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On a theorem of Baernstein

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Abstract. In the paper [B2], Baernstein constructs a simply connected domain Ω in the plane for which the conformal mapping f of Ω into the unit disc Δ satisfies

$$\int_{\mathbf{R}\cap\Omega} |f'(z)|^p |dz| = \infty,$$

for some $p\in(1,2)$, where $\mathbf{R}$ is the real line.

This gives a counterexample to a conjecture stating that for any simply connected domain Ω in the plane, all the above integrals are finite for any $1<p<2$.

In this paper, we give a conceptual proof of the basic estimate of Baernstein.

1. Introduction

Let us consider the following problem. Let Ω be a simply connected domain and f be the conformal mapping from Ω into the unit disc Δ. Assume that L is a straight line which intersects the domain Ω, Hayman and Wu [HW] showed that for any configuration as above,

$$\int_{L\cap\Omega} |f'(z)|\,|dz| \le C,$$

where C is a universal constant. Later Garnett, Gehring and Jones [GGJ] simplified Hayman and Wu's proof and gave an improved value for the constant C. Fernández, Heinonen and Martio in [FHM] gave another proof of the same result with a better constant $C=4\pi^2$, and a conjecture is offered for the best constant. In the same paper they showed that there exists a positive number p between 1 and 2, such that

$$\int_{L\cap\Omega} |f'(z)|^p |dz| \le C,$$

(1) I would like to thank Professor Albert Baernstein II for his helpful comments and suggestions concerning this work

where C and p are constants independent of the configuration. It is not difficult to see that the line L may be taken to be the real axis $\mathbf{R}$. The question is then for which exponents p is it true that $f'(z)\in L^p(\mathbf{R}\cap\Omega)$, for any f and Ω? Taking Ω to be $\Delta\setminus(-1,0]$ one sees that $f'(z)\in L^2(\mathbf{R}\cap\Omega)$ can fail. Baernstein [B1] conjectured that $f'(z)\in L^p(\mathbf{R}\cap\Omega)$ would be true for any $1<p<2$.

Baernstein in [B2] showed that his own conjecture is not true. He constructed a simply connected domain Ω such that if we consider the conformal mapping f from Ω into the unit disc, there exists a positive value p between 1 and 2, such that,

$$\int_{\mathbf{R}\cap\Omega} |f'(z)|^p|dz| = \infty.$$

We pass to describe briefly the work done by Baernstein in [B2]. His domain Ω is the complement of an infinite tree T clustering to the real line. The fixed aperture at every branching of the tree T is $\frac{1}{3}\pi$.

Let us consider the domain $\Theta=\mathbf{C}\setminus((-\infty,1]\cup(0,e^{i\pi/3}])$, where $(0,e^{i\pi/3}]$ is the segment joining these two points. We are going to call $a=e^{i\pi/3}$, and consider the conformal mappings $F_i(z)$, $i=1,2$; mapping Θ onto the domain $H=\mathbf{C}\setminus(-\infty,0]$, such that $F_1(1)=0$, $F_2(a)=0$ and $\lim_{z\to\infty}|F_i(z)/z|=1$, $i=1,2$.

If we consider,

$$\gamma = \lim_{z\to 1}\left|\frac{F_1(z)}{z-1}\right|, \quad \beta = \lim_{z\to a}\left|\frac{F_2(z)}{z-a}\right|,$$

then Baernstein's theorem states that,

Theorem.

$$\gamma^{1/2}+\beta^{1/2} > \sqrt{2}.$$

In his paper Baernstein proves this result after numerical evidence given to him by Donald Marshall, who computed the values of γ and β using Trefethen's program [T], see also [H, p. 422], for finding parameters for Schwarz–Christoffel transformations. He starts with the 4-place decimal approximation to the parameters given by the computer and confirm by Calculus the validity of the theorem, then mentions that it would be desirable to have a conceptual proof of the theorem.

In this paper, we present such a conceptual proof, in it our main tool is the method of the extremal metric. The idea of how to obtain lower bounds for γ and β using extremal metric was inspired by the paper of Jenkins and Oikawa [JO], in which they obtain a sharp version of Ahlfors' distortion theorem, and then use it to give simpler proofs of some well known results of Hayman.

2. Proof of the theorem

2.1. Estimating $\gamma=|F_1'(1)|$

Let ϱ be a small positive number and consider the discs $D_\varrho^{(1)}=\{z: |z-1|<\varrho\}$, and $D_{1/\varrho}^{(1)}=\{z: |z-1|<1/\varrho\}$. Let $\Theta_\varrho^{(1)}$ be the doubly connected domain

$$\Theta_\varrho^{(1)}=([\Theta\cap D_{1/\varrho}^{(1)}]\setminus\bar{D}_\varrho^{(1)}).$$

Let $H_\varrho^{(1)}$ be the image under $F_1(z)$ of $\Theta_\varrho^{(1)}$, by the normalization properties of the function $F_1(z)$, it is not difficult to show that for any positive ε, there exists a small positive $\varrho(\varepsilon)$ such that,

$$\{z: |z|<(1-\varepsilon)/\varrho(\varepsilon)\}\cap H\subset F_1(D_{1/\varrho(\varepsilon)}^{(1)})\subset\{z: |z|<(1+\varepsilon)/\varrho(\varepsilon)\}\cap H$$

and

$$\{z: |z|<|F_1'(1)|(\varrho(\varepsilon)-\varepsilon)\}\cap H\subset F_1(D_{\varrho(\varepsilon)}^{(1)})\subset\{z: |z|<|F_1'(1)|(\varrho(\varepsilon)+\varepsilon)\}\cap H.$$

Consider now the module problem for the family of curves Γ joining $\partial D_{\varrho(\varepsilon)}^{(1)}$ with $\partial D_{1/\varrho(\varepsilon)}^{(1)}$ in $\Theta_{\varrho(\varepsilon)}^{(1)}$. Using the conformal invariance of the module and the comparison property for the modules, we have that

$$M(\Gamma,\Theta_{\varrho(\varepsilon)}^{(1)})\leq\frac{2\pi}{\ln((1-\varepsilon)/\varrho(\varepsilon)(\varrho(\varepsilon)+\varepsilon)|F_1'(1)|)}.$$

This provides us with an upper bound for the module, our goal is to obtain a lower bound for the same module. For this we consider the conformal mapping $\Phi(z)=\ln(z-1)$,

$$\Phi(z)\colon\Theta_{\varrho(\varepsilon)}^{(1)}\to S_{\varrho(\varepsilon)}^{(1)},$$

where $S_{\varrho(\varepsilon)}^{(1)}$ is the quadrangle in the Figure 2.1a.

Let $\widetilde{\Gamma}$ be the family of curves in $S_{\varrho(\varepsilon)}^{(1)}$ joining the pair of sides opposite to the vertical sides. By the conformal invariance of the module we have the following equality

$$M\big(\Gamma,\Theta_{\varrho(\varepsilon)}^{(1)}\big)=M\big(\bar{\Gamma},S_{\varrho(\varepsilon)}^{(1)}\big),$$

where $\bar{\Gamma}$ is the family of curves in $S_{\varrho(\varepsilon)}^{(1)}$ joining the pair of vertical sides. Since the families of curves $\bar{\Gamma}$ and $\widetilde{\Gamma}$ are conjugate in the quadrangle $S_{\varrho(\varepsilon)}^{(1)}$, we have that

$$M\big(\bar{\Gamma},S_{\varrho(\varepsilon)}^{(1)}\big)=1/M\big(\widetilde{\Gamma},S_{\varrho(\varepsilon)}^{(1)}\big),$$

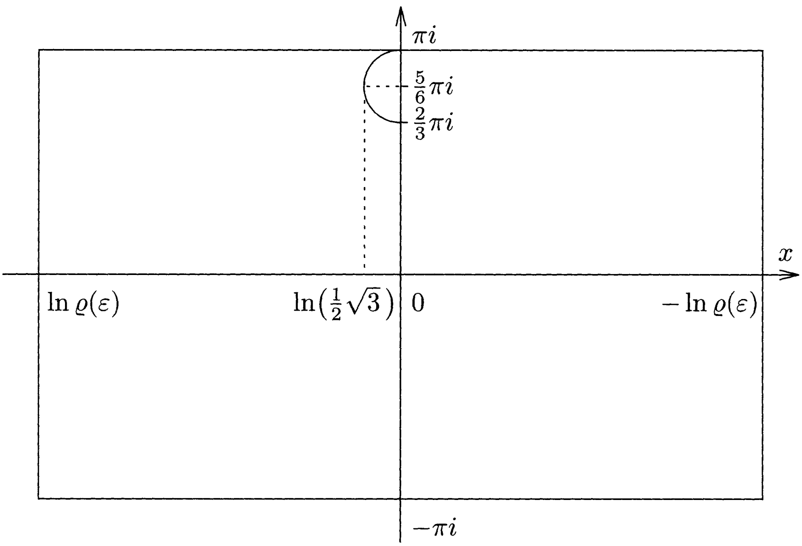

Figure 2.1a

therefore, to obtain a lower bound for $M(\Gamma, \Theta^{(1)}_{\varrho(\varepsilon)})$, we need an upper bound of $M(\widetilde{\Gamma}, S^{(1)}_{\varrho(\varepsilon)})$.

The idea of how to obtain the right upper bound for $M(\widetilde{\Gamma}, S^{(1)}_{\varrho(\varepsilon)})$ was suggested by [JO]. For any value of x in the interval $\ln \varrho(\varepsilon) < x < -\ln \varrho(\varepsilon)$, let $\sigma(x)$ denote the maximal open subinterval of $\mathrm{Re}\{z\} = x$ in $S^{(1)}_{\varrho(\varepsilon)}$ such that the two components of $(S^{(1)}_{\varrho(\varepsilon)} \setminus \sigma(x))$ have the two vertical sides as boundary components. Let $\theta(x)$ denote the length of $\sigma(x)$, $\theta_1(x)$ the length of the part of the segment $\sigma(x)$ below the x-axis, and $\theta_2(x)$ the length of the part above the x-axis. As it can be easily seen, $\theta_1(x) = \pi$ for any x in the interval $\ln \varrho(\varepsilon) < x < -\ln \varrho(\varepsilon)$. For $\theta_2(x)$ we have

$$\theta_2(x) = \begin{cases} \pi, & \text{if } \ln \varrho(\varepsilon) < x < \ln\left(\frac{1}{2}\sqrt{3}\right), \\ \theta_2(x), & \text{if } \ln\left(\frac{1}{2}\sqrt{3}\right) \le x < 0, \\ \pi, & \text{if } 0 \le x < -\ln \varrho(\varepsilon). \end{cases}$$

Let the interval $\left[\ln\left(\frac{1}{2}\sqrt{3}\right), 0\right)$ be divided into n consecutive half closed subintervals $\Delta_j = \left[\ln\left(\frac{1}{2}\sqrt{3}\right)(1 - j/n), \ln\left(\frac{1}{2}\sqrt{3}\right)(1 - (j+1)/n)\right)$, $j = 0, \dots, n-1$ of equal length, and for each $j = 0, \dots, n-1$ let

$$\theta^{(s)}_{2,j} = \min_{t \in \Delta_j} \theta_2(t),$$

and define for any $x \in \left[\ln\left(\frac{1}{2}\sqrt{3}\right), 0\right)$, $\theta^{(s)}_2(x) = \theta^{(s)}_{2,j}$ if $x \in \Delta_j$, $j = 0, \dots, n-1$. It is clear

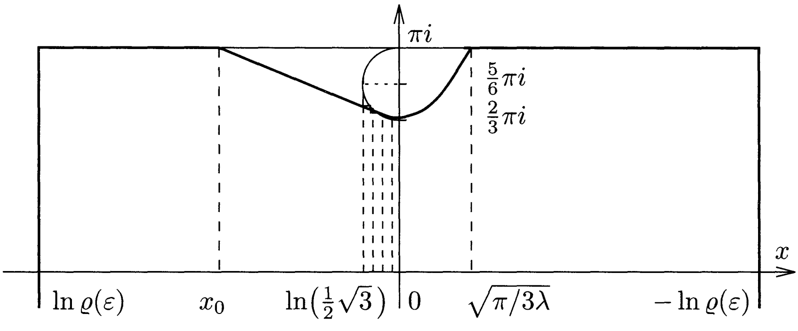

Figure 2.1b. $x_0=\ln(\frac{1}{2}\sqrt{3})+\Delta_1-2\pi+\theta_{2,2}^{(s)}$.

that such minimum exists and $\theta_2^{(s)}$ is a step function on the interval $[\ln(\frac{1}{2}\sqrt{3}),0)$. At the right end point $\bar{x}$ of any interval Δ_j the step function $\theta_2^{(s)}(x)$ has a negative jump, then we draw the ray given by $\bar{x}-\lambda$, $\theta_2^{(s)}(\bar{x})+\lambda$; $\lambda\geq 0$; $j=0\dots,n-1$. The lower envelope of these rays and the locus $y=\theta_2^{(s)}(x)$ defines on the interval $[\ln(\frac{1}{2}\sqrt{3}),0)$ a piecewise continuously differentiable function $\theta_2^{(t)}(x)$, which determines a decomposition of the interval into a finite number of subintervals on which the locus $y=\theta_2^{(t)}(x)$ has slope -1 or 0. We define $\theta_2^{(t)}(x)$ in the interval $(\ln\varrho(\varepsilon),-\ln\varrho(\varepsilon))$ by

$$\theta_2^{(t)}=\begin{cases}\pi, & \ln\varrho(\varepsilon)<x\leq\ln(\frac{1}{2}\sqrt{3})(1-n^{-1})-\pi+\theta_{2,2}^{(s)},\\ \theta_{2,2}^{(s)}-x+\ln(\frac{1}{2}\sqrt{3})(1-n^{-1}), & \\ & \ln(\frac{1}{2}\sqrt{3})(1-n^{-1})-\pi+\theta_{2,2}^{(s)}<x<\ln(\frac{1}{2}\sqrt{3})(1-n^{-1}),\\ \theta_2^{(t)}(x), & \ln(\frac{1}{2}\sqrt{3})(1-n^{-1})\leq x<0,\\ \frac{2}{3}\pi+\lambda x^2, & 0\leq x<\sqrt{\pi/3\lambda},\\ \pi, & \sqrt{\pi/3\lambda}\leq x<-\ln\varrho(\varepsilon),\end{cases}$$

where λ is a positive parameter to be determined later. The domain determined by

$$-\theta_1(x)<y<\theta_2^{(t)}(x);\quad \ln\varrho(\varepsilon)<x<-\ln\varrho(\varepsilon),$$

becomes a quadrangle $Q_{\varrho(\varepsilon)}^{(1)}$. The part of $Q_{\varrho(\varepsilon)}^{(1)}$ below the x-axis is the same as for $S_{\varrho(\varepsilon)}^{(1)}$ and the part above the x-axis is as in Figure 2.1b.

If we let $\widetilde{\Gamma}'$ be the family of curves $Q_{\varrho(\varepsilon)}^{(1)}$ joining the pair of sides complementary to the two vertical sides, we have that

$$M(\widetilde{\Gamma},S_{\varrho(\varepsilon)}^{(1)})\leq M(\widetilde{\Gamma}',Q_{\varrho(\varepsilon)}^{(1)}).$$

Thus it is enough to obtain an upper bound for $M(\widetilde{\Gamma}', Q^{(1)}_{\varrho(\varepsilon)})$. It is known that an upper bound for this module is given by the Dirichlet integral of any piecewise continuously differentiable function in $Q^{(1)}_{\varrho(\varepsilon)}$ taking the value 0 on the side given by $y=-\theta_1(x)$, and the value 1 on the side given by $y=\theta_2^{(t)}(x)$. A function like this is given by

$$u(x,y)=\frac{y+\theta_1(x)}{\theta^{(t)}(x)},$$

where $\theta^{(t)}(x)=\theta_1(x)+\theta_2^{(t)}(x)$. To estimate the Dirichlet integral of $u(x,y)$ we subdivide the domain $Q^{(1)}_{\varrho(\varepsilon)}$ into five pieces each corresponding to one of the following intervals in the x-axis:

$$\begin{aligned}
I&=\left(\ln\varrho(\varepsilon),\ln\left(\tfrac12\sqrt3\right)(1-n^{-1})-\pi+\theta^{(s)}_{2,2}\right];\\
II&=\left(\ln\left(\tfrac12\sqrt3\right)(1-n^{-1})-\pi+\theta^{(s)}_{2,2},\ln\left(\tfrac12\sqrt3\right)(1-n^{-1})\right);\\
III&=\left[\ln\left(\tfrac12\sqrt3\right)(1-n^{-1}),0\right);\\
IV&=\left[0,\sqrt{\pi/3\lambda}\right);\\
V&=\left[\sqrt{\pi/3\lambda}-\ln\varrho(\varepsilon)\right).
\end{aligned}$$

On the two pieces of the Dirichlet integral corresponding to the intervals I and V, the function $u(x,y)=(y+\pi)/2\pi$, and since when we take the limit as the number of subdivisions $n\to\infty$ then $\theta^{(s)}_{2,2}\to\frac56\pi$, we have that

$$\begin{aligned}
\iint_I+\iint_V|\nabla u(x,y)|^2\,dx\,dy&=\frac{1}{2\pi}\ln\left(\frac{1}{\varrho(\varepsilon)}\right)+\frac{1}{2\pi}\left[\ln\frac{\sqrt3}{2}-\frac{\pi}{6}\right]\\
&\quad+\frac{1}{2\pi}\ln\left(\frac{1}{\varrho(\varepsilon)}\right)-\frac{1}{2\pi}\sqrt{\frac{\pi}{3\lambda}}.
\end{aligned}$$

It is not difficult to see that the Dirichlet integral corresponding to II after we let $n\to\infty$ tends to

$$\begin{aligned}
&\int_{\ln(\sqrt3/2)-\pi/6}^{\ln(\sqrt3/2)}\int_{-\pi}^{5\pi/6-(x-\ln(\sqrt3/2))}\left|\nabla\left(\frac{y+\pi}{11\pi/6-\left(x-\ln\left(\sqrt3/2\right)\right)}\right)\right|^2dy\,dx\\
&\qquad=\int_{\ln(\sqrt3/2)-\pi/6}^{\ln(\sqrt3/2)}\int_{-\pi}^{5\pi/6-x+\ln(\sqrt3/2)}\left[\frac{1}{\left(11\pi/6-x+\ln\left(\sqrt3/2\right)\right)^2}\right.\\
&\qquad\quad\left.+\frac{(y+\pi)^2}{\left(11\pi/6-x+\ln\left(\sqrt3/2\right)\right)^4}\right]dy\,dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\ln(\sqrt{3}/2)-\pi/6}^{\ln(\sqrt{3}/2)} \frac{4}{3}\left[\frac{1}{11\pi/6-x+\ln(\sqrt{3}/2)}\right] dx \\
&= \frac{4}{3}\left[-\ln\left(\frac{11\pi}{6}-x+\ln\frac{\sqrt{3}}{2}\right)\right]_{\ln(\sqrt{3}/2)-\pi/6}^{\ln(\sqrt{3}/2)} \\
&= \frac{4}{3}\ln\frac{12}{11}.
\end{aligned}
$$

As for the piece corresponding to IV, we have that after some calculations

$$
\iint_{IV}\left|\nabla\left(\frac{y+\pi}{5\pi/3+\lambda x^2}\right)\right|^2 dx\,dy = \left[\sqrt{\frac{3}{5\pi}}\frac{1}{\sqrt{\lambda}}-\frac{4}{3}\sqrt{\frac{5\pi}{3}}\sqrt{\lambda}\right]\arctan\frac{1}{5}+\frac{4}{3}\sqrt{\frac{\pi}{3}}\sqrt{\lambda}.
$$

The estimate corresponding to III is more delicate, and we will treat it carefully.

$$
\begin{aligned}
\iint_{III}|\nabla u(x,y)|^2 dx\,dy &= \int_{\ln(\sqrt{3}/2)}^{0}\frac{dx}{\theta^{(t)}(x)}+\frac{1}{3}\int_{\ln(\sqrt{3}/2)}^{0}\frac{(\theta_1')^2-\theta_2^{(t)\prime}\theta_1'+(\theta_2^{(t)\prime})^2}{\theta^{(t)}(x)}\,dx \\
&= \int_{\ln(\sqrt{3}/2)}^{0}\frac{dx}{\theta^{(t)}(x)}+\frac{1}{3}\sum_{j=0}^{n-1}\int_{\Omega_j}\frac{1}{\theta^{(t)}(x)}\,dx \\
&= (\mathrm{i})+(\mathrm{ii}),
\end{aligned}
$$

where Ω_j is the subinterval of Δ_j over which $\theta_2^{(t)\prime}(x)$ is equal to -1. We proceed to estimate these two integrals (i) and (ii).

$$
(\mathrm{i}) = \int_{\ln(\sqrt{3}/2)}^{0}\frac{dx}{\theta^{(t)}(x)} \le -\frac{3}{5\pi}\ln\frac{\sqrt{3}}{2}.
$$

We estimate (ii) as follows,

$$
\begin{aligned}
(\mathrm{ii}) = \frac{1}{3}\sum_{j=0}^{n-1}\int_{\Omega_j}\frac{dx}{\theta^{(t)}(x)} &= \frac{1}{3}\sum_{j=0}^{n-1}\int_{\Omega_j}\frac{1}{\pi+\theta_2^{(s)}(x)+\ln(\sqrt{3}/2)(1-(j+1)/n)-x}\,dx \\
&= \frac{1}{3}\sum_{j=0}^{n-1}\int_{\Omega_j}\frac{1}{\pi+\theta_{2,j}^{(s)}+\ln(\sqrt{3}/2)(1-(j+1)/n)-x}\,dx \\
&= \sum_{j=0}^{n-1}\frac{-1}{3}\left[\ln\left(\pi+\theta_{2,j}^{(s)}+\ln\left(\frac{\sqrt{3}}{2}\right)\left(1-\frac{j+1}{n}\right)-x\right)\right]_{x_j^{(l)}}^{x_j^{(r)}},
\end{aligned}
$$

where $x_j^{(r)}$ is the right endpoint of the interval Ω_j and $x_j^{(l)}$ is the left endpoint. Hence, for n large enough,

$$\theta_{2,j}^{(s)}+\ln\left(\frac{\sqrt{3}}{2}\right)\left(1-\frac{j+1}{n}\right)-x_j^{(r)}=\theta_{2,j+1}^{(s)}+\ln\left(\frac{\sqrt{3}}{2}\right)\left(1-\frac{j+2}{n}\right)-x_{j+1}^{(l)}$$

for any $j=0,\dots,n-2$, thus

$$\begin{aligned}\sum_{j=0}^{n-1}\frac{-1}{3}&\left[\ln\left(\pi+\theta_{2,j}^{(s)}+\ln\left(\frac{\sqrt{3}}{2}\right)\left(1-\frac{j+1}{n}\right)-x\right)\right]_{x_j^{(l)}}^{x_j^{(r)}}\\ &=\frac{1}{3}\left[\ln\left(\pi+\theta_{2,0}^{(s)}+\ln\left(\frac{\sqrt{3}}{2}\right)\left(1-\frac{1}{n}\right)-x_0^{(l)}\right)-\ln(\pi+\theta_{2,n}^{(s)}-x_n^{(r)})\right],\end{aligned}$$

since $x_0^{(l)}=\ln\left(\frac{1}{2}\sqrt{3}\right)$, $\lim_{n\to\infty}\theta_{2,0}^{(s)}=\frac{5}{6}\pi$, $x_n^{(r)}=0$, and $\theta_{2,n}^{(s)}=\frac{2}{3}\pi$, letting n go to ∞, we obtain that

$$\text{(ii)}\to\frac{1}{3}\ln\frac{11}{10}.$$

This completes all our estimates, putting all of them together, we obtain that

$$\begin{aligned}M\left(\widetilde{\Gamma}',Q_{\varrho(\varepsilon)}^{(1)}\right)\le{}&\frac{1}{2\pi}\ln\left(\frac{1}{\varrho^2(\varepsilon)}\right)+\frac{1}{2\pi}\left[\ln\frac{\sqrt{3}}{2}-\frac{\pi}{6}\right]-\frac{1}{2\pi}\sqrt{\frac{\pi}{3\lambda}}+\frac{4}{3}\ln\frac{12}{11}-\frac{3}{5\pi}\ln\frac{\sqrt{3}}{2}\\ &+\left[\sqrt{\frac{3}{5\pi}}\frac{1}{\sqrt{\lambda}}-\frac{4}{3}\sqrt{\frac{5\pi}{3}}\sqrt{\lambda}\right]\arctan\frac{1}{5}+\frac{4}{3}\sqrt{\frac{\pi}{3}}\sqrt{\lambda}+\frac{1}{200}+\frac{1}{3}\ln\frac{11}{10}.\end{aligned}$$

Let us call $G(\lambda)$ the expression on the right hand side of the above inequality involving the positive parameter λ,

$$G(\lambda)=\left[\sqrt{\frac{3}{5\pi}}\frac{1}{\sqrt{\lambda}}-\frac{4}{3}\sqrt{\frac{5\pi}{3}}\sqrt{\lambda}\right]\arctan\frac{1}{5}+\frac{4}{3}\sqrt{\frac{\pi}{3}}\sqrt{\lambda}-\frac{1}{2\pi}\sqrt{\frac{\pi}{3}}\frac{1}{\sqrt{\lambda}},$$

and solve the equation $G(\lambda)=0$, hence

$$\left[\frac{4}{3}\sqrt{\frac{\pi}{3}}-\frac{4}{3}\sqrt{\frac{5\pi}{3}}\arctan\frac{1}{5}\right]\sqrt{\lambda}=\left[\frac{1}{2\pi}\sqrt{\frac{\pi}{3}}-\sqrt{\frac{3}{5\pi}}\arctan\frac{1}{5}\right]\frac{1}{\sqrt{\lambda}},$$

thus,

$$\lambda=\left(\frac{1}{2\pi}\sqrt{\frac{\pi}{3}}-\sqrt{\frac{3}{5\pi}}\arctan\frac{1}{5}\right)\Big/\left(\frac{4}{3}\sqrt{\frac{\pi}{3}}-\frac{4}{3}\sqrt{\frac{5\pi}{3}}\arctan\frac{1}{5}\right)=0.10050259.$$

Choosing λ to be this value the expression on the right hand side of the inequality involving λ is equal to zero, therefore

$$M(\widetilde{\Gamma}', Q^{(1)}_{\varrho(\varepsilon)}) \le \frac{1}{2\pi}\ln\left(\frac{1}{\varrho^2(\varepsilon)}\right)+\frac{1}{2\pi}\left(\ln\frac{\sqrt{3}}{2}-\frac{\pi}{6}\right) + \frac{1}{200}+\frac{1}{3}\ln\frac{11}{10}+\frac{4}{3}\ln\frac{12}{11}-\frac{3}{5\pi}\ln\frac{\sqrt{3}}{2}.$$

Putting together the two estimates of $M(\Gamma, \Theta^{(1)}_{\varrho(\varepsilon)})$ from above and from below, we obtain

$$\frac{2\pi}{\ln((1-\varepsilon)/\varrho(\varepsilon)(\varrho(\varepsilon)+\varepsilon)|F_1'(1)|)} \ge \left(\frac{1}{2\pi}\ln\left(\frac{1}{\varrho^2(\varepsilon)}\right)+\frac{1}{2\pi}\left(\ln\frac{\sqrt{3}}{2}-\frac{\pi}{6}\right) + \frac{1}{200}+\frac{1}{3}\ln\frac{11}{10}+\frac{4}{3}\ln\frac{12}{11}-\frac{3}{5\pi}\ln\frac{\sqrt{3}}{2}\right)^{-1}.$$

Taking inverses and exponentiating both sides, we obtain

$$\frac{1-\varepsilon}{(\varrho(\varepsilon)+\varepsilon)\varrho(\varepsilon)|F_1'(1)|} \le \frac{1}{\varrho^2(\varepsilon)}\exp\left\{\left(\ln\frac{\sqrt{3}}{2}-\frac{\pi}{6}\right) + 2\pi\left(\frac{1}{200}+\frac{1}{3}\ln\frac{11}{10}+\frac{4}{3}\ln\frac{12}{11}-\frac{3}{5\pi}\ln\frac{\sqrt{3}}{2}\right)\right\}.$$

It is not difficult to see that choosing ε conveniently and letting $\varepsilon\to 0$ we get that

$$\gamma=|F_1'(1)|\ge\exp\left\{-\left(\ln\frac{\sqrt{3}}{2}-\frac{\pi}{6}\right)-2\pi\left(\frac{1}{200}+\frac{1}{3}\ln\frac{11}{10}+\frac{4}{3}\ln\frac{12}{11}-\frac{3}{5\pi}\ln\frac{\sqrt{3}}{2}\right)\right\}=\alpha.$$

Hence,

$$\gamma^{1/2}=|F_1'(1)|^{1/2}\ge\alpha^{1/2}\ge 0.79249.$$

2.2. Estimating $\beta=|F_2'(a)|$

Let ϱ be a small positive number, and consider the discs $D^{(2)}_{\varrho}=\{z:|z-a|<\varrho\}$, and $D^{(2)}_{1/\varrho}=\{z:|z-a|<1/\varrho\}$. Let $\Theta^{(2)}_{\varrho}$ be the doubly connected domain

$$\Theta^{(2)}_{\varrho}=([\Theta\cap D^{(2)}_{1/\varrho}]\setminus\overline{D}^{(2)}_{\varrho}).$$

Let $H^{(2)}_{\varrho}$ be the image under $F_2(z)$ of $\Theta^{(2)}_{\varrho}$, by the same reason as in the first estimate 2.1, for any positive ε, there exists a small $\varrho(\varepsilon)$ positive such that;

$$\{z:|z|<|F_2'(a)|(\varrho(\varepsilon)-\varepsilon)\}\cap H\subset F_2(D^{(2)}_{\varrho(\varepsilon)})\subset\{z:|z|<|F_2'(a)|(\varrho(\varepsilon)+\varepsilon)\}\cap H$$

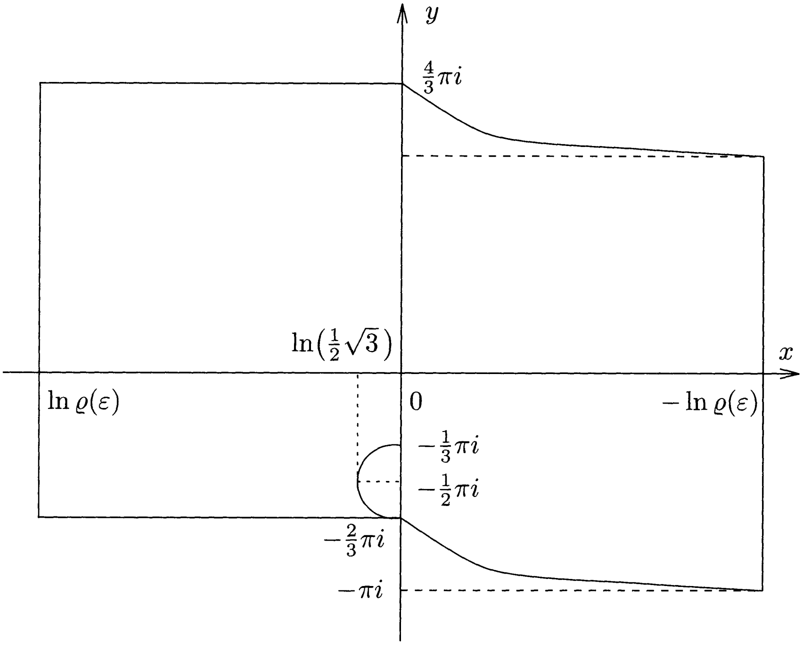

Figure 2.2a

and

$$\{z\colon |z|<(1-\varepsilon)/\varrho(\varepsilon)\}\cap H\subset F_2(D^{(2)}_{1/\varrho(\varepsilon)})\subset\{z\colon |z|<(1+\varepsilon)/\varrho(\varepsilon)\}\cap H.$$

Considering now the module problem for the family of curves Γ joining $\partial D^{(2)}_{\varrho(\varepsilon)}$ with $\partial D^{(2)}_{1/\varrho(\varepsilon)}$ in $\Theta^{(2)}_{\varrho(\varepsilon)}$, we have that

$$M\big(\Gamma,\Theta^{(2)}_{\varrho(\varepsilon)}\big)\le\frac{2\pi}{\ln((1-\varepsilon)/\varrho(\varepsilon)(\varrho(\varepsilon)+\varepsilon)|F_2'(a)|)}.$$

Our goal is to obtain a lower bound for the above module. For this we consider the conformal mapping $\Psi(z)=\ln(z-a)$,

$$\Psi(z)\colon \Theta^{(2)}_{\varrho(\varepsilon)}\to S^{(2)}_{\varrho(\varepsilon)},$$

where $S^{(2)}_{\varrho(\varepsilon)}$ is again a quadrangle as in Figure 2.2a above.

Let $\widetilde{\Gamma}$ be the family of curves in $S^{(2)}_{\varrho(\varepsilon)}$ joining the pair of sides opposite to the vertical sides of $S^{(2)}_{\varrho(\varepsilon)}$. By the conformal invariance of the module we have that,

$$M\big(\Gamma,\Theta^{(2)}_{\varrho(\varepsilon)}\big)=M\big(\bar{\Gamma},S^{(2)}_{\varrho(\varepsilon)}\big),$$

where $\bar{\Gamma}$ is the family of curves in $S^{(2)}_{\varrho(\varepsilon)}$ joining the pair of vertical sides. Since the families of curves $\bar{\Gamma}$ and $\widetilde{\Gamma}$ are conjugate in $S^{(2)}_{\varrho(\varepsilon)}$, we have that

$$M(\bar{\Gamma}, S^{(2)}_{\varrho(\varepsilon)}) = 1/M(\widetilde{\Gamma}, S^{(2)}_{\varrho(\varepsilon)}),$$

thus, to obtain a lower bound for $M(\Gamma, \Theta^{(2)}_{\varrho(\varepsilon)})$, all we need is an upper bound for the module $M(\widetilde{\Gamma}, S^{(2)}_{\varrho(\varepsilon)})$. To obtain this bound we proceed as in case 2.1. Our function $\theta_2(x)$ in this case is given by,

$$\theta_2(x) = \begin{cases} \frac{4}{3}\pi, & \text{if } \ln \varrho(\varepsilon) < x \le 0, \\ \pi + \arctan \sqrt{3/(4e^{2x}-3)}, & \text{if } 0 \le x < -\ln \varrho(\varepsilon). \end{cases}$$

We modify the function $\theta_1(x)$ in the same way we did with $\theta_2(x)$ in case 2.1 for values of x satisfying $\ln \varrho(\varepsilon) < x < 0$, and for values of x in the interval $[0, -\ln \varrho(\varepsilon))$ we are going to modify $\theta_1(x)$ as follows;

$$\theta_1^{(t)}(x) = \begin{cases} \frac{1}{3}\pi + \delta x, & \text{if } 0 \le x < \lambda, \\ \theta_1(x), & \text{if } \lambda \le x < -\ln \varrho(\varepsilon). \end{cases}$$

Where $\delta > 0$ is a free parameter and λ is implicitely defined by the equation

$$\delta\lambda + \arctan \sqrt{\frac{3}{4e^{2\lambda}-3}} = \frac{2\pi}{3}.$$

The domain determined by

$$-\theta_1^{(t)}(x) < y < \theta_2(x); \quad \ln \varrho(\varepsilon) < x < -\ln \varrho(\varepsilon),$$

becomes a quadrangle $Q^{(2)}_{\varrho(\varepsilon)}$ on assigning, as a pair of opposite sides, the segments

$$-\theta_1^{(t)}(\ln \varrho(\varepsilon)) < y < \theta_2(\ln \varrho(\varepsilon))$$

and

$$-\theta_1^{(t)}(-\ln \varrho(\varepsilon)) < y < \theta_2(-\ln \varrho(\varepsilon)).$$

The part of $Q^{(2)}_{\varrho(\varepsilon)}$ above the x-axis is the same as for $S^{(2)}_{\varrho(\varepsilon)}$ and the part below the x-axis is as in Figure 2.2b.

As in the case 2.1 we have that

$$M(\widetilde{\Gamma}, S^{(2)}_{\varrho(\varepsilon)}) \le M(\widetilde{\Gamma}', Q^{(2)}_{\varrho(\varepsilon)}),$$

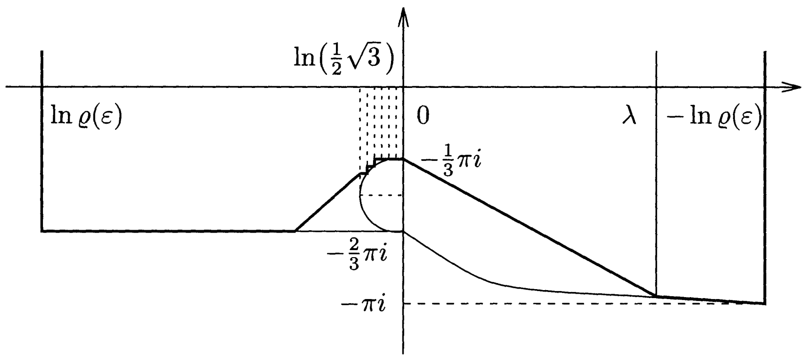

Figure 2.2b

where $\widetilde{\Gamma}'$ is the family of curves in $Q^{(2)}_{\varrho(\varepsilon)}$ joining the pair of sides complementary to the vertical sides. Thus, it is enough to obtain an upper bound for the module $M(\widetilde{\Gamma}', Q^{(2)}_{\varrho(\varepsilon)})$. An upper bound is given by the Dirichlet integral of the piecewise continuously differentiable function in $Q^{(2)}_{\varrho(\varepsilon)}$

$$u(x,y)=\frac{\theta_2(x)-y}{\theta^{(t)}(x)},$$

where $\theta^{(t)}(x)=\theta_1^{(t)}(x)+\theta_2(x)$. Hence,

$$\iint_{Q^{(2)}_{\varrho(\varepsilon)}} |\nabla u(x,y)|^2 dx\,dy = \iint_{Q^{(2)}_{\varrho(\varepsilon)}\cap\{\mathrm{Re}\{z\}\le 0\}} + \iint_{Q^{(2)}_{\varrho(\varepsilon)}\cap\{\mathrm{Re}\{z\}>0\}} |\nabla u(x,y)|^2 dx\,dy$$
$$= I + II.$$

The estimate of the integral I is the same as in case 2.1 because if we look at the left hand sides of the domains $Q^{(1)}_{\varrho(\varepsilon)}$ and $Q^{(2)}_{\varrho(\varepsilon)}$, they are the same up to a symmetry and a vertical translation. Thus,

$$I \le \frac{1}{2\pi}\ln\left(\frac{1}{\varrho(\varepsilon)}\right)+\frac{1}{3}\ln\frac{11}{10}+\frac{1}{200}+\frac{1}{2\pi}\left[\ln\frac{\sqrt{3}}{2}-\frac{\pi}{6}\right]+\frac{4}{3}\ln\frac{12}{11}-\frac{3}{5\pi}\ln\frac{\sqrt{3}}{2}.$$

We pass to estimate the second integral II,

$$II = \iint_{Q^{(2)}_{\varrho(\varepsilon)}\cap\{\mathrm{Re}\{z\}>0\}} |\nabla u(x,y)|^2 dx\,dy$$
$$= \int_0^{-\ln\varrho(\varepsilon)} \frac{dx}{\theta^{(t)}(x)} + \frac{1}{3}\int_0^{-\ln\varrho(\varepsilon)} \frac{\theta_1^{(t)\prime}(x)^2-\theta_1^{(t)\prime}(x)\theta_2'(x)+\theta_2'(x)^2}{\theta^{(t)}(x)}\,dx,$$

where

$$\theta_2(x)=\pi+\arctan\sqrt{\frac{3}{4e^{2x}-3}}\quad\text{for } 0<x<-\ln\varrho(\varepsilon),$$

and

$$\theta^{(t)}(x)=\theta_1^{(t)}(x)+\theta_2(x)=2\pi$$

for values of x such that $\lambda\le x<-\ln\varrho(\varepsilon)$. Thus II is equal to

$$II=\int_0^\lambda\frac{1}{\theta^{(t)}(x)}\,dx+\frac{1}{3}\int_0^\lambda\frac{\theta_1^{(t)\prime}(x)^2-\theta_1^{(t)\prime}(x)\theta_2'(x)+\theta_2'(x)^2}{\theta^{(t)}(x)}\,dx$$
$$+\frac{1}{2\pi}\ln\frac{1}{\varrho(\varepsilon)}-\frac{\lambda}{2\pi}+\frac{1}{6\pi}\int_\lambda^{-\ln\varrho(\varepsilon)}3(\theta_2'(x))^2dx.$$

Let us compute the last integral in the above equality,

$$\int_\lambda^{-\ln\varrho(\varepsilon)}(\theta_2'(x))^2dx=3\int_\lambda^{-\ln\varrho(\varepsilon)}\frac{dx}{4e^{2x}-3}=\frac{1}{2}\ln\left(\frac{4-3\varrho^2(\varepsilon)}{4-3e^{-2\lambda}}\right).$$

It remains to estimate

$$\int_0^\lambda\frac{dx}{\theta^{(t)}(x)}+\frac{1}{3}\int_0^\lambda\frac{\theta_1^{(t)\prime}(x)^2-\theta_1^{(t)\prime}(x)\theta_2'(x)+\theta_2'(x)^2}{\theta^{(t)}(x)}\,dx$$
$$=\int_0^\lambda\frac{1+\frac{1}{3}\delta^2+1/(4e^{2x}-3)+\frac{1}{3}\delta\sqrt{3/(4e^{2x}-3)}}{\frac{4}{3}\pi+\delta x+\arctan\sqrt{3/(4e^{2x}-3)}}\,dx$$
$$=\int_0^\lambda\frac{1+\frac{2}{3}\delta^2+1/(4e^{2x}-3)}{\frac{4}{3}\pi+\delta x+\arctan\sqrt{3/(4e^{2x}-3)}}\,dx$$
$$-\tfrac{1}{3}\delta\left[\ln\left(\delta x+\tfrac{4}{3}\pi+\arctan\sqrt{3/(4e^{2x}-3)}\right)\right]_0^\lambda.$$

The second term in the formula above is equal to $\frac{1}{3}\delta\ln\frac{6}{5}$. Thus, to complete our estimate, our final goal is to find a suitable bound for the following integral

$$\int_0^\lambda\left[\frac{1+\frac{2}{3}\delta^2+1/(4e^{2x}-3)}{\frac{4}{3}\pi+\delta x+\arctan\sqrt{3/(4e^{2x}-3)}}-\frac{1}{2\pi}\right]dx,$$

where $\delta\lambda+\arctan\sqrt{3/(4e^{2\lambda}-3)}=\frac{2}{3}\pi$. Our first observation is that

$$\frac{4\pi}{3}+\delta x+\arctan\sqrt{\frac{3}{(4e^{2x}-3)}}\ge 2\pi-\arctan\sqrt{\frac{3}{(4e^{2x}-3)}}$$

for $0<x<\lambda$. Therefore it is enough to estimate the integral

$$\int_0^\lambda \left[\frac{1+1/(4e^{2x}-3)}{2\pi-\arctan\sqrt{3/(4e^{2x}-3)}}-\frac{1}{2\pi}\right]dx$$
$$\le \int_0^\infty \frac{2\pi/(4e^{2x}-3)+\arctan\sqrt{3/(4e^{2x}-3)}}{2\pi\big(2\pi-\arctan\sqrt{3/(4e^{2x}-3)}\,\big)}\,dx.$$

In the above integral we have dropped the term $\frac{2}{3}\delta^2$ in the numerator, since when λ goes to ∞ then δ goes to 0 as $2\pi/3\lambda$, thus

$$\int_0^\lambda \frac{\frac{2}{3}\delta^2}{\frac{4}{3}\pi+\delta x+\arctan\sqrt{3/(4e^{2x}-3)}}\,dx\le\frac{\delta^2\lambda}{2\pi},$$

and this goes to $\frac{1}{3}\delta$ as λ goes to ∞, hence the term in the integral corresponding to $\frac{2}{3}\delta^2$ can be made as small as we please. Using the change of variable $u=\arctan\sqrt{3/(4e^{2x}-3)}$, the above integral becomes

$$-\int_{\pi/3}^0 \frac{\frac{2}{3}\pi(\tan u)^2+u}{2\pi(2\pi-u)}\frac{du}{\tan u}=\frac{1}{2\pi}\int_0^{\pi/3}\frac{2\pi}{3}\frac{\tan u}{(2\pi-u)}\,du+\frac{1}{2\pi}\int_0^{\pi/3}\frac{u}{(2\pi-u)}\cot u\,du$$
$$=A+B.$$

Standard numerical integration methods give us the following estimates from above for the two integrals A and B:

$$A\le\frac{1}{3}(0.126)=0.042$$

and

$$B\le\frac{1}{2\pi}(0.158)\le 0.0252.$$

Putting all these estimates together and letting δ go to 0, we obtain that

$$M(\widetilde{\Gamma}',Q^{(2)}_{\varrho(\varepsilon)})\le\frac{1}{2\pi}\ln\frac{1}{\varrho^2(\varepsilon)}+\frac{1}{3}\ln\frac{11}{10}+\frac{1}{200}+\frac{1}{2\pi}\left[\ln\frac{\sqrt{3}}{2}-\frac{\pi}{6}\right]$$
$$+\frac{4}{3}\ln\frac{12}{11}-\frac{3}{5\pi}\ln\frac{\sqrt{3}}{2}+\frac{1}{4\pi}\ln\left(\frac{4-3\varrho^2(\varepsilon)}{4}\right)+0.042+0.0252.$$

Putting together the estimates of $M(\Gamma,\Theta^{(1)}_{\varrho(\varepsilon)})$ from above and from below, taking inverses and exponentiating, as we did in case 2.1, and letting ε go to 0, we have that

$$\beta=|F_2'(a)|$$
$$\ge\exp\left(-2\pi\left\{\frac{1}{3}\ln\frac{11}{10}+\frac{1}{200}+\frac{1}{2\pi}\left(\ln\frac{\sqrt{3}}{2}-\frac{\pi}{6}\right)+\frac{4}{3}\ln\frac{12}{11}-\frac{3}{5\pi}\ln\frac{\sqrt{3}}{2}+0.0672\right\}\right)$$
$$=\eta.$$

Hence,

$$\beta^{1/2} = |F_2'(a)|^{1/2} \geq \eta^{1/2} \geq 0.6403.$$

Therefore putting together the two estimates 2.1 and 2.2, we have that

$$|F_2'(a)|^{1/2} + |F_1'(1)|^{1/2} = \beta^{1/2} + \gamma^{1/2} > 0.79249 + 0.6403 = 1.43279 > \sqrt{2},$$

and this proves the theorem. □

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