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Sums and products of commuting spectral operators[1]

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1. Introduction

Given two spectral operators T_1 and T_2, on a complex Banach space $\mathfrak{X}$, it is interesting to know if T_1+T_2 and T_1T_2 are spectral operators too. This problem is treated in [3] and [4]. It is proved there that if the space $\mathfrak{X}$ is weakly complete, and the operators T_1 and T_2 commute and the Boolean algebra of projections generated by the resolutions of the identity of T_1 and T_2 is bounded, then both T_1+T_2 and T_1T_2 are spectral operators. Moreover, if $T_1=S_1+N_1$, $T_2=S_2+N_2$, where S_1 and S_2 are scalar operators and N_1, N_2 are generalized nilpotents and S_1, N_1, S_2, N_2 commute, then S_1+S_2 and S_1S_2 are scalar operators and N_1+N_2, $S_1N_2+S_2N_1+N_1N_2$ are generalized nilpotents. The main problem in this paper will be to determine the resolutions of the identity of T_1+T_2 and T_1T_2. By the above remark it is enough to consider the case where T_1 and T_2 are scalar operators. A second problem treated here is to find the poles of the resolvents of T_1+T_2 and T_1T_2. In this part we do not assume that T_1 and T_2 are spectral operators.

2. Notation

We use here the notation and definitions of [3]. Let $\mathfrak{X}$ be a complex Banach space. A spectral measure is a set function $E(\cdot)$ defined on Borel sets in the complex plane whose values are projections on $\mathfrak{X}$ which satisfy:

1. For any two Borel sets σ and δ, $E(\sigma)E(\delta)=E(\sigma\cap\delta)$.
2. Let ϕ be the void set and P the complex plane, then

$$E(\phi)=0 \quad \text{and} \quad E(P)=I.$$

3. There exists a constant M such that $|E(\sigma)|\leq M$, for every Borel set σ.
4. The vector valued set function $E(\cdot)x$ is countably additive for each $x\in\mathfrak{X}$.

T is a spectral operator whose resolution of the identity is the spectral measure $E(\cdot)$ if

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(a) For any Borel set σ, $TE(\sigma)=E(\sigma)T$.
(b) If $T|E(\alpha)\mathfrak{X}$ is the restriction of T to the subspace $E(\alpha)\mathfrak{X}$, then

$$\sigma(T|E(\alpha)\mathfrak{X})\subset\bar{\alpha},$$

where $\sigma(A)$ denotes the spectrum of A.

If T is a spectral operator the operator $S=\int\lambda E(d\lambda)$ is its scalar part and $N=T-S$ its radical. The operator N is a generalized nilpotent (see [3], p. 333). The operators T, S, N, $E(\alpha)$ commute (α a Borel set). The operator T is called a scalar operator if $T=S$.

3. Sums and products of projections

In this section we shall find the resolutions of the identity of the sum and product of two scalar operators when one of them has a finite spectrum.

Lemma 1. *If S is a scalar operator whose resolution of the identity is $E(\cdot)$, and F a projection commuting with S, then $S+\mu F$ is a scalar operator whose resolution of the identity is given by the projection valued set function $G(\cdot)$*

$$G(\alpha)=E(\alpha-\mu)F+E(\alpha)F',$$

where F' is the complement of F, namely $I-F$, and α is a Borel set.

Proof. Let α and β be Borel sets.

$$\begin{aligned}G(\alpha)G(\beta)&=(E(\alpha-\mu)F+E(\alpha)F')(E(\beta-\mu)F+E(\beta)F')\\&=E(\alpha\cap\beta-\mu)F+E(\alpha\cap\beta)F'\\&=G(\alpha\cap\beta)\end{aligned}$$

because E and F commute. (See [3], p. 329.)

$$G(\phi)=0\quad\text{and}\quad G(P)=I\cdot F+IF'=I.$$

$$\begin{aligned}|G(\alpha)|&=|E(\alpha-\mu)F+E(\alpha)F'|\\&\le|E(\alpha-\mu)|\,|F|+|E(\alpha)|F'|\\&\le\operatorname{Sup}\{|E(\alpha)|\,|\alpha\text{ a Borel set}\}(|F|+|F'|).\end{aligned}$$

It is clear that $G(\alpha)x$ is countably additive. Now

$$\begin{aligned}\int\lambda E(d\lambda)+\mu F&=\int(\lambda+\mu)E(d\lambda)F+\int\lambda E(d\lambda)F'\\&=\int\lambda[E(d\lambda-\mu)F+E(d\lambda)F']\\&=\int\lambda G(d\lambda).\end{aligned}$$

Theorem 1. *Let* $T = S + \sum_{i=1}^{n} \mu_i F_i$, *where* S *is a scalar operator whose resolution of the identity is* $E(\cdot)$ *and*

$$F_i S = S F_i, \qquad \sum_{i=1}^{n} F_i = I, \qquad F_i F_j = 0 \quad i \neq j, \qquad F_i^2 = F_i,$$

then T *is a scalar operator whose resolution of the identity* G *is given by*

$$G(\alpha) = \sum_{i=1}^{n} E(\alpha - \mu_i) F_i$$

for any Borel set α.

Proof. By Lemma 1 the theorem holds for $m = 1$. Let us assume its validity for $m = n - 1$. On the space $Y = \sum_{i=1}^{n-1} F_i \mathfrak{X}$ by assumption

$$S + \sum_{i=1}^{n-1} \mu_i F_i = \int \lambda G_1(d\lambda),$$

$$G_1(\alpha) = \sum_{i=1}^{n-1} E(\alpha - \mu_i) F_i.$$

Thus for every $x \in \mathfrak{X}$

$$\begin{aligned} \left(S + \sum_{i=1}^{n-1} \mu_i F_i\right) x & \\ &= \left(S + \sum_{i=1}^{n-1} \mu_i F_i\right) \sum_{i=1}^{n-1} F_i x + S F_n x \\ &= \int \lambda G_1(d\lambda) \left(\sum_{i=1}^{n-1} F_i x\right) + \int \lambda E(d\lambda) F_n x \\ &= \int \lambda G_2(d\lambda) x, \end{aligned}$$

where

$$\begin{aligned} G_2(\alpha) x &= \left(\sum_{i=1}^{n-1} E(\alpha - \mu_i) F_i\right) \sum_{i=1}^{n-1} F_i x + E(\alpha) F_n x \\ &= \sum_{i=1}^{n-1} E(\alpha - \mu_i) F_i x + E(\alpha) F_n x. \end{aligned}$$

It is easy to verify that G_2 is a spectral measure. Using Lemma 1 again

$$\left(S + \sum_{i=1}^{n-1} \mu_i F_i\right) + \mu_n F_n = \int \lambda G(d\lambda),$$

where
$$G(\alpha) = G_2(\alpha - \mu_n) F_n + G_2(\alpha) F_n'$$
$$= E(\alpha - \mu_n) F_n + \sum_{i=1}^{n-1} E(\alpha - \mu_i) F_i$$
$$= \sum_{i=1}^{n} E(\alpha - \mu_i) F_i.$$

By a similar proof one can derive the following theorem.

Theorem 2. *Let* $T = \left(S \cdot \sum_{i=1}^{n} \mu_i F_i\right)$, *where* S *and* F_i *satisfy the conditions of Theorem* 1 *and* $\mu_i \neq 0$, *then* T *is a scalar operator whose resolution of the identity* G *is given by*

$$G(\alpha) = \sum_{i=1}^{n} E\left(\frac{\alpha}{\mu_i}\right) F_i.$$

The restriction $\mu_i \neq 0$ is not essential and is introduced here to simplify notation.

Corollary 1. Let S_1 and S_2 be two commuting scalar operators given by

$$S_1 = \sum_{i=1}^{n} \lambda_i E_i, \qquad S_2 = \sum_{i=1}^{m} \mu_i F_i, \qquad \sum_{i=1}^{n} E_i = I = \sum_{i=1}^{m} F_i,$$

then the spectrum of $S_1 + S_2$ is contained in the set $\{\lambda_i + \mu_j \mid 1 \leq i \leq n,\ 1 \leq j \leq m\}$ and the value of the resolution of the identity of $S_1 + S_2$ at the point δ is equal to $\sum_{\lambda_i + \mu_j = \delta} E_i F_j$. Also, the spectrum of $S_1 S_2$ is contained in the set $\{\lambda_i \mu_j \mid 1 \leq i \leq n,\ 1 \leq j \leq m\}$ and the value of the resolution of the identity of $S_1 S_2$ at the point δ is equal to $\sum_{\lambda_i \mu_i = \delta} E_i F_j$.

By Lemma 1 if E_1 and E_2 are two commuting projections then

$$E_1 + E_2 = 2 E_1 E_2 + (E_1 E_2' + E_2 E_1').$$

This can be generalized as follows.

Corollary 2. Let E_i $1 \leq i \leq n$ be n commuting projections then $\sum_{i=1}^{n} E_i$ is a scalar operator whose spectrum is contained in the set $\{0, 1, 2, \ldots, n\}$ and at the point i $1 \leq i \leq n$ the value of the spectral measure is

$$G_i = \sum_{1 \leq j_1 < j_2 < \cdots < j_{n-i} \leq n} \ \prod_{\substack{1 \leq K \leq n \\ K \neq j_v}} E_K E_{j_1}' \ldots E_{j_{n-i}}'$$

and
$$G_0 = \prod_{i=1}^{n} G_i'.$$

Proof. Let us prove by induction. For $m=2$ the equation holds. Assume that the theorem is true for $m=n$.

$$\sum_{j=1}^{n+1} E_j = 0\cdot G_0 + \sum_{i=1}^{n} i\, G_i + E_{n+1} + 0\cdot E'_{n+1}.$$

By Theorem 1, the value of the resolution of the identity of $\sum_{j=1}^{n+1} E_j$ at the point i, $1\leq i\leq n+1$, is

$$\begin{aligned} F_i &= G_{i-1} E_{n+1} + G_i E'_{n+1} \\ &= \sum_{1\leq j_1<\cdots<j_{n-i+1}\leq n} \prod_{\substack{1\leq K\leq n\\ K\neq j_v}} E_K E_{n+1} E'_{j_1}\ldots E'_{j_{n-i+1}} + \\ &\qquad + \sum_{1\leq j_1<\cdots<j_{n-i}\leq n} \prod_{\substack{1\leq K\leq n\\ K\neq j_v}} E_K E'_{j_1}\ldots E'_{j_{n-i}} E'_{n+1} \\ &= \sum_{1\leq j_1\leq j_2<\cdots<j_{n+1-i}\leq n+1} \prod_{\substack{1\leq K\leq n+1\\ K\neq j_v}} E_K E'_{j_1}\ldots E'_{j_{n+1-i}}. \end{aligned}$$

Now
$$F_0 + \sum_{i=1}^{n+1} F_i = I,$$
hence
$$F_0 = \prod_{i=1}^{n+1} F'_i.$$

4. Poles of the resolvents of the sum and product of two commuting operators

The operators discussed in this section are not assumed to be spectral. We shall say that λ is a pole of an operator T if λ is a pole of the resolvent of T. The following theorem will be used.

Theorem A. *If λ is an isolated point of the spectrum of an operator T, then there exists a projection E and a generalized nilpotent N such that*

$$E\,T = T\,E, \qquad N\,T = T\,N, \qquad N\,E = E\,N = N$$

$$T = (\lambda I + N)E + K \quad \text{with} \quad K = E'K = KE' \quad \text{and} \quad \lambda \notin \sigma(K).$$

The number λ is a pole of order n if and only if $N^n = 0$, $N^{n-1}\neq 0$. In addition, if an operator A commutes with T then A commutes with E, N, K.

See [4] Theorem VII.3.18. E is given by

$$E = \int_c R(\mu;\, T)\, d\mu,$$

where c is a circle around λ which does not contain any other point of the spectrum of T.

If $T = \sum_{i=1}^{m} T E_i$, where $E_i E_j = 0$ $i \neq j$, $E_i^2 = E_i$ and $T E_i = E_i T$, then

$$\sigma(T) = \bigcup_{i=1}^{n} \sigma(T \mid E_i \mathfrak{X}),$$

and if $\lambda \notin \sigma(T)$,

$$(\lambda I - T)^{-1} = \sum_{i=1}^{n} R_i(\lambda) E_i,$$

where $R_i(\lambda)$ is the inverse of $(\lambda I - T) E_i$ on the space $E_i \mathfrak{X}$.

Let T_1 and T_2 be two commuting operators. There exists an algebra $\mathfrak{a}$ of operators, containing T_1 and T_2 such that $\mathfrak{a}$ is a commutative algebra and if $U \in \mathfrak{a}$ and U^{-1} exists then $U^{-1} \in \mathfrak{a}$. By the Gelfand theory [5] if $\sigma_{\mathfrak{a}}(U)$ denotes the spectrum of U as an element of $\mathfrak{a}$ then $\sigma_{\mathfrak{a}}(T_1 + T_2) \subset \sigma_{\mathfrak{a}}(T_1) + \sigma_{\mathfrak{a}}(T_2)$, but for each $U \in \mathfrak{a}$, $\sigma_{\mathfrak{a}}(U) = \sigma(U)$, hence $\sigma(T_1 + T_2) \subset \sigma(T_1) + \sigma(T_2)$. If δ is an isolated point of $\sigma(T_1) + \sigma(T_2)$ then

$$\sigma(T_1) = \{\lambda_1, \ldots, \lambda_n\} \cup \sigma_1, \qquad \sigma(T_2) = \{\mu_1, \ldots, \mu_n\} \cup \sigma_2$$

with
$$\lambda_i + \mu_i = \delta \quad 1 \leq i \leq n$$

but
$$\delta \notin (\sigma_1 + \sigma(T_2)) \cup (\sigma_2 + \sigma(T_1)).$$

Theorem 3. *If T_1 and T_2 are two commuting operators, and δ is an isolated point of $\sigma(T_1) + \sigma(T_2)$, and λ_i is a pole of T_1 of order n_i, μ_i is a pole of T_2 of order m_i, where $\{\lambda_i\}$, $\{\mu_i\}$, σ_1, σ_2 are defined above, then δ is a pole of order not greater than* $\max [(m_i + n_i - 1) \;\; 1 \leq i \leq n]$.

Proof. By repeated application of Theorem A we have

$$T_1 = \sum_{i=1}^{n} (\lambda_i I + N_i) E_i + K_1$$

with
$$K_1 \sum_{i=1}^{n} E_i = \sum_{i=1}^{n} E_i K_1 = 0, \qquad E_i^2 = E_i, \qquad E_i E_j = 0 \quad i \neq j,$$

$$N_i^{n_i} = 0 \quad \sigma(K_1) = \sigma_1,$$

$$T_2 = \sum_{i=1}^{n} (\mu_i I + M_i) F_i + K_2$$

with

$$K_2 \sum_{i=1}^{n} F_i = \sum_{i=1}^{n} F_i K_2 = 0, \qquad F_i^2 = F_i, \qquad F_i F_j = 0 \quad i \neq j,$$

$$M_i^{m_i} = 0, \qquad \sigma(K_2) = \sigma_2.$$

The operators E_i, F_i, M_i, N_i, K_1, K_2 commute. Now

$$T_1+T_2=K_1+K_2+\left(\sum_{i=1}^{n}(\lambda_i I+N_i)E_i+\sum_{i=1}^{n}(\mu_i I+M_i)F_i\right). \tag{1}$$

By Corollary 1 of Theorem 1

$$\sum_{i=1}^{n}\lambda_i E_i+\sum_{i=1}^{n}\mu_i F_i=\delta\sum_{i=1}^{n}E_iF_i+\sum\delta_i A_i+\sum_{i=1}^{n}\lambda_i E_i F_0+\sum_{i=1}^{n}\mu_i F_i E_0,$$

where

$$\delta_i\neq\delta,\qquad A_i\left(\sum_{j=1}^{n}E_jF_j\right)=0,\qquad A_j^2=A_j,\qquad A_j\left(\sum_{i=1}^{n}E_i\right)\left(\sum_{i=1}^{n}F_i\right)=A_j,$$

and E_0 is $I-\sum_{i=1}^{n}E_i$, F_0 is $I-\sum_{i=1}^{n}F_i$.

Thus (1) takes the form

$$T_1+T_2=\delta\sum_{i=1}^{n}E_iF_i+\sum\delta_i A_i+\left(K_1+\sum_{i=1}^{n}\mu_i F_i E_0\right)+\left(K_2+\sum_{i=1}^{n}\lambda_i E_i F_0\right)+M. \tag{2}$$

The operator M is a nilpotent operator on the space

$$\left(\sum_{i=1}^{n}E_i\right)\left(\sum_{i=1}^{n}F_i\right)\mathfrak{X}.$$

Let

$$G_1=\sum_{i=1}^{n}E_i,\qquad G_2=\sum_{i=1}^{n}F_i,$$

$$E_0+G_1=F_0+G_2=I,\qquad E_0G_1=F_0G_2=0,$$

$$K_1E_0=K_1,\qquad K_2F_0=K_2.$$

Let

$$T=T_1+T_2,$$

$$T=TG_1G_2+TE_0G_2+TF_0G_1+TE_0F_0,$$

$$(\lambda I-T)^{-1}=((\lambda I-T)\,|\,G_1G_2\mathfrak{X})^{-1}G_1G_2+((\lambda I-T)\,|\,E_0G_2\mathfrak{X})^{-1}E_0G_2+$$
$$+((\lambda I-T)\,|\,F_0G_1\mathfrak{X})^{-1}F_0G_1+((\lambda I-T)\,|\,E_0F_0\mathfrak{X})^{-1}E_0F_0.$$

On the space $E_0F_0\mathfrak{X}$, T has the form

$$(K_1+K_2)E_0F_0,$$

$$\delta\notin\sigma_1+\sigma_2,\ \text{but}\ \sigma(K_1+K_2)\subset\sigma(K_1)+\sigma(K_2)=\sigma_1+\sigma_2.$$

Thus on $E_0F_0\mathfrak{X}$, $(\lambda I-T)^{-1}$ is regular at the point δ.

On the space $E_0 G_2 \mathfrak{X}$, T has the form

$$T \mid E_0 G_2 \mathfrak{X} = K_1 G_2 + \sum_{i=1}^{n} \mu_i F_i E_0 = \sum_{i=1}^{n} (K_1 + \mu_i I) F_i E_0.$$

Hence
$$\sigma (T \mid E_0 G_2 \mathfrak{X}) = \bigcup_{i=1}^{n} \sigma (K_1 + \mu_i I \mid E_0 F_i \mathfrak{X}) \subset \bigcup_{i=1}^{n} \sigma (K_1 + \mu_i I)$$
$$= \sigma_1 + \{\mu_1 \dots \mu_n\}.$$

Thus
$$\delta \notin \sigma (T \mid E_0 G_2 \mathfrak{X}).$$

By the same argument $\delta \notin \sigma (T \mid F_0 G_1 \mathfrak{X})$. The number δ is a regular point of T restricted to $G_1 G_2 \mathfrak{X}$ if and only if $\sum_{i=1}^{n} E_i F_i = 0$, in this case δ is a regular point of T. The nilpotent operator associated with the point δ is $\sum_{i=1}^{n} (N_i + M_i) E_i F_i$ by (1) and (2). Let $k = \max [(m_i + n_i - 1) \;\; 1 \leq i \leq n]$

$$\left(\sum_{i=1}^{n} (N_i + M_i) E_i F_i \right)^k = \sum_{i=1}^{n} (N_i + M_i)^k E_i F_i = 0.$$

Hence δ is a pole of $T \mid G_1 G_2 \mathfrak{X}$ of order at most k, but by the preceding discussion δ is a pole of T of the same order.

Using Theorem 2 and a similar proof we arrive at the following theorem.

Theorem 4. *If T_1 and T_2 are two commuting operators and*

$$\sigma (T_1) = \{\lambda_1, \dots, \lambda_n\} \cup \sigma_1, \qquad \sigma (T_2) = \{\mu_1, \dots, \mu_n\} \cup \sigma_2,$$
$$0 \neq \delta = \lambda_i \mu_i, \quad 1 \leq i \leq n, \qquad \delta \notin (\sigma_1 \cdot \sigma (T_2)) \cup (\sigma_2 \cdot \sigma (T_1))$$

(these conditions are equivalent to: $0 \neq \delta$ is an isolated point of $\sigma (T_1) \cdot \sigma (T_2)$), and if λ_i is a pole of order n_i of T_1, μ_i is a pole of order m_i of T_2, then δ is a pole of $T_1 T_2$ of order at most

$$\max [(m_i + n_i - 1), \; 1 \leq i \leq n].$$

5. Application

Let C be a linear operator in the space $\mathfrak{X}$. Define the operators T_1 and T_2 on the space $B(\mathfrak{X})$ of bounded linear operators in $\mathfrak{X}$ by

$$T_1 (A) = CA, \qquad T_2 (A) = AC, \qquad A \in B (\mathfrak{X}).$$

In this section we study the relations between the poles of C and those of $T = T_1 - T_2$. (This problem was raised by Professor E. Hille.) It is easy to see from the proof of Theorem 1 in [5] that $\sigma (C) = \sigma (T_1) = \sigma (T_2)$. By Theorem A λ is a pole of C if and only if it is a pole of T_1 and T_2 of the same order.

Theorem 5. *Let $\sigma(C)$ be decomposed*

$$\sigma(C) = \{\lambda_1, \ldots, \lambda_n\} \cup \sigma_1,$$

$$\sigma(C) = \{\mu_1, \ldots, \mu_n\} \cup \sigma_2,$$

with $\lambda_i - \mu_i = \delta$ $1 \le i \le n$, and $\delta \notin (\sigma_1 - \sigma(C)) \cup (\sigma(C) - \sigma_2)$, and λ_i, μ_i, are poles of order n_i, m_i respectively of C, then δ is a pole of the operator T on the space $B(\mathfrak{X})$ defined by

$$T(A) = CA - AC, \quad A \in B(\mathfrak{X})$$

of order $\max[(m_i + n_i - 1), \quad 1 \le i \le n]$.

Proof. In order to use Theorem 3 we note

1. $$T_1 T_2(A) = T_2 T_1(A) = CAC, \ A \in B(\mathfrak{X}).$$
2. If E and F are projections on $\mathfrak{X}$, then the operators $\bar{E}$, $\bar{E}(A) = EA$, and $\bar{F}$, $\bar{F}(A) = AF$, $A \in B(\mathfrak{X})$, are two commuting projections.
3. If N is a nilpotent operator on $\mathfrak{X}$ of order n then $\bar{N}$, $\bar{N}(A) = NA$, $A \in B(\mathfrak{X})$, is a nilpotent operator on $B(\mathfrak{X})$ of order n.

To show that δ is a singular point of T we prove the following lemma.

Lemma 2. *Let V_1 and V_2 be two non-zero operators on $\mathfrak{X}$, then the operator V on $B(\mathfrak{X})$ defined by*

$$V(A) = V_1 A V_2, \quad A \in B(\mathfrak{X})$$

is different from zero.

Proof. Let us choose $x \in \mathfrak{X}$ $y \in \mathfrak{X}$ such that $V_2 x \neq 0$, $V_1 y \neq 0$ and $x^* \in \mathfrak{X}^*$ such that $x^*(V_2 x) \neq 0$. Define A by $z \in \mathfrak{X}$, $Az = x^*(z) \cdot y$. Then

$$(V_1 A V_2)(x) = x^*(V_2 x) \cdot V_1 y \neq 0.$$

Now to conclude the proof of Theorem 5 let E_i, N_i be the projections and nilpotents respectively corresponding to λ_i, and F_i, M_i the ones associated with μ_i. By theorem 3 the projection associated with δ with respect to T is G

$$G(A) = \sum_{i=1}^{n} E_i A F_i \quad A \in B(\mathfrak{X}).$$

The corresponding nilpotent is given by

$$M(A) = \sum_{i=1}^{n} (N_i(E_i A F_i) - (E_i A F_i) M_i)$$

$E_j G(A) F_j = E_j A F_j \neq 0$ by Lemma 2, thus $G \neq 0$ and δ is a singular point of T. If $l = \max[(m_i + n_i - 2), \ 1 \le i \le n] = m_{j_0} + n_{j_0} - 2$, then

$$E_{j_0} M^l(A) F_{j_0} = \sum_{v=0}^{l} \alpha_v N_{j_0}^v E_{j_0} A F_{j_0} M_{j_0}^{l-v},$$

where α_v are non zero numbers, hence

$$E_{j_0} M^l(A) F_{j_0} = \alpha_{n_{j_0}-1} N_{j_0}^{n_{j_0}-1} A M_{j_0}^{m_{j_0}-1} \neq 0$$

if A is properly chosen.

6. Sum and product of two commuting scalar operators

Throughout this section we assume that the space $\mathfrak{X}$ is weakly complete.

Theorem 6. *Let $\{S_n\}$ be a sequence of commuting scalar operators which converge uniformly to the operator S. If the Boolean algebra of projections, generated by the resolutions of the identity of the operators S_n, is bounded, then S is a scalar operator, and if E_n, E, are the spectral measures of S_n and S respectively, then for each Borel set α*

(1) *If for some $x \in \mathfrak{X}$, E (boundary α) $x = 0$, then $E(\alpha) x = \lim E_n(\alpha) x$.*

(2) *If for some $x \in \mathfrak{X}$ and $x^* \in \mathfrak{X}^*$, $x^* E$ (boundary α) $x = 0$, then $x^* E(\alpha) x = \lim x^* E_n(\alpha) x$.*

Proof. By [4], Theorems XVII.2.5 and XVII.2.1, there exists a set function $F(\cdot)$ defined on a compact set Λ such that for each Borel set α in Λ, $F(\alpha)$ is a projection satisfying conditions 1, 2, 3, 4 of the Introduction (where the complex plane should be replaced by Λ), and for every T in the uniformly closed algebra generated by the resolutions of the identity of S_n, there exists a continuous function f defined on Λ such that

$$T = \int_{\Lambda} f(\lambda) F(d\lambda),$$

and this correspondence is an isomorphic homeomorphism if the norm of f is taken to be

$$|f| = \max_{\lambda \in \Lambda} |f(\lambda)|.$$

Let f_n and f correspond to S_n and S. By assumption $S_n \to S$, hence f_n tends to f uniformly.

$$S_n = \int_{\Lambda} f_n(\gamma) F(d\gamma), \qquad S = \int_{\Lambda} f(\gamma) F(d\gamma),$$

hence $$E_n(\alpha) = F\{\gamma \mid f_n(\gamma) \in \alpha\}, \qquad E(\alpha) = F\{\gamma \mid f(\gamma) \in \alpha\}$$

for every Borel set α in the complex plane. Let $A_n = \{\gamma \mid f_n(\gamma) \in \alpha\}$ and $A = \{\gamma \mid f(\gamma) \in \alpha\}$, then

$$E_n(\alpha) x = \int_{\Lambda} \chi_{A_n}(\gamma) F(d\gamma) x, \qquad E(\alpha) x = \int_{\Lambda} \chi_A(\gamma) F(d\gamma) x,$$

where χ_B denotes the characteristic function of B. In order to prove (1) it is enough to show that $\chi_{A_n}(\gamma)$ converges almost everywhere, with respect to the measure $F(\cdot)x$, to $\chi_A(\gamma)$. (See [4], Theorem IV.10.10.) If $f(\gamma)\in\alpha^0$ then for $n>n_0$ $f_n(\gamma)\in\alpha^0$. If $f(\gamma)\in(\Lambda-\alpha)^0$ then for $n>n_1$ $f_n(\gamma)\in(\Lambda-\alpha)^0$. By α^0 we mean the interior of α. Let $F\{\gamma\,|\,f(\gamma)\in \text{boundary}\,\alpha\}\,x=0$, then because of the multiplicity property of F

$$\sup\left\{\left|\sum_{i=1}^{n}\varepsilon_i F(B_i)\,x\right|,\qquad B_i\cap B_j=\phi\quad i\neq j,\right.$$

$$\left.B_i\subset\{\gamma\,|\,f(\gamma)\in\text{boundary}\,\alpha\},\quad \varepsilon_i \text{ complex number with } |\varepsilon_i|\leq 1\right\}=0.$$

Thus $E_n(\alpha)\,x\to E(\alpha)\,x$ whenever

$$F\{\gamma\,|\,f(\gamma)\in\text{boundary}\,\alpha\}\,x=0,$$

but
$$F\{\gamma\,|\,f(\gamma)\in\text{boundary}\,\alpha\}=E\{\text{boundary}\,\alpha\}$$

and so (1) is proved. (2) is proved in the same way.

Remark. This is a perturbation theorem similar to Rellich Theorem [6] and to Theorem 2.6 of [2], p. 402. By [1], p. 351, for each $x\in\mathfrak{X}$ there exists a functional x^* with the properties

1. $x^*E(\alpha)\,x\geq 0$ for any Borel set α.
2. If $x^*E(\alpha)\,x=0$, then $E(\alpha)\,x=0$.

Thus $E(\alpha)\,x=0$ if and only if $x^*E(\alpha)\,x=0$. Now for any collection of Borel sets $\{\alpha_t\}$ with $\alpha_{t_1}\cap\alpha_{t_2}=\phi$ whenever $t_1\neq t_2$, only a countable number of terms in the set $\{x^*E(\alpha_t)\,x\}$ are different from zero, because of countable additivity. This shows that there are enough Borel sets α with $E(\text{boundary}\,\alpha)\,x=0$ to compute the value of the Riemann Integral $\int g(\lambda)\,E(d\lambda)\,x$ for every continuous function g.

Theorem 7. *Let S_1 and S_2 be two commuting scalar operators with resolutions of the identity $E(\cdot)$ and $F(\cdot)$ respectively. If the Boolean algebra of projections generated by $E(\alpha)$, $F(\beta)$ is bounded then $T_1=S_1+S_2$ and $T_2=S_1S_2$ are scalar operators whose resolutions of the identity $G_1(\cdot)$ and $G_2(\cdot)$ respectively, are*

$$G_1(\alpha)\,x=\int E(\alpha-\mu)\,F(d\mu)\,x\quad \textit{if}\ G_1(\textit{boundary}\,\alpha)\,x=0,$$

$$G_2(\alpha)\,x=\int E\left(\frac{\alpha}{\mu}\right)F(d\mu)\,x\quad \textit{if}\ G_2(\textit{boundary}\,\alpha)\,x=0,$$

where the integrals exist in the sense of Riemann, provided that in the sums $\sum_{i=1}^{n}E\left(\frac{\alpha}{\mu_i}\right)F(\Delta_i)\,x$ approximating $G_2(\alpha)\,x$ we take $\mu_i\neq 0$. The integrals are evaluated over any rectangle containing $\sigma(S_2)$.

Proof. Let K be a rectangle containing $\sigma(S_2)$ and let $\{\pi_n\}$ be a sequence of partitions of K.

$$\pi_n = \{\Delta_1^n, \ldots, \Delta_n^n\}, \qquad \Delta_i^n \cap \Delta_j^n = \phi \quad i \neq j,$$

$$\bigcup_{i=1}^n \Delta_i^n = K, \qquad \max\,[\text{diam}\,(\Delta_i^n),\ 1 \leq i \leq n] \to 0.$$

Let

$$\mu_i^n \in \Delta_i^n, \qquad \mu_i^n \neq 0.$$

$$S_1 + S_2 = \lim \left(S_1 + \sum_{i=1}^n F(\Delta_i^n)\,\mu_i^n \right) = \lim R_n.$$

By Theorem 1, R_n is a scalar operator whose resolution of the identity G_1^n, is

$$G_1^n(\alpha) = \sum_{i=1}^n E(\alpha - \mu_i^n)\, F(\Delta_i^n).$$

By Theorem 6, $G_1^n(\alpha)\,x \to G_1(\alpha)\,x$ if

$$G_1\,(\text{boundary}\ \alpha)\,x = 0.$$

Similarly $S_1 S_2 = \lim U_n$, and the resolution of the identity of U_n, G_2^n is by Theorem 2

$$G_2^n(\alpha) = \sum_{i=1}^n E\left(\frac{\alpha}{\mu_i^n}\right) F(\Delta_i^n).$$

Thus if $G_2\,(\text{boundary}\ \alpha)\,x = 0$ then $G_2(\alpha)\,x = \int E\left(\frac{\alpha}{\mu}\right) F(d\mu)\,x.$

Remark. Let S be a scalar operator with resolution of the identity $E(\cdot)$, and let the set function $E(\cdot)$ be chosen in such a way that for every point λ_0, $E((\lambda_0)) = 0$. Let $S_1 = I + S$ and $S_2 = I - S$ then $S_1 + S_2 = 2I$ the spectral measure of the operator $2I$ is concentrated at the point 2. The resolution of the identity of S_1 is given by

$$E_1(\alpha) = E(\alpha - 1) \quad \text{for any Borel set } \alpha.$$

The resolution of the identity of S_2 is

$$E_2(\alpha) = E(1 - \alpha) \quad \text{for any Borel set } \alpha.$$

The Boolean algebra of projections generated by E_1, E_2 is bounded, but $E_1((\lambda_0)) = 0$ for every point λ_0, hence

$$\int E_1((2-\mu))\, E_2(d\mu) = 0 \neq I.$$

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