following three properties of a set S in R^n $(n > 1)$ are equivalent*:

(1) *There is a compact convex set* $K \subset R^n$, *such that* $S = E(K)$.
(2) (a) $\bar{S}$ *is compact*,
(b) $\bar{S} \subset H(S)$, *and*
(c) $S \cap H(A) \subset A$ *for all* $A \subset S$.
(3) (a) $\bar{S}$ *is compact*,
(b) $\bar{S} \subset H_{n-1}(S)$, *and*
(c) $S \cap H(B) \subset B$ *for all* $B \subset \bar{H}(S)$.

Proof: Evidently (3) $\Rightarrow$ (2), so that we only have to prove that (2) $\Rightarrow$ (1) $\Rightarrow$ (3). First, suppose (2) is true. From (2 a), $H(\bar{S})$ is compact [1, exc. 2]. But from (2 b), $H(\bar{S}) \subset H(S)$, and so $H(\bar{S}) = H(S)$. Further, from (2 c), Lemma 1.6 and Lemma 1.3, we get $S = E(H(S))$. Thus (1) is proved with $K = H(S)$.

Finally, suppose (1) is true. Then $K = \bar{H}(S) = H(S)$, (see e.g. [1, exc. 9]). (3 a) is evident and (3 c) follows from Lemma 1.6. Since $H(S)$ is closed, we get $\bar{S} \subset H(S)$, or by Carathéodory's theorem, $\bar{S} \subset H_{n+1}(S)$. If S is closed, (3 b) is evident. If not, let $x \in \bar{S} - S$, and let L be a supporting plane of $H(S)$ through x. Since $x \in H(S) \cap L = H(S \cap L)$, we get from Carathéodory's theorem that $x \in H_n(S \cap L) \subset \subset H_n(S)$. First, if $S \subset L$, we replace R^n by L, repeat the argument and find $x \in H_{n-1}(S)$. Next, if S contains points outside of L, let a be such a point, and let $L_a \ni a$ and L_b be the open halfspaces produced by L. Let b be any point $\in L_b$. Suppose $x \notin H_{n-1}(S)$. Then x must be an interior point of a simplex T in

L with its vertices in $S \cap L$. Let O be the interior of $H(T \cup \{a\} \cup \{b\})$. Since O is a neighborhood of x, there is in O a point $y \in S \subset L_a \cup L$. Then y is an interior point of $H(T \cup \{a\})$ or of T. This violates condition (3 c) with $B = \{a\} \cup \{\text{vertices of } T\}$, and the proof of (3 b) and of Theorem 2.1 is complete.

2.2. Corollary. From (3 b), we get the well-known result that if K is a compact convex set in R^2, then $E(K)$ is closed.

2.3. Remark. In Theorem 2.1, condition (3 b) cannot be replaced by $\bar{S} \subset H_{n-2}(S)$. Indeed, we shall construct a set S, satisfying e.g. condition (1) but not the proposed condition. (For $n = 3$, the same example is given in [1, exc. 8].) We consider R^n as $R^{n-2} \times R^2$ (with the coordinate spaces imbedded). In R^{n-2} we take a simplex W with the origin o as an interior point. Let V be the set of vertices of W. Then $o \notin H_{n-2}(V)$. The two remaining dimensions will assist in making o an accumulation point of extreme points. In R^2, let C be the circumference of a circle through o. Finally, take $S = V \cup (C - \{o\})$. Then, since $o \in H(V) \subset H(S)$, we find $H(S) = H(V \cup C)$. But $V \cup C$ is compact, and so $H(S)$ is compact, and $E(H(S)) \subset V \cup C$. On the other hand, a point $x \in C - \{o\}$ obviously cannot be the barycenter of masses which are partly distributed in W. Hence, since $x \in E(C)$, we get $x \in E(H(S))$, and similarly for the points of V. Thus $S = E(H(S))$. Finally, we see that $o \notin H_{n-2}(S)$.

3. Compact convex sets in a Fréchet space

3.1. Theorem. *The following three properties of a set S in a real Fréchet space X* (i.e. a complete, metrizable, locally convex, real vector space, e.g. a real Banach space) *are equivalent*:

(1) *There is a compact convex set $K \subset X$, such that $S = E(K)$.*

(2) (a) *$\bar{S}$ is compact,*
  (b) *$\bar{S} \subset H_C(S)$, and*
  (c) *$S \cap H_C(A) \subset A$ for all $A \subset S$.*

(3) (a) = (2 a)
  (b) = (2 b)
  (c) *$S \cap H_C(B) \subset B$ for all $B \subset \bar{H}(S)$.*

Proof: As in Theorem 2.1, we shall prove that (1) ⇒ (3) and (2) ⇒ (1). Suppose (1) is true. Then $K = \overline{H_C(S)} = \bar{H}(S)$ [5, théorème 1, or 6, théorème 1]. Then (3 b) follows from $S \subset \overline{H_C(S)} = H_C(S)$, and (3 a) is obvious. To prove (3 c), let $B \subset K$ and let $x \in S \cap H_C(B)$. Hence $x = \int_{(B)} t \, d\mu$. But $x \in E(K)$, and so μ must consist of one unity pointmass, which must then be in B. Consequently, $x \in B$, which completes the proof of (3).

Finally, suppose (2) is true. Since $\bar{S}$ is compact and X is complete, $\bar{H}(\bar{S})$ is compact [1, N° 1]. From $\bar{S} \subset \bar{H}(S)$ we get $\bar{H}(S) = \bar{H}(\bar{S})$, and so $\bar{H}(S)$ is compact. Next we shall prove that $E(\bar{H}(S)) \subset S$. Let $x \in E(\bar{H}(S)) = E(\bar{H}(\bar{S}))$. Then $x \in \bar{S}$ [1, prop. 4], and by (2 b), $x \in H_C(S)$. As in the first part of the proof, from $x = \int_{(S)} t \, d\mu$ it follows that $x \in S$. We have thus proved the inclusion $E(\bar{H}(S)) \subset S$, and we shall now use this to prove the reversed inclusion. In (2 c), we take $A = E(\bar{H}(S))$. From Choquet's theorem, $H_C(A) = \bar{H}(S)$. Hence from (2 c), we get $S \subset A$, which completes the proof of (1) and of theorem 3.1.

In sections 3.3–3.5 we shall discuss the possibilities of making certain changes in the conditions of Theorem 3.1. We shall need the following lemma, which states that in general $H_{st}(E(K)) \neq H_C(E(K))$.

3.2. Lemma. *There exist in some Fréchet space Y a compact convex set W and a point $a \in W$ such that $a \notin H_{st}(E(W))$.*

In fact, the following example is given by Choquet [6, p. 17]: Let U (with elements u) be the closed interval $[0, 1]$ and let V (with elements v) be the set of reals modulo 1. Then Y is the space of Radon measures m on $U \times V$ and W is the set of $m \in Y$ such that $m \geq 0$, $m(U \times V) = 1$ and $dm(u,v) = dm(u,v+u)$. Let $a \in W$ be the homogeneous distribution of total mass one on $U \times V$. We find that $E(W)$ is the set of $m \in W$ satisfying:

(1) the support of m is contained in the circle $u = u_m$, where u_m is a constant (depending on m);

(2) if u_m is irrational, then m is the homogeneous distribution of total mass one on the circle $u = u_m$; and

(3) if $u_m = p/q$, where p and $q > 0$ are relatively prime integers, then m consists of q symmetrically distributed point-masses, each with mass $1/q$.

Then it is easily seen[1] that with a certain μ with the support of μ not contained in $E(W)$, we have $a = \int_{(E(W))} t\, d\mu$. From a uniqueness theorem [4, théorème 1, or 6, théorème 2), it follows that $a \notin H_{st}(E(W))$.

3.3. Remark. In Theorem 3.1, condition (3 b) cannot be replaced by $\bar{S} \in H_{st}(S)$. In fact, let Y, W and a be as in Lemma 3.2. If we take $X = Y \times R^2$ and proceed in a way similar to that used in section 2.3 (with o replaced by a), we find that $S = E(W) \cup (C - \{a\})$ satisfies condition (1) of Theorem 3.1 and that $\bar{S} \notin H_{st}(S)$.

3.4. Remark. In Theorem 3.1, condition (3 c) may be replaced by the apparently weaker condition:

$$(3\,c_1) \qquad S \cap H_2(B) \subset B \text{ for all } B \subset H_C(S).$$

In fact, (3 c_1) is exactly the statement $S \subset E(H_C(S))$. To prove that (3 a, b, c_1) implies (1), we observe that we have as in the second part of the proof of Theorem 3.1, that $E(\bar{H}(S)) \subset S$. If we take the H_C-hull of both members of the last inclusion, we get $\bar{H}(S) \subset H_C(S)$. Consequently these two sets are equal, which completes the proof that $S = E(\bar{H}(S))$.

3.5 Remark. In contrast with Remark 3.4, condition (2 c) of Theorem 3.1 cannot be weakened to

$$(2\,c_1) \qquad S \cap H_{st}(A) \subset A \text{ for all } A \subset S.$$

We consider the following example. Let $X = Y$ and $S = \{a\} \cup E(W)$ with Y, W, and a as in Lemma 3.2. Clearly S satisfies conditions (2 a, b) but not (1) of Theorem 3.1. We shall now prove that S satisfies (2 c_1). Let $A \subset S$ and $x \in S \cap H_{st}(A)$. First, suppose $x = a$. If $a \in A$, there is nothing to prove. If $a \notin A$, it follows that $A \subset S - \{a\}$, and hence $x \in H_{st}(S - \{a\})$, or in the notation of Lemma 3.2, $a \in H_{st}(E(W))$, which is not true. Finally, suppose $x \neq a$. Hence

$$x \in (S - \{a\}) \cap H_{st}(A) = E(W) \cap H_{st}(A) \subset E(W) \cap H_C(A) \subset A,$$

by (3 c) of Theorem 3.1.

[1] For the concept of integral of a family of measures, see [3, § 3, N° 1].

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