Communicated 13 March 1957 by ÅKE PLEIJEL

A non-closed relative spectrum

By EDGAR ASPLUND

Let E be a linear topological space, T a continuous linear transformation from E into itself, I the identity transformation on E and put $T_\lambda = T - \lambda I$. If, for a complex λ, there exists a continuous linear transformation R_λ from E into E, which satisfies the condition $T_\lambda R_\lambda T_\lambda = T_\lambda$, we say that T_λ is relatively regular for this value of λ and that R_λ is the relative inverse of T_λ. Those values of λ for which T_λ is not relatively regular constitute the relative spectrum of T. The relative spectrum is contained in the intersection of the right spectrum and the left spectrum of T. Unlike this set, however, the relative spectrum need not be a closed set even if E is complete and metrizable as we will show by an example. (A statement to the contrary has appeared in the literature and has not, to my knowledge, been disproved earlier.) Our tool in constructing this example will be the following proposition.

Let the complex number α belong to the relative spectrum of the linear continuous transformation T from E into E. Then the linear continuous transformation T' from the direct sum $E+E$ of E with itself into $E+E$, represented by the (block) matrix

$$T' = \begin{pmatrix} \alpha I & 0 \\ I & T \end{pmatrix},$$

where I denotes the identity transformation in E, has for relative spectrum the relative spectrum of T with exception of the point α.

Firstly, we show that α does not belong to the relative spectrum of T'. This is settled by the identity

$$T' - \alpha I' = \begin{pmatrix} 0 & 0 \\ I & T-\alpha I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ I & T-\alpha I \end{pmatrix} \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ I & T-\alpha I \end{pmatrix}.$$

Secondly, we show that the points of the relative spectrum of T other than α belong to the relative spectrum of T'. Suppose that $R'_\lambda = \begin{pmatrix} A_\lambda & B_\lambda \\ C_\lambda & D_\lambda \end{pmatrix}$ is a relative inverse of $T'_\lambda = T' - \lambda I'$ for one of these values of λ. The identity $T'_\lambda R'_\lambda T'_\lambda = T'_\lambda$ becomes

$$\begin{pmatrix} (\alpha-\lambda)^2 A_\lambda + (\alpha-\lambda) B & (\alpha-\lambda) B_\lambda (T-\lambda I) \\ (\alpha-\lambda)(A_\lambda + (T-\lambda I) C_\lambda) + B_\lambda + (T-\lambda I) D_\lambda & B_\lambda (T-\lambda I) + (T-\lambda I) D_\lambda (T-\lambda I) \end{pmatrix} = \begin{pmatrix} (\alpha-\lambda) I & 0 \\ I & T-\lambda I \end{pmatrix}. \quad (1)$$

As $\lambda \neq \alpha$, the identities of the right column give $T_\lambda D_\lambda T_\lambda = T_\lambda$, that is T_λ is relatively regular, contrary to our supposition.

Lastly, if R_λ satisfies $T_\lambda R_\lambda T_\lambda = T_\lambda$, then the linear continuous transformation

$$R'_\lambda = \begin{pmatrix} \dfrac{T_\lambda R_\lambda}{\alpha - \lambda} & I - T_\lambda R_\lambda \\ -\dfrac{R_\lambda}{\alpha - \lambda} & R_\lambda \end{pmatrix}$$

will satisfy the relation $T'_\lambda R'_\lambda T'_\lambda = T'_\lambda$, proving that the relative spectrum of T includes that of T'. Indeed, as α does not belong to the relative spectrum of T', this gives the opposite inclusion to that proved above and thus proves the identity of the relative spectrum of T minus α and the relative spectrum of T'.

Therefore, we have only to exhibit a transformation whose relative spectrum does not consist of isolated points. Such a transformation T is given by the relations

$$T x_1 = 0$$
$$T x_{n+1} = x_n \qquad n = 1, 2 \ldots$$

on the set x_n, $n = 1, 2 \ldots$ of orthonormal base vectors of a separable Hilbert space E. This transformation has been discussed by Beurling ([1], p. 242) and by Hamburger ([2], p. 504). For our purpose it will be enough to show that its relative spectrum is the circle $|\lambda| = 1$. The transformation T itself is right regular with right inverse R determined by $R x_n = x_{n+1}$. As the series $R_\lambda = \sum_0^\infty \lambda^n R^{n+1}$ converges for $|\lambda| < 1$, $T_\lambda = T - \lambda I$ is right regular with right inverse R_λ in this region. For $|\lambda| > 1$ the series for $(T_\lambda)^{-1} = -\sum (T^n/\lambda^{n+1})$ converges, thus the resolvent set includes the set $|\lambda| > 1$. Also, it is easily established that the transpose T^*_λ of T_λ (which, by the way, is given by $R - \bar{\lambda} I$ with R defined above) is such that for any $a \in E$, $T^*_\lambda a = 0$ implies $a = 0$ for all values of λ. For suppose there existed a vector $a = \sum_0^\infty a_n x_n$ in E, orthogonal to all vectors in the range of $T - \lambda I$. Then one obtains

$$(-\lambda x_1, a) = -\lambda \bar{a}_1 = 0,$$
$$(x_1 - \lambda x_2, a) = \bar{a}_1 - \lambda \bar{a}_2 = 0, \text{ etc.},$$

which assures that $a = 0$.

Now, suppose that for some value μ on the unit circle, T_μ were relatively regular with relative inverse R_μ so that $T_\mu R_\mu T_\mu = T_\mu$. Then, transposing this relation, one readily finds that the range of the transformation $I - R^*_\mu T^*_\mu$ is included in $T^{*-1}_\mu(O)$ which is the origin. Accordingly $T_\mu R_\mu = I$, i.e. T_μ is right regular with right inverse R_μ. Then, for all values of λ in the open disc $|\lambda - \mu| > \|R_\mu\|^{-1}$, $R_\lambda = \sum_0^\infty (\lambda - \mu)^n R_\mu^{n+1}$ is a right inverse of T_λ. But this implies a contradiction, since by means of this "right resolvent" and the right resolvent $R_\lambda = \sum_0^\infty \lambda^n R^{n+1}$ in the interior of the unit circle one would be able to continue the resolvent $(T_\lambda)^{-1} =$

$-\sum_0^\infty \frac{T^n}{\lambda^{n+1}}$ analytically into the whole interior of the unit circle, whereas $\lambda=0$ plainly is an eigen-value of T.

My deep gratitude is due to Professor Göran Borg for his suggestions, help and encouragement.

REFERENCES

BEURLING, A., On two problems concerning linear transformations in Hilbert space. Acta Mathem. *81* (1949), 239–255.

HAMBURGER, H. L., Five notes on a generalization of quasi-nilpotent transformations in Hilbert space. Proc. London Math. Soc. (3) *1* (1951), 494–512.

Tryckt den 7 maj 1957

Uppsala 1957. Almqvist & Wiksells Boktryckeri AB