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On measurement of velocity by Pitot tube

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With 4 figures in the text

1. We know several paradoxal results from theoretical hydrodynamics which are incompatible with practical experience. Obviously such a paradox can be explained by an unpermitted simplification in the theorizer's assumptions. If e.g. all boundary conditions in a problem are stationary or the boundary has some symmetry properties, it is plausible to assume the same character for the solution of the hydrodynamical equations. Sometimes it is a good approximation to neglect the viscosity, e.g. by calculation of the velocity outside an airfoil in a homogeneous flow, but the supposition is an over-simplification if we seek the resistance of the body (d'Alembert paradox). For further examples of this kind I refer to a famous book by GARRETT BIRKHOFF [1]. Here I shall discuss a problem where the effects of viscosity are problematical.

A real fluid is viscous, and this fact may cause accumulation of fluid in "wakes". In some cases a wake may have a rather well defined boundary zone which on idealization to non-viscous fluid tends to a surface, called a "free" boundary, where the velocity is discontinuous. Sometimes the wake is bounded by a turbulent "mixing zone", and the turbulence produces motions in the wake, which are practically inaccessible for theoretical analysis. Evidently it is difficult to predict the existence of wakes, and without a condition of stability, we get an infinite number of solutions to a given problem. On account of the difficulty of surveying the stability problem for all conceivable cases, we must in general supplement the theoretical speculations with experimental experiences.

2. A long, straight, circular tube with thin walls is closed by a wall inside the tube. The tube is immersed in an incompressible fluid, and at a great distance from the end of the tube the flow is homogeneous, stationary and parallel to the tube axis. The velocity is so great that we may, as a first approximation, neglect the viscosity.

If no wakes were accumulated the calculation of the flow (Fig. 1) should be a classical problem for harmonic functions. In reality we may expect that fluid is accumulated in the tube and eventually forms a stable wake before the wall. Perhaps it is plausible that the free boundary has rotational symmetry about the tube axis and forms a peak on the axis (Fig. 2). We shall not try to make a stability analysis of this flow, but it is obvious that internal friction will break down the peak at an arbitrary small deviation from the symmetric arrangement.

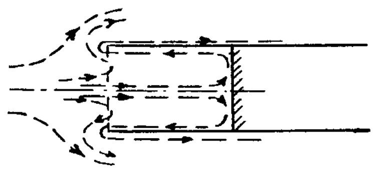

Fig. 1.

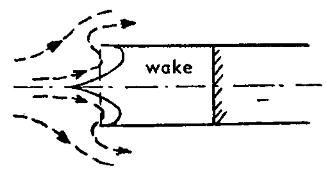

Fig. 2.

In order to establish the real behaviour of this flow the following experiment was carried out. The tube, made of glass and filled with coloured water, was immersed in a tank with flowing water and suitably arranged. A stationary state was quickly attained and it was found that the wake was not symmetric about the tube axis but apparently symmetric about a plane through the axis (Fig. 3). Small disturbances of the direction of the tube caused a rotation of the symmetry plane round the axis.

It seems to me that the formation of the wake continues until the free boundary reaches the front edge of the tube. If the free boundary extended beyond the opening of the tube we should obtain a peak like that in Fig. 2 and this peak might be broken down by the turbulence that always exists in the surrounding flow.

3. We study the two-dimensional case, obtained by substituting for the tube two thin, parallel plates of infinite extension (Fig. 4). At a great distance from the edges the flow is parallel to the plates. We assume that the formation of a stable two-dimensional wake is completed and the free boundary reaches the edge of one plate. We introduce curtesian coordinates x, y according to Fig. 4. Let $\bar{u}=u\hat{x}+v\hat{y}$ be the velocity vector and put $z=x+iy$, $\tau=u-iv$. If $\varphi(z)$ is the real velocity potential, $\bar{u}=\operatorname{grad}\varphi$, and $\psi(z)$ is the conjugate harmonic function to φ, the "stream function", then $W=\varphi+i\psi$ is an analytic function, regular in the flow region A_z outside the wake. Further we know that $\tau=W'(z)$.

We can assume that $\psi=0$ on the free boundary and $\tau=1$ at $z=-\infty$. According to Bernoulli's theorem the velocity is constant on the free boundary.

In the W-plane, the region A_z is represented "schlicht" on a region A_W, which is the W-plane cut along the positive real axis, and in the τ-plane "schlicht" on the region

$$A_\tau=\{\tau : I\,\tau<0,\ |\tau|>k\}$$

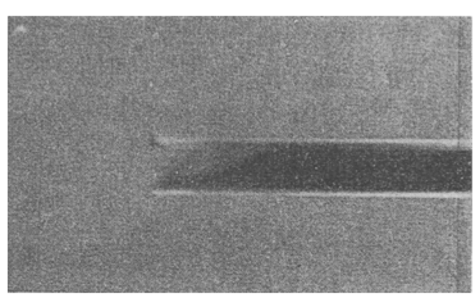

Fig. 3.

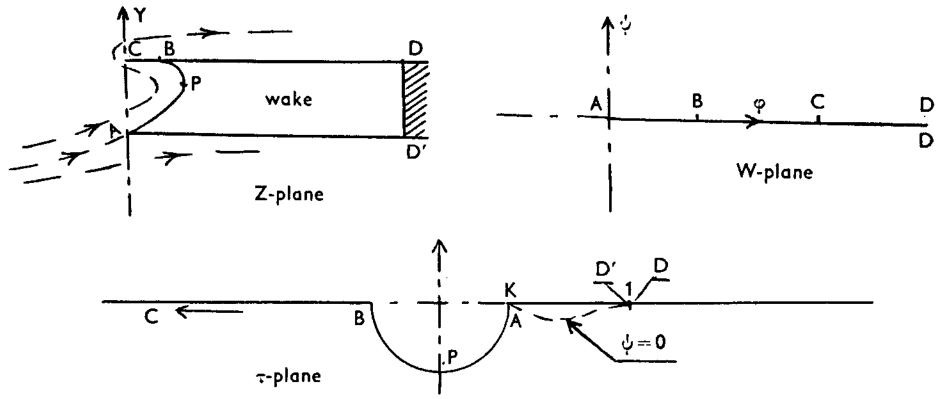

Fig. 4.

where k is the constant velocity on the free boundary. The detailed correspondence of the boundaries is clear from Fig. 4.

The mapping $A_\tau \to A_W$ is given by

$$W(\tau) = K \frac{(\tau - k)^4}{(\tau - 1)^2 (\tau - k^2)^2}$$

where K is a positive constant. Hence follows

$$\left.\begin{aligned} z(\tau) = \int_k^\tau \tau^{-1} W'(\tau)\, d\tau = K \left\{ \frac{(\tau-k)^4}{\tau(\tau-1)^2(\tau-k^2)^2} + \frac{\tau-k}{k\tau} + \frac{1-k}{(1+k)^2} \cdot \frac{\tau-k}{1-\tau} + \right. \\ + \frac{1-k}{k(1+k)^2} \cdot \frac{\tau-k}{\tau-k^2} + 2\frac{k(2+k)}{(1+k)^3}(1-\tau)\log\frac{1-\tau}{1-k} + 2\frac{(1-k)^2}{k^2}\log\frac{\tau}{k} - \\ \left. - 2\frac{(1-k)(1+2k)}{k^2(1+k)^3}\log\frac{\tau-k^2}{k(1-k)} \right\}. \end{aligned}\right\} \quad (1)$$

At the edge C we have $\tau = \infty$ and $z = z_C = iH$, where H is the distance between the plates. We obtain from (1)

$$z_C = K \left\{ \frac{2(1+k^2)}{k(1+k)^2} - 2\frac{k(1-k)(2+k)}{(1+k)^3}\log\frac{1}{k} - 2\frac{(1-k)^2}{k^2}\log\frac{1}{1-k} + \right. \\ \left. + 2\pi i \frac{k(1-k)(2+k)}{(1+k)^3} \right\}.$$

From $R z_C = 0$ follows $k = 0.3270$ and hence $H = 1.377\ K$. The free boundary is given by $z = z(k e^{i\beta})$, $0 \geq \beta \geq \pi$. Especially if $\beta = -\frac{\pi}{2}$ we get the point z_P where the distance from the opening to a point on the free boundary has a maximum, and we get $z_P = (0.896 + 0.713 \cdot i) H$.

4. According to Bernoulli's theorem we get the pressure in the wake

$$p_{\text{wake}} = p_0 + \frac{\varrho}{2}(1-k^2) = p_0 + \frac{\varrho}{2}\cdot 0.8931,$$

p_0 being the pressure at $z = -\infty$ and ϱ the density of the fluid.

The tube studied in section 2 might, for instance, be the tube in a Pitot head for velocity measurement, in which case it is desired to measure the stagnation pressure. The two-dimensional case studied in section 3 indicates that, if the pressure in the wake is erroneously taken equal to the stagnation pressure $= p_0 + \frac{\varrho}{2} v^2$, where v is the velocity at $z = -\infty$, the velocity obtained will be about 5 per cent too low. In the three-dimensional case it is therefore to be expected that there will be an error of at least one or two per cent in the velocity determined.

In a lecture in 1950, Professor ARNE BEURLING, Uppsala, treated some hydrodynamic problems connected with wakes, and it was at that lecture the author got the idea for this study.

REFERENCE

1. GARRETT BIRKHOFF, Hydrodynamics, a Study in Logic, Fact and Similitude. Princeton University Press 1950.

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