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On the connection between Hausdorff measures and capacity

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1. The metrical characterization of pointsets has been carried out along two different lines. Hausdorff (1919) introduced what is now called Hausdorff measures and the concept of capacity was first given a general sense by Polya–Szegö (1931).[1] The first general result on the connection between the two concepts was given by Frostman [3] (1935). He proved that if a closed set has capacity zero, then its Hausdorff measure vanishes for every increasing function $h(r)$, $h(0)=0$, such that

$$\int_0^1 \frac{h(r)}{r}\,dr < \infty. \tag{1}$$

It has since then been an open question whether or not a converse of this result holds true: given a closed set E of positive capacity, does there exist a measure function $h(r)$ such that (1) holds and such that the corresponding Hausdorff measure is positive? This is known to be true e.g. for Cantor sets. The main object of this note is to exhibit a set E for which it fails to hold. This will make it clear that the two ways of measuring sets E are fundamentally different.

In the other direction it has been proved by Erdös and Gillis [2] that if a set E has finite Hausdorff measure with respect to $(\log(1/r))^{-1}$, then its capacity vanishes. We shall give a new and very simple proof of this result. The method will also permit us to prove, for sets of positive capacity, the existence of a uniformly continuous potential, a result that does not seem to have been observed before.

2. Let I be a subinterval of $(0,1)$. By $(m,q)\,I$, m an integer, we denote a subdivision of I into smaller intervals in the following way. The subintervals cover I and have lengths (from left to right): $e^{-m}, e^{-q}, e^{-m-1}, e^{-q}, \ldots, e^{-q}, e^{-2m}$. We assume that m and q are so chosen that this actually gives a covering of I, and we speak of the m-intervals and the q-intervals. We shall construct E applying this kind of subdivision on intervals, and we shall each time let the m q-intervals of length e^{-q} belong to the complement of E.

Let us assume that we have applied the above method n times and in this way obtained the set E_n of m-intervals. Let μ_n be a distribution of unit mass with constant density on each interval of E_n and let $u_n(x)$ be the corresponding potential. Let I be the interval of E_n to be subdivided. We distribute the mass $\mu_n(I)/(m+1)$ uni-

[1] For definitions see [4], pp. 114 ff.

formly on each arising m-interval but do not change μ_n on the rest of E_n, and we call the corresponding distribution μ_{n+1} and potential $u_{n+1}(x)$. It is obvious that

$$u_{n+1}(x) \to u_n(x), \quad m \to \infty, \quad \text{uniformly on } E_n - I. \tag{2}$$

Also, if we put

$$u_k'(x) = \int_I \log \left| \frac{1}{x-t} \right| d\mu_k(t),$$

$$u_n(x) - u_n'(x) = u_{n+1}(x) - u_{n+1}'(x). \tag{3}$$

By (2) and (3) and the maximum principle it is sufficient to give an estimate of $u_{n+1}'(x)$ for x belonging to an m-interval of I. Since the m-intervals have lengths $\geq e^{-2m}$, the following estimate holds:

$$u_{n+1}'(x) \leq (m+1)^{-1} \mu_n(I)\, 2 e^{2m} \int_0^{\frac{1}{2}e^{-2m}} \log \frac{1}{t}\, dt + u_n'(x)$$

$$\leq 2\mu_n(I) + u_n'(x) + O\left(\frac{1}{m}\right).$$

Hence

$$\overline{\lim_{m=\infty}}\, u_{n+1}'(x) \leq 2\mu_n(I) + u_n'(x), \quad x \in m\text{-interval of } I. \tag{4}$$

3. We now construct the set E in the following way. We make an arbitrary division of (0,1) by use of the operation (m_1, q_1) and get the m-intervals $I_1, I_2, \ldots, I_{m_1+1}$. Each interval I_k carries the mass $1/(m_1+1)$. On I_1 we use the operation (m_2, q_2), $m_2 > 2m_1$, and we choose m_2 so large that the potential $u_2(x)$ on $I_2, \ldots, I_{m_1+1}$ increases by less than a given positive number. On I_2 we use (m_3, q_3), $m_3 > 2m_2$, and make an analogous requirement concerning $u_3(x)$. Finally we have subdivided all intervals I_ν, each time choosing $m_{\nu+1} > 2m_\nu$. On all the m-intervals of (0,1) that have now been constructed, we perform in succession the same kind of subdivision, and this process is then continued indefinitely. The (m, q)'s are so chosen that the sum of the increases of $u_{n+1}(x)$, $x \notin I_n$, for all the resulting potentials is uniformly bounded. The set of points, not belonging to any q-interval during this process, constitutes our set E.

Let us now choose a point x in E. It belongs to an infinity of m-intervals, I_1^*, $I_2^*, \ldots, I_n^*, \ldots$, where I_{n+1}^* is an m-interval resulting from I_n^* under the operation (m_n^*, q_n^*). It is now easy to see that the sequence of potentials $u_n^*(x)$ is bounded at this point. The increases from subdivisions of other intervals than I_ν^* have already been dealt with; the increases under the operations (m_n^*, q_n^*) are by (4) bounded by the series const. $\sum_1^\infty \frac{1}{m_n^*}$, $m_{n+1}^* > 2m_n^*$, and hence bounded. From this follows that $u_n^*(x)$ is bounded on E. It is obvious that $\mu_n(e)$ is a convergent sequence of set functions converging to a distribution $\mu(e)$ on E. The potential corresponding to μ, $u(x)$, is bounded on E and hence, by the maximum principle, everywhere. We have thus proved that E has positive capacity.

We now turn the attention to the Hausdorff measure of E. Let $h(r)$ satisfy condition (1). Let us consider a set of m-intervals, $\omega_1, \omega_2, \ldots, \omega_p$, which cover E. We can assume their lengths $e^{-s_1}, \ldots, e^{-s_p}$ to be $< e^{-s}$, where s is arbitrarily large. Furthermore—and this is the crucial point of our construction—all the s_i are different from each other. Hence

$$\sum_{i=1}^{p} h(e^{-s_i}) \leq \sum_{\nu=s+1}^{\infty} h(e^{-\nu}) < \int_0^{e^{-s}} \frac{h(r)}{r}\, d\,r.$$

The last expression is arbitrarily small and we have proved that E has vanishing h-measure. Let us summarize our result in the following theorem.

Theorem 1. *There exists a closed set of positive capacity such that its Hausdorff measure vanishes for every measure function $h(r)$ for which (1) holds.*

4. In the other direction the following theorem of Erdös-Gillis [2] holds. It is an improvement of a classical result of Lindeberg.

Theorem 2. *If a set E has finite Hausdorff measure with respect to* $(\log(1/r))^{-1}$, *then its capacity vanishes.*

Remark: The construction in Theorem 1 easily yields that Theorem 2 fails for any function $h(r)$ such that $h(r)\log(1/r)\to 0$.

Let us suppose that E has positive capacity and let μ be a distribution with bounded energy integral:

$$\int_E\int_E \log\left|\frac{1}{x-y}\right| d\mu(x)\,d\mu(y) = \int_0^{\infty} \log\frac{1}{r}\, d_r \int_E \mu(r;x)\,d\mu(x) < \infty,$$

where $\mu(r;x)$ is the value of μ for the circle $|z-x|<r$. From the last formula it follows that there is a positive, decreasing function $K(r)$, such that the corresponding integral with $\log(1/r)$ replaced by $K(r)$ also converges and $K(r)\log(1/r)\to\infty$, $r\to 0$. This can be written

$$\int_E\int_E K(|x-y|)\,d\mu(x)\,d\mu(y) < \infty,$$

and it follows that, for a restriction μ_1 of μ to a suitable closed subset E_1 of E, $\mu_1(e)\not\equiv 0$, the potential

$$\int_{E_1} K(|x-y|)\,d\mu_1(y) \tag{5}$$

is bounded, $\leq V$, on E_1. Let $C_1, C_2, \ldots, C_n$ be open circles of diameters $l_\nu \leqq \varepsilon$ which cover E_1. We have for $x_\nu \in C_\nu \cap E_1$

$$K(l_\nu)\cdot\mu_1(C_\nu) \leqq \int_{E_1} K(|x_\nu - y|)\,d\mu_1(y) \leq V.$$

Hence
$$0 < \mu_1(E_1) \leq \sum_{\nu=1}^{n}\mu_1(C_\nu) \leqq V\sum_{\nu=1}^{n} K(l_\nu)^{-1}.$$

Since $K(l_\nu)\log(1/l_\nu) \geqq A(\varepsilon)$, $A(\varepsilon)\to\infty$, $\varepsilon\to 0$, we finally get

$$\sum_1^n \left(\log\frac{1}{l_\nu}\right)^{-1} \geqq \frac{\mu_1(E_1)}{V}A(\varepsilon),$$

and the assertion follows.

5. Let us finally observe that the above method can be used to prove the following theorem.

Theorem 3. *If a set E has positive capacity then there exists a uniformly continuous potential of a distribution of unit mass on E. Hence there exists a uniformly continuous harmonic function in the complement of E.*

The last statement follows if we divide E into two disjoint closed subsets of positive capacity and form the difference between the two uniformly continuous potentials which correspond to the two subsets. The situation should be compared with the analogous problem for analytic function [1]. There the existence of bounded and of uniformly continuous functions is not equivalent.

For the proof of Theorem 3, we simply construct the largest convex minorant $H(r)$ of $K(r)$. It is obvious that $H(r)\log(1/r)\to\infty$, $r\to 0$. Since by (5)

$$\int_{E_1} H(|x-y|)\,d\mu_1(y)\le V \tag{6}$$

on E_1, it follows from the maximum principle for $H(r)$ that (6) holds everywhere. The logarithmic potential of μ_1 is then evidently uniformly continuous.

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