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Directional operators and radial functions on the plane

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Abstract. Let $E\subset S^1$ be a set with Minkowski dimension $d(E)<1$. We consider the Hardy–Littlewood maximal function, the Hilbert transform and the maximal Hilbert transform along the directions of E. The main result of this paper shows that these operators are bounded on $L^p_{\text{rad}}(\mathbf{R}^2)$ for $p>1+d(E)$ and unbounded when $p<1+d(E)$. We also give some end-point results.

1. Introduction

Given a one-dimensional operator T and a direction $u\in S^{n-1}$, we define the directional operator T_u in $\mathbf{R}^n$ as follows: for a smooth function f and given $x\in\mathbf{R}^n$, set $x=\langle x,u\rangle u+x'$ and $g(t)=f(tu+x')$; then, $T_uf(x)=Tg(\langle x,u\rangle)$. If T is either bounded in $L^p(\mathbf{R})$ or of weak type (p,p), the same is true for T_u in $\mathbf{R}^n$ with the same norm (in particular, the norm does not depend on u).

In this paper we shall consider directional operators in the plane. Associated to the angle θ which determines the direction given by the point $e^{i\theta}=(\cos\theta,\sin\theta)$ in S^1 we define the maximal operator M_θ, the Hilbert transform H_θ and the maximal Hilbert transform H^*_θ as follows,

$$M_\theta f(x)=\sup_{h>0}\frac{1}{2h}\int_{-h}^{h}|f(x-te^{i\theta})|\,dt,$$

$$H_\theta f(x)=\lim_{\varepsilon\to0}\frac{1}{\pi}\int_{|t|>\varepsilon}\frac{f(x-te^{i\theta})}{t}\,dt:=\lim_{\varepsilon\to0}H_{\theta,\varepsilon}f(x)$$

and

$$H^*_\theta f(x)=\sup_{\varepsilon>0}|H_{\theta,\varepsilon}f(x)|.$$

Thus, all of them are bounded operators in $L^p(\mathbf{R}^2)$, $1<p<\infty$, and of weak type $(1,1)$ with norms independent of θ.

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Let E be a closed subset of S^1. Associated to E we define the operators

$$\mathfrak{M}_E f(x) = \sup_{\theta\in E} M_\theta f(x),$$
$$\mathcal{H}_E f(x) = \sup_{\theta\in E} |H_\theta f(x)|,$$
$$\mathcal{H}^*_E f(x) = \sup_{\theta\in E} H^*_\theta f(x)$$

(we shall write $\theta\in E$ instead of $e^{i\theta}\in E$). We are interested in the boundedness properties of these operators as functions of E. When E is a lacunary set, that is, $E=\{\theta_j\}$ is a decreasing sequence converging to 0 such that $\limsup(\theta_{j+1}/\theta_j)<1$, then $\mathfrak{M}_E$ is bounded in $L^p(\mathbf{R}^2)$, $1<p\le\infty$ ([NSW]), but if E has positive measure, $\mathfrak{M}_E$ is unbounded for $p<\infty$.

We consider radial functions. Denote by $L^p_{\mathrm{rad}}(\mathbf{R}^2)$ the set of radial functions belonging to L^p. We associate to each E a number, $d(E)$, as follows:

$$d(E) = \limsup_{\delta\to 0^+} \frac{\log \mathcal{N}(\delta)}{-\log\delta},$$

where $\mathcal{N}(\delta)$ is the minimum number of closed intervals of length δ needed to cover E. If E has positive Lebesgue measure, $d(E)=1$. If E has zero measure and we write $S^1\setminus E$ as the union of a sequence of disjoint open intervals, $\{I_j\}$, then

$$d(E) = \inf\Big\{\alpha\ge 0 : \sum_j |I_j|^\alpha < \infty\Big\}$$

where $|I_j|$ denotes the length of I_j. Our main theorem is the following:

Theorem 1.
(i) $\mathfrak{M}_E$ *is bounded on* $L^p_{\mathrm{rad}}(\mathbf{R}^2)$ *if* $p>1+d(E)$ *and unbounded if* $p<1+d(E)$.
(ii) *The same holds for* $\mathcal{H}_E$ *and* $\mathcal{H}^*_E$ *if* $d(E)<1$.

There is no standard name for $d(E)$ and one can find the names Minkowski dimension, box-counting dimension, entropy dimension and logarithmic density among others. There are also many different equivalent definitions (twelve of them appear in [T], including those mentioned above). The number $d(E)$ is an upper bound for the Hausdorff dimension of E but they are different in general. Nevertheless, they coincide for self-similar sets like the Cantor ternary set (see [F, p. 118]).

When E is lacunary, $d(E)=0$ and the operators are bounded in $L^p_{\mathrm{rad}}(\mathbf{R}^2)$ for all $p>1$; if E is the set of directions given by the sequence $k^{-\gamma}$, then $d(E)=1/(\gamma+1)$ and we prove boundedness if and only if $p>(\gamma+2)/(\gamma+1)$. We remark that acting on general functions, the operators are unbounded for all $1<p<\infty$. If E is the Cantor ternary set, $d(E)=\log 2/\log 3$ and the boundedness on $L^p_{\mathrm{rad}}(\mathbf{R}^2)$ holds if and only if $p>1+\log 2/\log 3$. In this case the question for general functions remains open. When $E=S^1$ the boundedness for $p>2$ is proved in [CHS2]. A result slightly more restrictive than our Theorem 2, but still enough to deduce Theorem 1, was claimed in [CHS1], but it was obtained using an erroneous estimate. The correction in [CHS2] does not supply a proof to our theorem.

2. Maximal operators

Due to the result in [CHS2] we can limit ourselves to considering the case $d(E)<1$. Decompose $S^1\setminus E$ as the union of $\{I_j\}$ as in the introduction and assume that there are N_k intervals of length $2^{-k-1}<|I_j|\le 2^{-k}$. With this notation,

$$d(E)=\inf\left\{\alpha\ge 0:\sum_{k=1}^{\infty}N_k 2^{-k\alpha}<\infty\right\}.$$

To prove the necessity of $p\ge 1+d(E)$, take as f the characteristic function of the unit ball. On $2^l\le|x|\le 2^{l+1}$, $l>0$, $\mathfrak{M}_E$ is of the order 2^{-l} at least on the union of $\sum_{k=1}^{l}N_k$ rectangles of sides 1×2^l which correspond to the directions defined by end-points of intervals of length greater than 2^{-l-1}. Thus, if $\mathfrak{M}_E$ is bounded in $L^p_{\mathrm{rad}}(\mathbf{R}^2)$ we have

$$\sum_{l=1}^{\infty}2^{-lp}\left(\sum_{k=1}^{l}N_k\right)2^l\le C,$$

which is the same as

$$\sum_{k=1}^{\infty}N_k\sum_{l=k}^{\infty}2^{l(1-p)}\le C.$$

Hence, $p-1\ge d(E)$.

Theorem 1 will be a consequence of the following:

Theorem 2. *If $p<2$ and $\sum_j|I_j|^{p-1}(\log(1/|I_j|))^{p-1+\varepsilon}<\infty$ for some $\varepsilon>0$, then $\mathfrak{M}_E$ is bounded in $L^p_{\mathrm{rad}}(\mathbf{R}^2)$.*

Let ϕ be a nonnegative one-dimensional Schwartz function such that $\phi(0)\neq 0$ and set $\phi_j(t)=2^{-j}\phi(2^{-j}t)$. Define the operator

$$\widetilde{M}_\theta f(x)=\sup_{j\in\mathbf{Z}}\left|\int_{\mathbf{R}}\phi_j(t)f(x-te^{i\theta})\,dt\right|.$$

Thus, M_θ and $\widetilde{M}_\theta(|f|)$ are equivalent in the sense that their quotient is bounded between two strictly positive constants, so that we can study $\widetilde{\mathfrak{M}}_E f=\sup_{\theta\in E}\widetilde{M}_\theta f$ instead of $\mathfrak{M}_E$. The following lemma is used in a crucial way.

Lemma 3. *Let $f\in L^p_{\mathrm{rad}}(\mathbf{R}^2)$ and $1<p\leq 2$. Then*

$$\|\widetilde{M}_\theta f-\widetilde{M}_{\theta'}f\|_p\leq C_p|\theta-\theta'|^{1/p'}\|f\|_p,$$

where $|\theta-\theta'|$ denotes the length of the shortest arc joining $e^{i\theta}$ and $e^{i\theta'}$.

Proof. Consider first the case $p=2$, the others following by interpolation. For the sake of simplicity, we can assume that $\theta'=-\theta$ (that is, the directions θ and θ' are symmetric with respect to the OX_1-axis). For every function g we have

$$\begin{aligned}|\widetilde{M}_\theta g(x)-\widetilde{M}_{\theta'}g(x)|&\leq\sup_j\left|\int\phi_j(t)\{g(x-te^{i\theta})-g(x-te^{-i\theta})\}\,dt\right|\\&:=\sup_j|T^j_\theta g(x)|:=T_\theta g(x)\leq\left[\sum_j|T^j_\theta g(x)|^2\right]^{1/2}.\end{aligned}$$

The Fourier transform of $T^j_\theta g(x)$ has an explicit expression,

$$(T^j_\theta g)\hat{}(\xi)=[\hat\phi(2^j\langle e^{i\theta},\xi\rangle)-\hat\phi(2^j\langle e^{-i\theta},\xi\rangle)]\hat g(\xi).$$

Let χ_θ be the characteristic function of the (double) sector centred at the OX_2-axis of width 4θ and P_θ the multiplier operator associated to χ_θ. For a radial function f, the inequality

$$\|P_\theta f\|_2\leq C|\theta|^{1/2}\|f\|_2,$$

is a consequence of Plancherel's theorem and the fact that $\hat f$ is also radial. Hence,

$$\|T_\theta(P_\theta f)\|_2\leq\|M_\theta(P_\theta f)\|_2+\|M_{\theta'}(P_\theta f)\|_2\leq C|\theta|^{1/2}\|f\|_2.$$

Therefore, we are reduced to estimate

$$\begin{aligned}\|T_\theta(f-P_\theta f)\|_2^2&\leq\left\|\left(\sum_j|T_{j,\theta}(f-P_\theta f)|^2\right)^{1/2}\right\|_2^2\\&=\int\sum_{j=-\infty}^{\infty}[\hat\phi(2^j\langle e^{i\theta},\xi\rangle)-\hat\phi(2^j\langle e^{-i\theta},\xi\rangle)]^2|\hat f(\xi)|^2(1-\chi_\theta(\xi))\,d\xi.\end{aligned}$$

Using the bound

$$|\hat\phi(2^j\langle e^{i\theta},\xi\rangle)-\hat\phi(2^j\langle e^{-i\theta},\xi\rangle)|\leq C\frac{2^j|\xi_2\sin\theta|}{(2^j|\xi_1\cos\theta|)^2+1},$$

valid when $1-\chi_\theta(\xi)\neq 0$, we add the series in j and integrate in polar coordinates (ϱ,τ) to get

$$C\int_0^\infty|\hat f(\varrho)|^2\varrho\int_0^{\pi/2-2\theta}\tan^2\tau\,d\tau\tan^2\theta\,d\varrho\leq C|\theta|\|f\|_2^2.$$

The result for $1<p<2$ is an almost straightforward application of the Marcinkiewicz interpolation theorem. There is only a minor technical problem: $\widetilde{M}_\theta-\widetilde{M}_{\theta'}$ is not sublinear. It is not difficult to go around this failure. We have defined a new operator T_θ satisfying $|\widetilde{M}_\theta g - \widetilde{M}_{-\theta} g| \le T_\theta g$, and T_θ is sublinear, of weak type $(1,1)$ and $\|T_\theta g\|_2 \le C|\theta|^{1/2}\|g\|_2$. We can perform the interpolation over T_θ. □

We use the preceding lemma to prove

Lemma 4. *Let $\theta_1, \theta_2, \dots, \theta_N$ be N directions in an interval $I \subset S^1$. Then, for radial f we have*

$$\Big\|\sup_j |\widetilde{M}_{\theta_j} f - \widetilde{M}_{\theta_N} f|\Big\|_2 \le C(\log N)|I|^{1/2}\|f\|_2$$

and

$$\Big\|\sup_j |\widetilde{M}_{\theta_j} f - \widetilde{M}_{\theta_N} f|\Big\|_p \le C N^{2/p-1}|I|^{1/p'}\|f\|_p, \quad 1<p<2.$$

Proof. Without loss of generality we can assume that $N=2^l-1$ (adding points if necessary) and label the directions consecutively. Decompose the index set $\mathcal{J} = \{1, 2, \dots, 2^l-1\}$ into l subsets containing 1, 2, 2^2, ..., 2^{l-1} points as follows: $\mathcal{J}_1 = \{2^{l-1}\}$, $\mathcal{J}_m = \mathcal{J}_{m-1} \pm 2^{l-m}$ $(m \ge 2)$. This definition associates to each point θ_j with $j \in \mathcal{J}_m$ a unique point $\theta_{\sigma(j)} \in \mathcal{J}_{m-1}$ in a canonical way. Now, we have

$$\sup_{j\in\mathcal{J}_m} |\widetilde{M}_{\theta_j} f - \widetilde{M}_{\theta_N} f| \le \sup_{j\in\mathcal{J}_m} |\widetilde{M}_{\theta_j} f - \widetilde{M}_{\theta_{\sigma(j)}} f| + \sup_{j\in\mathcal{J}_m} |\widetilde{M}_{\theta_{\sigma(j)}} f - \widetilde{M}_{\theta_N} f|,$$

and using Lemma 3

$$\begin{aligned}
&\Big\|\sup_{j\in \bigcup_{i\le m}\mathcal{J}_i} |\widetilde{M}_{\theta_j} f - \widetilde{M}_{\theta_N} f|\Big\|_p \\
&\qquad \le \Big\|\sup_{j\in \bigcup_{i\le m-1}\mathcal{J}_i} |\widetilde{M}_{\theta_j} f - \widetilde{M}_{\theta_N} f|\Big\|_p + \Big(\sum_{j\in\mathcal{J}_m} |\theta_j - \theta_{\sigma(j)}|^{p/p'}\Big)^{1/p} \|f\|_p.
\end{aligned}$$

Applying Hölder's inequality, the last term is bounded by $2^{(m-1)(2/p-1)}|I|^{1/p'}\|f\|_p$. The lemma follows by induction. □

Proof of Theorem 2. By a limiting argument we only need to consider the directions determined by the end-points of the intervals I_j. Denote by E_k the set of end-points of intervals of length $2^{-k-1} < |I| \le 2^{-k}$ which are not end-points of intervals of length greater than 2^{-k}. The connected components of the complement in S^1 of the union of these latter intervals (of length greater than 2^{-k}) will be denoted $J_{k,l}$; let $E_{k,l}$ be the subset of E_k contained in $J_{k,l}$ and $N_{k,l}$ the cardinality of $E_{k,l}$. Then $\sum_l N_{k,l} \le 2N_k$ and $\sum_l |J_{k,l}| \le \sum_{j\ge k} N_j 2^{-j}$.

If θ_l is anyone of the end-points of $J_{k,l}$ (which are in $\bigcup_{j<k} E_j$), according to Lemma 4, we have for $p<2$

$$\Big\| \sup_{\theta\in E_{k,l}} |\widetilde{M}_\theta f - \widetilde{M}_{\theta_l} f| \Big\|_p \le C N_{k,l}^{2/p-1} |J_{k,l}|^{1/p'} \|f\|_p.$$

Since

$$\sup_l \sup_{\theta\in E_{k,l}} |\widetilde{M}_\theta f(x)| \le \sup_l |\widetilde{M}_{\theta_l} f(x)| + \bigg(\sum_l \sup_{\theta\in E_{k,l}} |\widetilde{M}_\theta f(x) - \widetilde{M}_{\theta_l} f(x)|^p \bigg)^{1/p},$$

we deduce

$$\Big\| \sup_{\theta\in\bigcup_{j\le k} E_j} |\widetilde{M}_\theta f| \Big\|_p \le \Big\| \sup_{\theta\in\bigcup_{j\le k-1} E_j} |\widetilde{M}_\theta f| \Big\|_p + C\bigg(\sum_l N_{k,l}^{2-p} |J_{k,l}|^{p-1} \bigg)^{1/p} \|f\|_p.$$

Applying Hölder's inequality to the sum in l with exponents $1/(2-p)$ and $1/(p-1)$, the coefficient of $\|f\|_p$ in the last term can be bounded by

$$C\bigg(\sum_l N_{k,l} \bigg)^{(2-p)/p} \bigg(\sum_l |J_{k,l}| \bigg)^{(p-1)/p} = C N_k^{2/p-1} \bigg(\sum_{j\ge k} N_j 2^{-j} \bigg)^{1/p'}.$$

By induction, $\sup_{\theta\in E} \widetilde{M}_\theta f$ is bounded in $L^p_{\mathrm{rad}}(\mathbf{R}^2)$ if

$$\sum_{k=1}^{\infty} N_k^{2/p-1} \bigg(\sum_{j\ge k} N_j 2^{-j} \bigg)^{1/p'} < \infty. \tag{1}$$

Assume that $\sum |I_j|^{p-1} (\log(1/|I_j|))^b < \infty$ for some $b>0$, i.e.,

$$\sum_{k=1}^{\infty} N_k 2^{-k(p-1)} k^b < \infty. \tag{2}$$

To check that $\sum_{k=1}^\infty N_k^{2/p-1} (\sum_{j\ge k} N_j 2^{-j})^{1/p'}$ is finite, apply first Hölder's inequality with exponents $p/(2-p)$ and $p/(2p-2)$, after including $2^{-k(p-1)(2-p)/p} k^{b(2-p)/p}$ in the sum and the same with opposite exponent. We get a factor like (2) and we are left with

$$\sum_{k=1}^{\infty} \bigg(\sum_{j\ge k} N_j 2^{-j} \bigg)^{1/2} 2^{k(2-p)/2} k^{-b(2-p)/2(p-1)}.$$

Applying the Cauchy–Schwarz inequality, we get

$$\bigg(\sum_{k=1}^{\infty} \bigg(\sum_{j\ge k} N_j 2^{-j} \bigg) 2^{k(2-p)} k^b \bigg)^{1/2} \bigg(\sum_{k=1}^{\infty} k^{-b/(p-1)} \bigg)^{1/2}.$$

The second term is finite if and only if $b>p-1$. Rearranging the first term we again get (2). $\square$

3. Hilbert transforms

The counterexample to show the necessary condition is the same as above. To prove the theorem for Hilbert transforms we need first two lemmas like in the case of maximal functions. The first one is:

Lemma 5. *If f is a radial function, then*

$$\|H_\theta f - H_{\theta'} f\|_p \le C_p |\theta-\theta'|^{1/p'} \|f\|_p, \qquad 1<p<2,$$
$$\|H_\theta f - H_{\theta'} f\|_p \le C_p |\theta-\theta'|^{1/p} \left(\log \frac{2\pi}{|\theta-\theta'|}\right)^{1-2/p} \|f\|_p, \quad 2\le p<\infty.$$

Proof. The multiplier of the operator $H_\theta - H_{\theta'}$ is supported in a (double) sector of width $|\theta-\theta'|$. This gives the L^2-result. For $1<p<2$ we interpolate with the weak type $(1,1)$ estimate, and for $p>2$ with a uniform estimate for big p_0. □

Applying this lemma we get the following counterpart of Lemma 4:

Lemma 6. *Let $\theta_1, \theta_2, \dots, \theta_N$ be N directions in an interval $I\subset S^1$. Then, for radial f we have*

$$\Big\|\sup_j |H_{\theta_j} f - H_{\theta_N} f|\Big\|_p \le C N^{2/p-1} |I|^{1/p'} \|f\|_p, \quad 1<p<2$$

and

$$\Big\|\sup_j |H_{\theta_j} f - H_{\theta_N} f|\Big\|_p \le C(\log N)|I|^{1/p} \sup_{i<j}\left(\log \frac{2\pi}{|\theta_i-\theta_j|}\right)^{1-2/p} \|f\|_p, \quad 2\le p<\infty.$$

The proof of the theorem now follows the same scheme as before for $1+d(E)<p<2$. But we cannot interpolate with $p=\infty$ as in the maximal function case and the range $p>2$ also needs a proof. Minor changes using the second estimate in Lemma 6 lead to the following substitute of condition (1):

$$\sum_{k=1}^{\infty} \log N_k \left(\sum_{j\ge k} N_j 2^{-j}\right)^{1/p} k^{1-2/p} < \infty \tag{3}$$

which is satisfied if $d(E)<1$ since $N_j \le C2^{j\alpha}$ for some $\alpha<1$.

To treat the maximal Hilbert transform we apply Cotlar's inequality. Usually it is written as

$$H^* f(x) \le C[M(Hf)(x) + Mf(x)]$$

with M the Hardy–Littlewood maximal function, but it is easy to see ([S, p. 67]) that the first term can be taken as $\widetilde{M}(Hf)$. Then, for directional operators

$$H_\theta^* f(x) \le C[\widetilde{M}_\theta(H_\theta f)(x) + M_\theta f(x)]$$

and

$$\mathcal{H}_E^* f(x) \le C\Big[\sup_{\theta\in E} \widetilde{M}_\theta(H_\theta f)(x) + \mathfrak{M}_E f(x)\Big].$$

We shall need a new modification of the key lemma:

Lemma 7. *Let f be a radial function. Then*

$$\|\widetilde{M}_\theta(H_\theta f) - \widetilde{M}_{\theta'}(H_{\theta'} f)\|_p \le C_p |\theta-\theta'|^{1/p'} \|f\|_p, \qquad 1<p\le 2$$

$$\|\widetilde{M}_\theta(H_\theta f) - \widetilde{M}_{\theta'}(H_{\theta'} f)\|_p \le C_p |\theta-\theta'|^{1/p} \left(\log \frac{2\pi}{|\theta-\theta'|}\right)^{1-2/p} \|f\|_p, \quad 2<p<\infty.$$

Proof. Again, we only need to prove the case $p=2$. From it, the estimate for $p<2$ is obtained by interpolation with the weak-type $(1,1)$ estimate (which holds because $\widetilde{M}(Hf) \le C(H^* f + Mf)$) and for $p>2$ we interpolate with an estimate for big p_0.

To prove the case $p=2$ write first

$$\widetilde{M}_\theta(H_\theta f) - \widetilde{M}_{\theta'}(H_{\theta'} f) = [\widetilde{M}_\theta(H_\theta f) - \widetilde{M}_\theta(H_{\theta'} f)] + [\widetilde{M}_\theta(H_{\theta'} f) - \widetilde{M}_{\theta'}(H_{\theta'} f)].$$

For the first term we have

$$\|\widetilde{M}_\theta(H_\theta) f - \widetilde{M}_\theta(H_{\theta'}) f\|_2 \le C\|H_\theta f - H_{\theta'} f\|_2 \le C|\theta-\theta'|^{1/2} \|f\|_2$$

and we only need to control the second one. Observe then that in the proof of Lemma 3 we can replace f with Sf where S is a bounded multiplier operator, without modifying the estimate. $\square$

With this lemma we prove the theorem for $\mathcal{H}_E^*$ exactly as we did it for $\mathcal{H}_E$. Notice also that if $p<2$ the condition in Theorem 2 is sufficient for the boundedness of $\mathcal{H}_E$ and $\mathcal{H}_E^*$.

4. Further results

4.1. End-point boundedness

The proof of the necessity in Theorem 1 shows that if the operators are bounded in $L^{1+d(E)}_{\mathrm{rad}}(\mathbf{R}^2)$, then

$$\sum_j |I_j|^{d(E)} < \infty. \tag{4}$$

In the case where E is given by a decreasing sequence θ_j converging to 0, such that $\theta_j - \theta_{j+1}$ is also decreasing (i.e., the intervals are ordered by their lengths), condition (4) is also sufficient. Indeed, in the proof of Theorem 2, $J_{k,l}$ is reduced to just one interval of length $\le N_k 2^{-k}$ and $N_{k,l} = N_k$. Then,

$$\Big\| \sup_{\theta \in E_k} |\widetilde{M}_\theta f - \widetilde{M}_{\theta_k} f| \Big\|_p \le C N_k^{1/p} 2^{-k/p'} \|f\|_p,$$

where θ_k is the direction in $\bigcup_{j<k} E_j$ closest to E_k. Then

$$\Big\| \sup_{\theta \in E} |\widetilde{M}_\theta f| \Big\|_p \le \Big\| \sup_k |\widetilde{M}_{\theta_k} f| \Big\|_p + C \Big(\sum_{k=1}^{\infty} N_k 2^{-k(p-1)} \Big)^{1/p} \|f\|_p.$$

The last sum is bounded for $p = 1 + d(E)$, assuming (4). Moreover $|\theta_k - \theta_{k+1}| \le N_k 2^{-k} \le C 2^{-k(1-d(E))}$ so that $\sum_k |\theta_k - \theta_{k+1}|^\beta < \infty$ for all $\beta > 0$, if $d(E) < 1$ and, therefore, the first term is bounded for $1 < p < \infty$.

4.2. Rearrangement of directions

When E is finite and has N points, $\mathfrak{M}_E$ is bounded in L^p, $1 < p \le \infty$, but then the interesting problem is to study the dependence on N of the L^p norm of the operator. It is known that

$$\|\mathfrak{M}_E f\|_p \le C(\log N + 1)^\alpha \|f\|_p \quad (\alpha = \alpha(p) > 0,\ 2 \le p < \infty)$$

and

$$\|\mathfrak{M}_E f\|_p \le C_p N^{2/p-1} (\log N + 1)^\alpha \|f\|_p \quad (\alpha = \alpha(p) \ge 0,\ 1 < p < 2),$$

if the directions are equidistributed (see [Co1] and [St]). Without this hypothesis it is an open problem to determine the above estimates. When f is radial, no matter what the directions are, both results are given by Lemma 4 and even more, if the N directions are distributed in a small arc I, the norm can be made smaller, more precisely

$$\|\mathfrak{M}_E f\|_2 \le C(1 + (\log N)|I|^{1/2}) \|f\|_2,$$
$$\|\mathfrak{M}_E f\|_p \le C(1 + N^{2/p-1} |I|^{1/p'}) \|f\|_p, \quad 1 < p < 2.$$

An estimate of this type is impossible for general f, a change of variables showing that the norm is independent of the length of the arc in that case.

This observation is sometimes useful in order to show that an operator is unbounded in the general case. In fact, if there are N_k consecutive intervals of length $2^{-k-1}<|I|\leq 2^{-k}$ and the sequence $\{N_k\}$ is unbounded, the associated operator cannot be bounded in L^p for $p<\infty$.

We have shown in Theorem 1 that the boundedness of $\mathfrak{M}_E$, $\mathcal{H}_E$ and $\mathcal{H}_E^*$ on $L^p_{\text{rad}}(\mathbf{R}^2)$ depends only on the length of the intervals in $S^1\setminus E$ (except maybe the end-point). In the general case the distribution of the intervals can modify the boundedness of the operator. More precisely, we can give two sets E and E' such that $S^1\setminus E$ and $S^1\setminus E'$ are decomposed as unions of sequences $\{I_j\}$ and $\{I'_j\}$ such that $|I_j|=|I'_j|$, and $\mathfrak{M}_E$ is bounded in $L^p(\mathbf{R}^2)$, $1<p\leq\infty$, while $\mathfrak{M}_{E'}$ is unbounded if $p<\infty$. To this end, decompose S^1 dyadically and decompose then each dyadic interval into dyadic pieces again. Take as E the set of end-points of these intervals. Rearrange the intervals so that they are ordered according to their length and take as E' the complement of their union. Then $\mathfrak{M}_E$ and $\mathfrak{M}_{E'}$ satisfy the above claim. For the positive result see [SjSj], for the negative one apply the observation in the preceding paragraph.

Added on March 24, 1995. The maximal operators studied in this paper are of restricted weak type in L^p, $p>1$, if and only if there exists a constant C such that $N_k\leq C2^{k(p-1)}$, where N_k denotes the number of intervals I_j whose length is between 2^{-k-1} and 2^{-k} (or equivalently, the minimum number of intervals of length 2^{-k} needed to cover E).

This result is particularly interesting at the end-point $p=1+d(E)$. The counterexample is the same as in Section 2 and to prove the sufficiency decompose the operator into two parts according to the length of the intervals I_j. Then apply to each one of them an L^2 estimate or an L^1 estimate. The result follows by choosing the decomposition in such a way that both terms are of equivalent size.

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