

## On linear estimates defined by a continuous weight function

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With 1 figure in the text

### 1. Introduction

Let  $X$  be a random variable having the cumulative distribution function (cdf)  $F\left(\frac{x-\mu}{\sigma}\right)$ , where  $F(y)$  is a known cdf and where  $\mu$  and  $\sigma$  are unknown parameters. From  $n$  independent observations of  $X$ , we obtain by rearranging them in order of magnitude the order statistics  $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$ . A linear estimate of  $\mu$  or  $\sigma$  is a systematic statistic

$$\theta^* = \sum_{\nu=1}^n h_{\nu}^{(n)} x_{(\nu)}.$$

Assuming the means and the covariances of the standardized order statistics  $y_{(\nu)} = \frac{x_{(\nu)} - \mu}{\sigma}$  to be known, the constants  $h_{\nu}^{(n)}$  may be determined so as to make  $\theta^*$  unbiased and of minimum variance. (LLOYD [4].) In a few special cases explicit solutions of the constants  $h_{\nu}^{(n)}$  are obtained (DOWNTON [3], SARHAN [6]). In the general case much numerical work is needed.

In this note, we put  $h_{\nu}^{(n)} = \frac{1}{n} h\left(\frac{\nu}{n+1}\right)$ , where  $h(u)$  is a continuous function, depending on  $F(y)$  only, and we shall prove that  $h(u)$  may often be chosen so as to make the corresponding estimate  $\theta^*(h)$  consistent and asymptotically efficient.

In section 2 we study the properties of the statistic  $\theta^*(h)$  for an unspecified function  $h(u)$ . In the next section we solve a special minimum problem, and the solution is applied to the determination of the best weight function in section 4, where we also make the transformations necessary for practical use. The last sections deal with the asymptotic efficiency of the estimates obtained and some applications.

### 2. Asymptotic expressions for the mean and the variance of a certain type of linear systematic statistics

Let  $y_{(1)} < y_{(2)} < \dots < y_{(n)}$  be the order statistics from a sample of  $n$  independent observations of the random variable  $Y$ . We assume that  $Y$  has the cdf  $F(y)$ , a continuous probability density function (pdf)  $f(y)$  and a finite second moment.

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If  $h(u)$  is a real function defined on  $(0, 1)$  we are going to study the random variable

$$Z_n = \frac{1}{n} \sum_{\nu=1}^n h\left(\frac{\nu}{n+1}\right) y_{(\nu)}. \quad (1)$$

In the sequel we need the following

**Lemma 1.** *If  $h(u)$  satisfies the condition*

**C 1:**  *$h(u)$  is bounded and has at least four derivatives, which are bounded in  $(0, 1)$ ,*  
then

$$E(Z_n) = \int_0^1 G(u) h(u) du + O(n^{-1}) \quad (2)$$

$$D^2(Z_n) = \frac{2}{n} \int_0^1 \int_0^v u(1-v) G'(u) h(u) G'(v) h(v) du dv + O(n^{-2}), \quad (3)$$

where  $G(u)$  is the inverse function

$$F^{-1}(u) \text{ of } F(y). \quad (4)$$

**Proof of (2).** The expectation of  $y_{(\nu)}$  can be written

$$E[y_{(\nu)}] = n \int_0^1 G(u) \binom{n-1}{\nu-1} u^{\nu-1} (1-u)^{n-\nu} du \quad (\text{cf. CRAMÉR [1], p. 370}).$$

Thus from (1) the expectation of  $Z_n$  is

$$E(Z_n) = \int_0^1 G(u) S(u) du, \quad (5)$$

where we have (substituting  $\nu+1$  for  $\nu$ )

$$S(u) = \sum_{\nu=0}^{n-1} h\left(\frac{\nu+1}{n+1}\right) \binom{n-1}{\nu} u^{\nu} (1-u)^{n-1-\nu}. \quad (6)$$

By C 1 we may expand  $h\left(\frac{\nu+1}{n+1}\right)$  in a Taylor series about  $u_n = \frac{(n-1)u+1}{n+1}$

$$h\left(\frac{\nu+1}{n+1}\right) = \sum_{k=0}^3 \frac{[\nu-(n-1)u]^k}{k!(n+1)^k} h^{(k)}(u_n) + \frac{[\nu-(n-1)u]^4}{4!(n+1)^4} h^{(4)}(\theta_\nu), \quad (7)$$

where  $\theta_\nu$  depends on  $n, u$  and  $\nu$  and satisfies  $\frac{1}{n+1} < \theta_\nu < \frac{n}{n+1}$ .

If we change the order of summation in (6) and (7), we shall need the sums

$$b_k = \sum_{\nu=0}^{n-1} [\nu - (n-1)u]^k \binom{n-1}{\nu} u^\nu (1-u)^{n-1-\nu}$$

which are the well-known central moments of a binomial distribution, whence

$$b_0 = 1 \quad b_1 = 0 \quad b_2 = (n-1)u(1-u) \quad b_3 = (n-1)u(1-u)(1-2u) \quad \text{and} \\ b_4 = 3(n-1)^2 u^2 (1-u)^2 + (n-1)u(1-u)[1-6u(1-u)] < \frac{3}{16}(n-1)^2.$$

Combining (6) and (7) we thus have

$$S(u) = h(u_n) + \frac{(n-1)u(1-u)}{2(n+1)^2} h''(u_n) + \frac{(n-1)u(1-u)(1-2u)}{6(n+1)^3} h'''(u_n) + R_1,$$

where, supposing  $|h^{(4)}(u)| < M_4$  (cf. C 1),

$$|R_1| = \left| \sum_{\nu=0}^{n-1} \frac{[\nu - (n-1)u]^4}{4!(n+1)^4} h^{(4)}(\theta_\nu) \binom{n-1}{\nu} u^\nu (1-u)^{n-1-\nu} \right| \leq \frac{M_4}{4!(n+1)^4} b_4 \leq \frac{M_4}{128n^2}.$$

Renewed Taylor expansion about  $u$  gives

$$S(u) = h(u) + \frac{1}{n} [(1-2u)h'(u) + \frac{u(1-u)}{2} h''(u)] + R_2 \tag{8}$$

where  $R_2$  depends on the upper bounds of the derivatives  $h''(u)$ ,  $h'''(u)$  and  $h^{(4)}(u)$  and is of the form  $O(n^{-2})$ .

As  $S(u)$  is bounded, and as  $\int_0^1 G(u) du = E(Y)$  exists, eq. (2) follows a fortiori.

The approximation of  $E(Z_n^2)$  is a straightforward generalisation of the above method to two dimensions, and will not be carried out here. Subtracting the square of  $E(Z_n)$  as given by (5) and (8) we get

$$D^2(Z_n) = \frac{2}{n} \int_0^1 G(v) \frac{d}{dv} [(1-v)h(v)] \int_0^v G(u) \frac{d}{du} [uh(u)] du dv + \\ + \frac{1}{n} \int_0^1 G^2(u) h^2(u) du + O(n^{-2}).$$

By three successive integrations by part, the right integral is cancelled, the integrated parts vanish, and we get eq. (3).

Introduce in (3) the symmetrical kernel

$$k(u, v) = \begin{cases} u(1-v) & 0 \leq u \leq v \leq 1 \\ (1-u)v & 0 \leq v \leq u \leq 1 \end{cases} \tag{9}$$

$$\therefore D^2(Z_n) = \frac{1}{n} \int_0^1 \int_0^1 k(u, v) G'(u) h(u) G'(v) h(v) du dv + O(n^{-2}). \tag{10}$$

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**Corrollary.** If  $Z_{1n} = \frac{1}{n} \sum_{v=1}^n h_1\left(\frac{v}{n+1}\right) y_{(v)}$  and  $Z_{2n} = \frac{1}{n} \sum_{v=1}^n h_2\left(\frac{v}{n+1}\right) y_{(v)}$ , where  $h_1(u)$  and  $h_2(u)$  satisfy C 1, then

$$\mathbf{Cov} (Z_{1n}, Z_{2n}) = \frac{1}{n} \int_0^1 \int_0^1 k(u, v) G'(u) h_1(u) G'(v) h_2(v) du dv + O(n^{-2}). \quad (11)$$

Putting  $Z_n = Z_{1n} + Z_{2n}$  we have

$$\mathbf{D}^2(Z_n) = \mathbf{D}^2(Z_{1n}) + 2 \mathbf{Cov} (Z_{1n}, Z_{2n}) + \mathbf{D}^2(Z_{2n}).$$

Applying (10) to both members of the equation, the corrollary follows.

### 3. On a special minimum problem associated with the asymptotically best linear estimates

Let  $g_1(u)$  and  $g_2(u)$  be continuous functions in  $(0, 1)$  and  $k(u, v)$  be the kernel defined in (9).

Put

$$I(\varphi, \psi) = \int_0^1 \int_0^1 k(u, v) \varphi(u) \psi(v) du dv.$$

**Lemma 2.** If  $g_1(u)$  and  $g_2(u)$  satisfy the conditions for  $g(u)$

**C 2:**  $g(0) = g(1) = 0$

$g''(u)$  exists, is continuous and finite in the open interval  $(0, 1)$

$|g''(u)| = O\left(\frac{1}{u}\right)$  as  $u \rightarrow 0$ ,  $|g''(u)| = O\left(\frac{1}{1-u}\right)$  as  $u \rightarrow 1$

then there exists one unique function  $\varphi(u) = \varphi_0(u)$ , that makes  $I(\varphi, \varphi)$  a minimum subject to

$$\int_0^1 \varphi(u) g_1(u) du = \alpha_1, \quad \int_0^1 \varphi(u) g_2(u) du = \alpha_2. \quad (12)$$

**Proof.** From the calculus of variations we quote, that if a solution exists, it must be of the form

$$\varphi_0(u) = a_1 \varphi_1(u) + a_2 \varphi_2(u), \quad (13)$$

where  $\varphi_1(u)$  and  $\varphi_2(u)$  satisfy

$$\int_0^1 k(u, v) \varphi_i(v) dv = g_i(u) \quad (i=1, 2) \quad (14)$$

and where  $a_1$  and  $a_2$  are determined by (12).

Inserting (9) in (14) and differentiating twice with respect to  $u$  we obtain a necessary condition for  $\varphi_i(u)$ :

$$\varphi_i(u) = -g_i''(u). \quad (15)$$

By partial integration of the left member of (14) with  $\varphi_i(u) = -g_i''(u)$ , the integrated terms vanish by C 2, and thus (15) is a unique solution of (14).

In the sequel we need the integrals

$$\begin{aligned} d_{\mu\nu} = I(\varphi_\mu, \varphi_\nu) &= \int_0^1 \varphi_\mu(u) \int_0^1 k(u, v) \varphi_\nu(v) \, du \, dv = \int_0^1 \varphi_\mu(u) g_\nu(u) \, du = \\ &= \int_0^1 -g_\mu''(u) g_\nu(u) \, du = \int_0^1 g_\mu'(u) g_\nu'(u) \, du, \quad (\mu, \nu = 1, 2) \end{aligned} \quad (16)$$

The successive equalities follow from equations (14), (15) and (C 2).

Introducing the matrix notations

$$\alpha = (\alpha_1, \alpha_2) \quad \mathbf{a} = (a_1, a_2) \quad \mathbf{D} = \begin{Bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{Bmatrix}, \quad D = |\mathbf{D}| > 0 \quad (17)$$

we get from (12), (13) and (16)

$$\mathbf{D} \mathbf{a} = \alpha, \quad \therefore \mathbf{a} = \mathbf{D}^{-1} \alpha. \quad (18)$$

Having solved  $\mathbf{a}$ , we have easily

$$I(\varphi_0, \varphi_0) = \alpha' \mathbf{D} \mathbf{a} = \alpha' \mathbf{D}^{-1} \alpha. \quad (19)$$

That  $I(\varphi_0, \varphi_0)$  is a minimum value can be proved in the usual way.

#### 4. Asymptotically best linear estimates of location and scale parameters

$X$  is a random variable with cdf  $F\left(\frac{x-\mu}{\sigma}\right)$ , where  $\mu$  and  $\sigma$  are unknown parameters, and where  $F(y)$  has a finite second moment, a continuous pdf  $f(y)$  and an inverse function  $G(u) = F^{-1}(u)$ . Having observed the order statistics  $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$ , we want to estimate an arbitrary linear combination  $\theta = \alpha_1 \mu + \alpha_2 \sigma$  of the parameters  $\mu$  and  $\sigma$  by an estimate of the form

$$\theta^*(h) = \frac{1}{n} \sum_{v=1}^n h\left(\frac{v}{n+1}\right) x_{(v)}, \quad (19)$$

where the function  $h$  is to be conveniently chosen. As the inverse of  $F\left(\frac{x-\mu}{\sigma}\right)$  is  $\mu + \sigma G(u)$  we obtain, when Lemma 1 is applicable,

$$E[\theta^*(h)] = \int_0^1 [u + \sigma G(u)] h(u) \, du + O(n^{-1}) \equiv M(h) + O(n^{-1}) \quad (20)$$

$$\begin{aligned} D^2[\theta^*(h)] &= \frac{\sigma^2}{n} \int_0^1 \int_0^1 k(u, v) G'(u) h(u) G'(v) h(v) \, du \, dv + O(n^{-2}) \equiv \\ &\equiv \frac{\sigma^2}{n} V(h) + O(n^{-2}), \end{aligned} \quad (21)$$

where we have defined  $M(h)$  and  $V(h)$ , and where  $k(u, v)$  is defined by (9).

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The "best" estimate of this kind is defined by a function  $h = h_0$ , that makes  $V(h)$  a minimum when  $M(h) \equiv \theta = \alpha_1 \mu + \alpha_2 \sigma$ .

By introduction of the functions

$$\varphi(u) = G'(u) h(u) \quad g_1(u) = \frac{1}{G'(u)} \quad g_2(u) = \frac{G(u)}{G'(u)} \quad (22)$$

we get identically the problem of section 3.

As  $G'(u) > 0$ , we have the following

**Theorem 1.** *Subject to the conditions C 1 and C 2 of the lemmata 1 and 2, the asymptotically best estimate is defined by the function*

$$h_0(u) = -\frac{a_1}{G'(u)} \frac{d^2}{du^2} \frac{1}{G'(u)} - \frac{a_2}{G'(u)} \frac{d^2}{du^2} \frac{G(u)}{G'(u)} \equiv a_1 h_1(u) + a_2 h_2(u)$$

where  $a_1$  and  $a_2$  are defined by (16), (18) and (22).

The asymptotical variance of  $\theta^*(h)$  is

$$D^2[\theta^*(h)] = \frac{\sigma^2}{n} (\alpha' \mathbf{D}^{-1} \alpha) + O(n^{-2}),$$

with  $\alpha$  and  $\mathbf{D}$  from (16), (17) and (22).

As a rule the function  $G(u)$  is difficult to handle, and therefore in practical applications it is easier first to calculate the functions  $H_i(y) \equiv h_i[F(y)]$ , and then to calculate  $h_i(u)$  from  $F(y)$ .

The transformations, given without proof, are elementary, but somewhat tedious.

Introduce the auxiliary functions

$$\gamma_1(y) = -\frac{d \log f(y)}{dy} \quad \gamma_2(y) = -1 - y \frac{d \log f(y)}{dy}. \quad (23)$$

Then we have for the matrix  $\mathbf{D}$

$$d_{\mu\nu} = \int_{-\infty}^{\infty} \gamma_1(y) \gamma_2(y) f(y) dy. \quad (24)$$

For the functions  $H_i$  we get

$$H_1(y) = \gamma_1'(y) \quad H_2(y) = \gamma_2'(y) \quad (25)$$

$$H_0(y) = a_1 H_1(y) + a_2 H_2(y), \quad \text{where as before} \quad (26)$$

$$\alpha = \mathbf{D}^{-1} \alpha.$$

For the special cases of  $\mu^*$  and  $\sigma^*$  we obtain

when  $\theta \equiv \mu: H_0(y) \equiv B(y) = \frac{d_{22}}{D} H_1(y) - \frac{d_{21}}{D} H_2(y)$  (27)

$$D^2(\mu^*) = \frac{\sigma^2}{n} \frac{d_{22}}{D} + 0(n^{-2});$$

when  $\theta \equiv \sigma: H_0(y) \equiv C(y) = -\frac{d_{12}}{D} H_1(y) + \frac{d_{22}}{D} H_2(y)$  (28)

$$D^2(\sigma^*) = \frac{\sigma^2}{n} \frac{d_{11}}{D} + 0(n^{-1}).$$

Note. If  $\mu$  or  $\sigma$  is known, we may from the estimate (19) subtract the known part of (20), and get only one condition on  $h_0(u)$ . We then obtain the estimates and weight functions

$$\mu_1^* = \frac{1}{n} \sum_{v=1}^n b_1 \left( \frac{v}{n+1} \right) x_{(v)} - \sigma \int_0^1 G(u) b_1(u) du, \tag{29}$$

where

$$B_1(y) \equiv b_1[F(y)] = \frac{1}{d_{11}} H_1(y), \quad D^2(\mu_1^*) = \frac{\sigma^2}{n} \frac{1}{d_{11}} + 0(n^{-2})$$

$$\sigma_1^* = \frac{1}{n} \sum_{v=1}^n c_1 \left( \frac{v}{n+1} \right) x_{(v)} - \mu \int_0^1 c_1(u) du, \tag{30}$$

where

$$C_1(y) \equiv c_1[F(y)] = \frac{1}{d_{22}} H_2(y), \quad D^2(\sigma_1^*) = \frac{\sigma^2}{n} \frac{1}{d_{22}} + 0(n^{-2}).$$

The estimate  $\sigma_1^*$  is asymptotically equivalent with the estimate

$$\sigma_2^* = \frac{1}{n} \sum_{v=1}^n c_1 \left( \frac{v}{n+1} \right) (x_{(v)} - \mu). \tag{30 a}$$

### 5. On the asymptotical efficiency of the estimates

If  $g(x; \theta_1 \dots \theta_k)$  is a pdf depending on  $k$  parameters  $\theta_1 \dots \theta_k$  the Cramér-Rao theorem (cf. CRAMÉR [2]) gives the minimum variance of any regular unbiased estimate  $\theta_i^*$  or the minimum generalized variance of any joint estimate  $(\theta_{i_1}, \theta_{i_2}, \dots, \theta_{i_r})$  in terms of the matrix  $D$  defined by

$$d_{\mu\nu} = E \left[ \frac{\partial \log g}{\partial \theta_\mu} \frac{\partial \log g}{\partial \theta_\nu} \right] \quad (\mu, \nu = 1, \dots, k). \tag{31}$$

In our case  $g(x; \theta_1, \theta_2) = \frac{1}{\theta_2} f\left(\frac{x - \theta_1}{\theta_2}\right)$ , and it is easy to prove, that the elements  $d_{\mu\nu}$  of (24) are identical with those of (31). A simple comparison of the variances then proves

**Theorem 2.** *If the conditions C 1 and C 2 of the lemmata 1 and 2 are satisfied, then*

- 1)  $(\mu^*, \sigma^*)$  as defined by (19), (27) and (28) are asymptotically joint efficient;
- 2)  $\mu_1^*$  and  $\sigma_1^*$  defined by (29) and (30) are asymptotically efficient;
- 3)  $\mu^*$  and  $\sigma^*$  are each of asymptotical minimum variance if the other is regarded as an unknown nuisance parameter. They are each asymptotically efficient if the matrix element

$$\begin{aligned} d_{12} = d_{21} &= \int_{-\infty}^{\infty} \frac{f'(y)}{f(y)} \left[ 1 + y \frac{f'(y)}{f(y)} \right] f(y) dy = (\text{by C 2}) \\ &= \int_{-\infty}^{\infty} y \frac{f'^2(y)}{f(y)} dy = 0. \end{aligned}$$

(This applies e.g. to all symmetric distributions.)

## 6. Applications

a. *Student's distribution with m d. of f.* Put

$$F(y) = S_m(y) \quad \text{and} \quad f(y) = s_m(y) = c_m \cdot \left( 1 + \frac{y^2}{m} \right)^{-\frac{m+1}{2}},$$

where

$$c_m = \frac{1}{\sqrt{m\pi}} \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}.$$

From eq. (23) we get

$$\gamma_1(y) = \frac{m+1}{m} \cdot \frac{y}{1 + \frac{y^2}{m}} \quad \gamma_2(y) = \frac{m+1}{m} \cdot \frac{y^2}{1 + \frac{y^2}{m}} - 1. \quad (32)$$

From (24) we have

$$d_{\mu\nu} = \int_{-\infty}^{\infty} \gamma_\mu(y) \gamma_\nu(y) f(y) dy.$$

Introducing  $1 + \frac{y^2}{m} = \frac{1}{\xi}$  the integrals  $d_{\mu\nu}$  are easily evaluated, and we get

$$d_{11} = \frac{m+1}{m+3} \quad d_{12} = d_{21} = 0 \quad d_{22} = \frac{2m}{m+3}.$$

From (25) we get



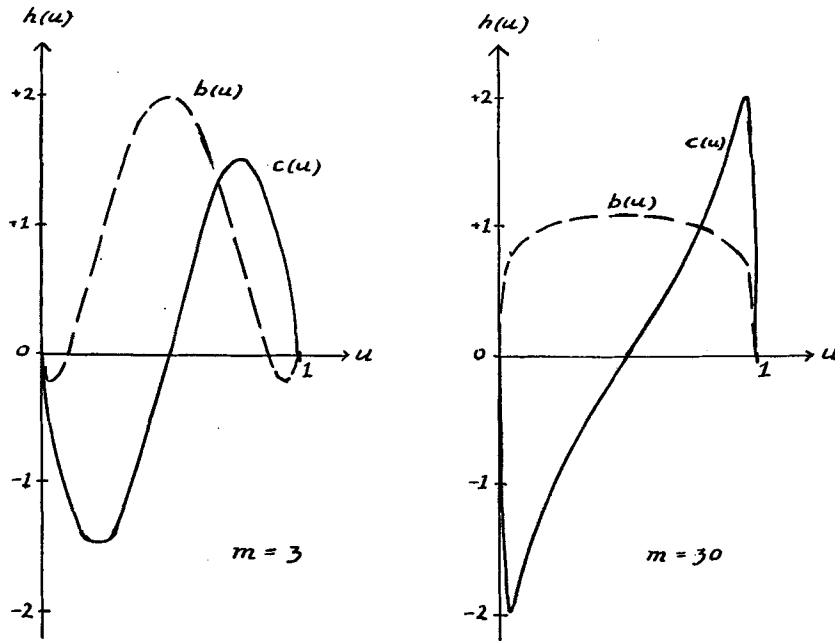


Fig. 1. Weight functions  $b(u)$  of  $\mu^*$  (---) and  $c(u)$  of  $\sigma^*$  (—) for Student's distribution with 3 and 30 degrees of freedom.

$$H_1(y) = \frac{m+1}{m} \cdot \frac{1 - \frac{y^2}{m}}{\left(1 + \frac{y^2}{m}\right)^2}$$

$$H_2(y) = 2 \frac{m+1}{m} \cdot \frac{y}{\left(1 + \frac{y^2}{m}\right)^2}$$

and finally from (27) and (28)

$$\text{for } \mu^*: B(y) = \frac{m+3}{m} \cdot \frac{1 - \frac{y^2}{m}}{\left(1 + \frac{y^2}{m}\right)^2}, \quad D^2(\mu^*) = \frac{\sigma^2}{n} \frac{m+3}{m+1} + 0(n^{-2})$$

$$\text{for } \sigma^*: C(y) = \frac{(m+3)(m+1)}{m^2} \cdot \frac{y}{\left(1 + \frac{y^2}{m}\right)^2}, \quad D^2(\sigma^*) = \frac{\sigma^2}{n} \frac{m+3}{2m} + 0(n^{-2}).$$

Introducing in  $B(y)$  and  $C(y)$   $u = S_m(y)$  we get the weight functions  $b(u)$  and  $c(u)$  to be used in (19).

In Fig. 1  $b(u)$  and  $c(u)$  are shown for  $m=30$  and  $m=3$ .

For  $m=1$  or  $2$ , the second moment of  $X$  is infinite, and therefore the finite expression (21) does not represent the variance of  $\mu^*$  or  $\sigma^*$ . (Leaving out the 4 extreme observations we can, however, obtain consistent estimates for  $\mu$  and  $\sigma$ .) For  $m \rightarrow \infty$ ,  $b(u) \rightarrow 1$  in the interior of  $(0, 1)$  and hence  $\bar{x}$  is an asymptotically efficient estimate of the mean of a normal distribution, which is no overstatement.

When  $m \rightarrow \infty$ ,  $|c(u)|$  tends to infinity at  $u=0$  or  $1$ . Thus we cannot apply Lemma 1 and are not justified to use as weight function for  $\sigma^*$  the limit  $\Phi^{-1}(u)$ , as  $m \rightarrow \infty$ , of  $c(u)$  for  $m$  d. of f.

b. *Distributions of the Pearson type III.* With  $f(y) = \frac{1}{\Gamma(\lambda)} y^{\lambda-1} e^{-y}$  for  $0 \leq y$ , we obtain formally  $H_1(y) = \frac{\lambda-1}{y^2}$ ,  $H_2(y) = 1$ ,

$$d_{11} = \frac{1}{\lambda-2}, \quad d_{21} = d_{12} = 1, \quad d_{22} = \lambda.$$

As soon as  $\mu$  is unknown, we cannot avoid the use of  $H_1(y)$  or  $h_1(u)$ , which do not satisfy condition C 1. But in the common case where  $\mu$  is known, we could apply (30) or (30 a) and obtain the asymptotically efficient estimate

$$\sigma_2^* = \frac{1}{n} \sum_{v=1}^n c_1 \left( \frac{v}{n+1} \right) [x_v - \mu], \quad \text{where } c_1(u) = \frac{1}{\lambda}$$

$$\sigma_2^* = \frac{\bar{x} - \mu}{\lambda} \quad \text{with the variance}$$

$$D^2(\sigma_2^*) = \frac{\sigma^2}{n\lambda} + O(n^{-2}).$$

### 7. Remarks

It should be stressed that the conditions C 1 and C 2 are not necessary conditions. The essential part of C 2 is the condition  $\lim_{u \rightarrow 0} g_1(u) = \lim_{u \rightarrow 1} g_2(u) = 0$  as  $u \rightarrow 0$  or  $u \rightarrow 1$ . In terms of  $f(y)$  this implies that  $\lim_{y \rightarrow 0} y f(y) = \lim_{y \rightarrow 1} y f(y) = 0$  as  $y$  tends to the endpoints of the distribution. If these endpoints are finite, this is an essential condition, reminding of the conditions for the existence of a regular estimate. (Cf. CRAMÉR [2], p. 485, ex. 4.)

The condition C 1 appears to be unnecessary restrictive. If it were possible to allow  $h(u)$  to tend to infinity at  $u=0$  or  $1$  and still have a remainder term in (3) which tends to zero not too slowly, some of the difficulties in the applications could be removed.

Among all the problems left open, the most important one seems to be the conditions on  $h$  in order that  $\theta^*$  should be asymptotically normal. If such a condition could be derived, we should have a restricted class of BAN (Best Asymptotically Normal) estimates (cf. NEYMAN [5]).

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