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A note on an inequality

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The following is a supplement to an earlier paper [9], where we have given a "triangular condition" which the exponents must fulfill in order that an inequality

$$\int_0^\infty x^\alpha |f(x)|^\beta dx \le K\left(\int_0^\infty x^{\alpha_1}|f|^{\beta_1} dx\right)^{\varkappa_1}\left(\int_0^\infty x^{\alpha_2}|f|^{\beta_2} dx\right)^{\varkappa_2} \tag{1}$$

should hold true. A number of authors have discussed the best possible value for K.

In this note we observe that the simple method we used in a special case in the cited note [9] gives—in the general case also—the extremal function and so the value of K.

By the transformations $x \to x^p$, $|f| \to x^q |f|^r$ we first bring (1) into the form

$$\int_0^\infty |f|\, dx \le K(\alpha, \beta_1, \beta_2)\left(\int_0^\infty x^\alpha |f|^{\beta_1} dx\right)^{\varkappa_1}\left(\int_0^\infty x^\alpha |f|^{\beta_2} dx\right)^{\varkappa_2}. \tag{2}$$

For brevity we here do not consider the simplest case $\beta_1 = \beta_2$. To satisfy the conditions in [9] we must have $\alpha \ge 0$; but $\alpha = 0$ corresponds to Hölder's inequality and thus is of no interest in this connection.

Set $\varphi = |f|$ and form

$$L(\varphi) = \varphi - \lambda x^\alpha \varphi^{\beta_1} - \mu x^\alpha \varphi^{\beta_2}, \tag{3}$$

where λ and μ are positive parameters at our disposal. Take the maximum of $L(\varphi)$ for fixed x and variable φ; let it be attained for $\varphi = \varphi_0(x)$. If we put $\int_a^b x^\alpha \varphi_0^{\beta_1} dx = A_1$ and $\int_a^b x^\alpha \varphi_0^{\beta_2} dx = A_2$, it is evident that among all functions φ giving the same values to these integrals the function $\varphi_0(x)$ gives the maximum of $\int_a^b \varphi\, dx$. The maximum of $L(\varphi)$ for fixed x is either $0 = L(0)$ or positive; in the latter case φ_0 is a solution of the equation

$$1 - \lambda \beta_1 x^\alpha \varphi_0^{\beta_1 - 1} - \mu \beta_2 x^\alpha \varphi_0^{\beta_2 - 1} = 0 \quad \text{or} \quad \lambda \beta_1 \varphi_0^{\beta_1 - 1} + \mu \beta_2 \varphi_0^{\beta_2 - 1} = x^{-\alpha}. \tag{4}$$

In the case $1 \leq \beta_1 < \beta_2$ (3) and (4) yield that the extremal function $\varphi_0(x)$ is continuous and steadily decreasing from $+\infty$ to 0 in $(0, \infty)$. After having taken φ_0 as the independent variable, the integrals of (2), and thus the constant K, can be explicitly expressed in terms of Γ-functions. The expression for K is given by Levin [10].

We then consider the remaining case $\beta_1 < 1 < \beta_2$. As is seen by a glance at (3), it holds true for x not too large that $L(\varphi)$ (for φ in $(0, \infty)$) first decreases from 0 to a negative minimum, then increases up to a positive maximum $L(\varphi_0)$ and then decreases again. Thus $\varphi_0(x)$ is the largest of the two solutions of eq. (4) which now exist. For x exceeding a certain value x_0, easy to calculate, the maximum of $L(\varphi)$ is obtained for $\varphi = 0$. The extremal function $\varphi_0(x)$ thus steadily decreases in the interval $(0, x_0)$ from $+\infty$ to the positive value $\varphi_0(x_0)$; at x_0 there is discontinuity, since $\varphi_0 = 0$ for $x > x_0$. After the same substitutions as in the first case, it is again possible to obtain an explicit but now complicated expression for K.

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