A CONVERSE OF CAUCHY'S THEOREM AND APPLICATIONS TO EXTREMAL PROBLEMS

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1. Introduction

1.1. In recent years many papers have been concerned with pairs of extremal problems which are conjugate in the sense that the extremal values are the same [1, 5, 6, 7, 11, 12, 13, 16]. The conjugacy is usually related to the conjugacy of the Lebesgue classes L_p and L_q where $p^{-1}+q^{-1}=1$; in the one problem one is maximizing an L_p norm, in the other minimizing an L_q norm. It is now well known that the conjugacy of such problems and the existence of extremals can be derived from the Hahn–Banach Theorem and related results. In this process the following converse of Cauchy's Theorem is of great assistance: If W is a region whose boundary C consists of a finite number of analytic Jordan curves, and if g is a bounded measurable function defined on C such that $\int_C g\,\omega=0$ for all differentials ω analytic in the closure of W, then g represents p.p. the boundary values of a function analytic in W. A proof of this theorem is given by Rudin [12] for the case of plane regions, and the theorem can be extended to Riemann surfaces. In the present paper a rather more general theorem of this type is proved for Riemann surfaces (Theorem 2.2 and its corollary), enabling a greater variety of extremal problems to be handled.

In studying the conjugate extremal problems it is convenient to consider separately the cases $1<p<\infty$, $p=1$ and $p=\infty$. The last two cases seem to have the more direct significance on Riemann surfaces, but special difficulties are apt to arise in their discussion. An account of the case in which the maximal problem is of type $p=1$ has been given in [11]. The present paper deals with the case in which the maximum problem is of type $p=\infty$, but its results are of greater variety owing to the use of the more powerful form of the converse of Cauchy's Theorem which is referred to above.

1.2. We begin by introducing terminology and notation which will be used throughout the paper. W is a region of a Riemann surface W_0: the closure $\overline{W}$ of W is compact, and its boundary consists of a finite number of analytic Jordan curves γ_j. It is known that W is conformally equivalent to the surface obtained by identifying pairs of sides in a polygon

$$x_1 x_2 x_1' x_2' \ldots x_{2p-1} x_{2p} x_{2p-1}' x_{2p}' c_1 m_1 c_1' \ldots c_q m_q c_q'$$

in the complex plane, whose sides are analytic arcs [8]. To the sides x_i there correspond on W certain closed curves which we shall denote by α_i ($i=1$ to p): to the sides x_i' there correspond closed curves which whe shall denote by α_{i+p} ($i=1$ to p): and to the sides m_j there correspond the boundary curves γ_j. It is easy to construct a closed Riemann surface of genus p which contains $\overline{W}$ and for which the α_i ($i=1$ to $2p$) form a canonical homology basis. It therefore involves no further limitation on W to assume from the start that W_0 is a closed Riemann surface of genus p. We shall refer to curves α_i, α_{i+p} as conjugate members of the canonical homology basis.

In what follows it is fundamental that for each value of j ($j=1$ to q) there exist (i) a region G_j of W_0 containing γ_j, (ii) an annulus $a<|z|<b$ in the complex plane with $a<1<b$, (iii) a one-one conformal mapping of G_j onto the annulus such that γ_j is mapped on the circumference U of the unit circle. To show this we remark that the standard theorems on the mapping of planar Riemann surfaces [8] ensure that any planar region of W_0 containing γ_j can be mapped one-one conformally onto a plane region. Such a mapping takes γ_j onto an analytic Jordan curve J in the plane. The interior of J can now be mapped on the interior of the unit circle, and since J is analytic this mapping can be extended across the boundary. Combining the two mappings we obtain one which has all the required properties.

The region G_j and the associated mapping are not in any way unique, but it will be convenient to fix our attention on one such mapping in relation to each of the γ_j. Taking τ to be a variable point of G_j we shall always denote this mapping by $\tau=\lambda_j(z)$ and call it the *j-th annular mapping.* We may clearly suppose that the points of G_j in W correspond to points inside rather than outside the circle U, and that the curves α_i do not intersect any of the G_j. The region G_j together with the mapping $\tau=\lambda_j(z)$ define a local coordinate system valid on γ_j. This will be referred to as the *j-th annular coordinate system,* and G_j will be called the *j-th parametric annulus.*

We shall describe a function $g(\tau)$ defined on γ as being of L-character on γ if for each of the annular mappings $\tau=\lambda_j(z)$ the function $g[\lambda_j(e^{i\theta})]$ belongs to the Lebesgue class $L(0,2\pi)$. Moreover $g(\tau)$ will be said to have bounded variation on γ if each of the functions $g[\lambda_j(e^{i\theta})]$ has bounded variation over $(0,2\pi)$: and $g(\tau)$ will be called absolutely continuous on γ if each of the functions $g[\lambda_j(e^{i\theta})]$ is absolutely continuous in $(0,2\pi)$. Given a function $f(\tau)$ defined on W and a function $g(\tau)$ defined on γ_j we shall say that $f(\tau)$ takes the boundary values $g(\tau)$ on γ_j if

$$\lim_{r\to 1-} f[\lambda_j(r e^{i\theta})]=g[\lambda_j(e^{i\theta})]$$

for almost all θ in $(0,2\pi)$. If this happens for all the γ_j we shall say that $f(\tau)$ takes the boundary values $g(\tau)$ on γ.

We recall that a function $f(z)$ is said to belong to the class H_p $(0<p<\infty)$ if it is analytic in $|z|<1$ and if $\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta$ is bounded for $0\leqslant r<1$: and that $f(z)$ is said to belong to the class H_∞ if it is bounded and analytic in $|z|<1$. Now let $f(\tau)$ be a function defined on W. We shall say that $f(\tau)$ belongs to the class $H_p(\gamma_j)$ if $f[\lambda_j(z)]=g(z)+h(z)$ where $g(z)$ is analytic on U and $h(z)\in H_p$. It is to be observed that in making this definition we are interested only in the behaviour of f near the boundary γ_j and do not insist on analyticity throughout W. To that extent it is not a direct generalization of the class H_p. An equivalent formulation of the defining property is that $f[\lambda_j(z)]$ is analytic in some annulus $a<|z|<1$ and

$$\int_0^{2\pi}|f[\lambda_j(re^{i\theta})]|^p\,d\theta=O(1)$$

as $r\to 1$: for a function analytic in such an annulus is, by Laurent's Theorem, of the form $g(z)+h(z)$ where $g(z)$ is analytic in $|z|>a$ and $h(z)$ in $|z|<1$, and the equivalence follows from Minkowski's and related inequalities ([15] p. 67). If $f(\tau)\in H_p(\gamma_j)$ for all $j=1$ to q we shall say that $f(\tau)\in H_p(\gamma)$; and if in addition f is analytic in W then we shall say that $f\in H_p(W)$.

We shall also say that a differential dg, defined on W, belongs to the class $K_p(\gamma_j)$ if $dg=G\,dh$, where dh is a differential analytic on γ_j and $G\in H_p(\gamma_j)$. This is, in fact, equivalent to saying that $dg\in K_p(\gamma_j)$ if in terms of the jth annulus parameter

$$dg=[F(z)+G(z)]\,dz$$

where $G(z)$ is analytic on U and $F(z)\in H_p$. If $dg\in K_p(\gamma_j)$ for all $j=1$ to q we shall say that $dg \in K_p(\gamma)$. In the applications to Riemann surfaces considered in the present paper we are concerned only with the cases $p=1$ and ∞.

1.3. The following properties of functions $f(z)$ belonging to H_p are known or are corollaries of known theorems (see, for instance, [15] pp. 157–163).

(i) $\lim_{r\to 1-} f(re^{i\theta})$ exists p.p. and is an integrable function of θ over $(0, 2\pi)$. We denote it by $f(e^{i\theta})$.

(ii) If $f(z) \in H_p$ and $f(e^{i\theta}) \in L^q$ where $p<q\leqslant\infty$ then $f(z)\in H_q$.

(iii) If $f(e^{i\theta})$ is of bounded variation and $f(z)=\sum_{n=0}^{\infty} a_n z^n$ when $|z|<1$, then the real and imaginary parts of $\sum a_n e^{in\theta}$ are Fourier series of functions of bounded variation. Hence ([15] p. 158) $f(e^{i\theta})$ is absolutely continuous.

(iv) $\lim_{r\to 1-} \int_0^{2\pi} |f(re^{i\theta}) - f(e^{i\theta})|^p d\theta = 0$ if $0<p<\infty$, and as a corollary $\int_0^{2\pi} f(e^{i\theta}) e^{i\theta} d\theta = 0$ if $1\leqslant p\leqslant\infty$ (that is, we may use the boundary function $f(e^{i\theta})$ in applying Cauchy's Theorem).

(v) If $p\geqslant 1$, the function $F(z)=\int_0^z f(z)\,dz$ has continuous boundary values $F(e^{i\theta})$, and $f(e^{i\theta}) = dF(e^{i\theta})/d(e^{i\theta})$.

It is convenient also to quote here a criterion of Smirnoff [14] for a function to belong to H_p.

(vi) If $f(z)=\int_U \dfrac{g(w)\,dw}{w-z}$, where g is integrable on the unit circle U, then $f(z)$ belongs to H_p for all $p<1$. (A proof of this result will be found in [10].)

The following properties of the classes $H_p(\gamma_j)$ are immediate consequences of (i), (ii) and (iii). If $f(\tau)\in H_p(\gamma_j)$ then

(vii) $f(\tau)$ has boundary values on γ_j in the sense defined above, and the boundary function is of L-character if $p\geqslant 1$ (when τ is on γ_j we shall use $f(\tau)$ to denote the boundary function);

(viii) if $f(\tau)$ is bounded on γ_j then $f(\tau)\in H_\infty(\gamma_j)$;

(ix) if $f(\tau)$ is of bounded variation on γ_j then it is absolutely continuous on γ_j.

If $df \in K_p(\gamma_j)$, then by definition $df = G\,dh$ where $G\in H_p(\gamma_j)$ and dh is analytic on γ. When τ is on γ_j we shall use $df(\tau)$ to denote $G(\tau)\,dh(\tau)$ and call this the boundary differential of df. We derive from (iv):

(x) if $df\in K_p(\gamma_j)$ and $p\geqslant 1$ then $\int_{\gamma_j} df$ may be handled by Cauchy's Theorem as if df were analytic on γ_j.

2. Boundary value theorems

2.1. In this section we prove the criterion, referred to in the Introduction, for functions defined on γ to represent the boundary values of analytic functions; and we derive some more specialized results from it. The proofs depend on properties of normal differentials of the third kind on closed Riemann surfaces, which we summarize in the following statement (for more detailed information see [9]).

On the closed Riemann surface W_0 with the canonical homology basis α_i ($i=1$ to $2p$) let $\omega_{\tau\tau_0}$ be the differential of the third kind which is analytic except for poles of residue 1, -1 at τ, τ_0 respectively and which has vanishing α_i-periods ($i=1$ to p). Then the Principle of Exchange of Argument and Parameter states that $\int_{\sigma_0}^{\sigma} \omega_{\tau\tau_0} = \int_{\tau_0}^{\tau} \omega_{\sigma\sigma_0}$: and it follows that, in a region obtained from W_0 by making suitable cuts, both sides of this equation represent a function of τ which is analytic except for logarithmic singularities at σ, σ_0. The period of $\omega_{\tau\tau_0}$ around an arbitrary analytic Jordan curve has a constant discontinuity across the curve but is otherwise analytic as a function of τ. In particular the α_i-period of $\omega_{\tau\tau}$ is a branch of an Abelian integral of the first kind. Finally, suppose that in terms of a local coordinate system $\omega_{\tau\tau_0} = M(\tau, z)\, dz$. Then for each value of z the function $M(\tau, z)$ has a removable singularity at τ_0 (where it is undefined) and a simple pole at the point with local parameter z; it is otherwise analytic in the region obtained by cutting W_0 along the α_i: and $dM(\tau, z)$ is an analytic differential in W_0 except for a pole of the second order at z. In the applications of this section we take τ_0 to be a fixed point in $W_0 - \overline{W}$, and we denote by ω_τ the differential $\omega_{\tau\tau_0}$ of the above summary.

2.2. THEOREM. *Let C be the set of differentials analytic in W_0 except for poles in $W_0 - \overline{W}$, and let $f_1, f_2, \ldots, f_N$ be linear functionals on C such that, for each i and j, $f_i(\omega_\tau) \in H_1(\gamma)$ and $f_i(\omega_\tau)$ jumps by a constant as τ crosses γ_j normally. Let g be a function of integrable character on γ such that $\int_\gamma g\,\omega = 0$ for all differentials ω, analytic in $\overline{W}$, which satisfy $f_i(\omega)=0$ for all $i=1$ to N. Then there exists a function $H(\tau)$, vanishing when $\tau \in W_0 - \overline{W}$, which has the form*

$$H(\tau) = h(\tau) + \sum_{i=1}^{N} a_i f_i(\omega_\tau), \tag{2.2.1}$$

where the a_i are constants and $h(\tau)$ is a member of $H_1(W)$ which decreases p.p. by g as τ crosses γ in the direction of the outward normal from W.

Proof. We first remark that there exists a differential Ω_τ of the form

$$\Omega_\tau = \omega_\tau + \sum_{i=1}^{N} f_i(\omega_\tau)\,\Omega_i, \tag{2.2.2}$$

where Ω_i belongs to C and is independent of τ, such that $f_i(\Omega_\tau)=0$ for all $i=1$ to N when $\tau \in W_0 - \overline{W}$. It is clearly sufficient to prove this on the assumption that for each value of j from 1 to N there is a differential $\omega_j \in C$ such that $f_j(\omega_j) \neq 0$ but $f_k(\omega_j)=0$ for all $k \neq j$; for otherwise the problem could be reduced to one with fewer functionals. But then, if we take

$$\Omega_i = -\frac{\omega_i}{f_i(\omega_i)} \tag{2.2.3}$$

the differential Ω_τ defined by (2.2.2) has the required property.

We next define $H(\tau)$ by

$$H(\tau) = \frac{1}{2\pi i}\int_\gamma g\,\Omega_\tau. \tag{2.2.4}$$

It then follows from the hypothesis of the theorem that $H(\tau)=0$ if $\tau \in W_0 - \overline{W}$. Also

$$\begin{aligned} 2\pi i\,H(\tau) &= \int_\gamma g\,\omega_\tau + \sum_{i=1}^{N}\int_\gamma f_i(\omega_\tau)\,g\,\Omega_i \\ &= \int_\gamma g\,\omega_\tau + \sum_{i=1}^{N} f_i(\omega_\tau)\int_\gamma g\,\Omega_i, \end{aligned}$$

which is of the form (2.2.1) where the a_i are constants and $h(\tau) = \frac{1}{2\pi i}\int_\gamma g\,\omega_\tau$.

To prove that $h(\tau)$ has the required properties let us first suppose that $\tau \notin \gamma_j$. Let $g_j(z)$ denote $g[\lambda_j(z)]$. If t is a local parameter for τ then $\int_{\gamma_j} g\,\omega_\tau$ is of the form

$$\int_{|z|=1} g_j(z)\,f(t,z)\,dz \tag{2.2.5}$$

where $f(t,z)$ is an analytic function of t for each value of z and an analytic function of z for each value of t. From this it follows by a theorem of Hartogs ([4] p. 119) that $f(t,z)$ is expressible as a double power series in t and z and consequently that it is a continuous function of t and z and that it possesses partial derivatives which are also analytic functions of t and z. Under these circumstances differentiation under the integral sign is justified and the integral (2.2.5) represents an analytic function of t. Thus $h(\tau)$ is analytic in W.

When τ is in the kth parametric annulus and $j \neq k$ it follows from the above that $\int_{\gamma_j} g\,\omega_\tau$ is an analytic function of τ. However $\int_{\gamma_k} g\,\omega_\tau$ is of the form

$$\int_{|z|=1} g_k(z)\left(\frac{1}{z-t}+f(t,z)\right)dz$$

where $t=\lambda_k^{-1}(\tau)$ and $f(t,z)$ again represents an analytic function of both variables. Thus in the kth parametric annulus $h(\tau)$ is of the form

$$\frac{1}{2\pi i}\int_{|z|=1}\frac{g_k(z)\,dz}{z-t}+G(t) \tag{2.2.6}$$

where $G(t)$ is analytic on $|t|=1$. Now the jump in the value of the integral as t crosses $|t|=1$ is known ([2] p. 116): in fact as t crosses the circle at the point $e^{i\alpha}$ in the direction of the outward normal, the value decreases by $g_k(e^{i\alpha})$ for almost all α. Thus $h(\tau)$ decreases by g as τ crosses γ from W into its complement.

Finally we deduce from 1.3 (vi) that $h(\tau)$, being of the form (2.2.6), belongs to $H_p(\gamma_k)$ for all $0<p<1$: and it then follows from 1.3 (ii) that in fact $h(\tau)\in H_1(\gamma_k)$. Since this is true for all k, and since we have proved $h(\tau)$ to be analytic in W, therefore $h(\tau)\in H_1(W)$.

COROLLARY. *If the functions $f_i(\omega_\tau)$ are all analytic on γ, then there exists a function $H(\tau)\in H_1(\gamma)$, with boundary values g on γ, which is analytic in every region of W where all the $f_i(\omega_\tau)$ are analytic.*

2.3. DEFINITION. Let P and Z be finite (possibly empty) disjoint sets of points which lie on W but not on any of the α_i; let S be a subset of the integers 1 to $2p$; and let T be a subset of the integers 1 to $q-1$. We shall say that ω belongs to the class $\mathfrak{D}(P, Z, S, T)$ if ω satisfies the following conditions.

(i) ω is the boundary form of a differential $\Omega\in K_1(\gamma)$.

(ii) Ω is analytic in W with the possible exception of simple poles at points of the set P.

(iii) Ω has zeros at the points of the set Z.

(iv) If $i\in S$, the α_i-period of Ω vanishes.

(v) If $j\in T$, the γ_j-period of Ω vanishes.

Furthermore we shall say that a function $g(\tau)$ defined on γ belongs to the class $\mathfrak{J}(P, Z, S, T)$ if $g(\tau)$ satisfies the following conditions.

(i) There exists a function $H(\tau) \in H_1(\gamma)$ which on each contour γ_j differs p.p. from $g(\tau)$ by a constant c_j.

(ii) $H(\tau)$ vanishes at the points of P.

(iii) At a point of Z, $H(\tau)$ is analytic or has a simple pole. It is otherwise analytic in the region obtained by cutting W along the α_i: and $dH(\tau)$ is analytic in W provided that $\tau \notin Z$.

(iv) If $i \notin S$, $dH(\tau)$ has zero period around the curve α_k which is conjugate to α_i.

(v) If $j \notin T$ then $c_j = 0$.

2.4. LEMMA. *If* $g \in \mathfrak{J}(P, Z, S, T)$, $\omega \in \mathfrak{D}(P, Z, S, T)$ *and either* $g \in H_\infty(\gamma)$ *or* $\omega \in K_\infty(\gamma)$, *then* $\int_\gamma g\,\omega = 0$.

Proof. Let H be the analytic function which differs from g by a constant on certain of the γ_j. Then $\int_\gamma H\omega = \int_\gamma g\,\omega$, since ω has zero periods around the curves γ_j on which H and g differ. We now apply Cauchy's Theorem to the region obtained by cutting W along the α_i. $H\omega$ has no singularities in the cut region, and hence (compare [8] p. 174)

$$\int_\gamma g\,\omega = \int_\gamma H\omega = \sum_{i=1}^{p} \left[\int_{\alpha_i} dH \int_{\alpha_{i+p}} \omega - \int_{\alpha_{i+p}} dH \int_{\alpha_i} \omega \right],$$

which is zero because in each product of two integrals one factor vanishes.

2.5. LEMMA. *There exists a single-valued function which is analytic and non-zero in* $\overline{W}$ *except for poles and zeros of prescribed orders at prescribed points.*

Proof. There exists a differential of the third kind on W_0 whose only poles in $\overline{W}$ are prescribed poles of residue ± 1. We cannot completely prescribe the periods. However, there are differentials which are analytic in $\overline{W}$ and have their α-periods and $q-1$ of their γ-periods prescribed ([1] p. 110). Combining these results we see that there are differentials which are analytic on $\overline{W}$, except for prescribed simple poles of residue ± 1, whose α-periods vanish and whose γ-periods are multiples of $2\pi i$. Let ω be such a differential. Then the function $\exp\left(\int \omega\right)$ is a rational function on $\overline{W}$ with prescribed simple poles and zeros. The general case follows at once by multiplication.

2.6. LEMMA. (i) *If* $\sigma \in W$ *there exists a differential which is analytic in* W *except for a simple pole of residue* 1 *at* σ, *and vanishes at the points of* Z.

(ii) *If* $1 \leqslant k \leqslant 2p$ *there exists a function, analytic in* $\overline{W}$ *except on* α_k, *which vanishes at the points of* P *and whose differential is analytic in* $\overline{W}$ *with non-vanishing period around the curve* α_i *which is conjugate to* α_k.

Proof. (i) Let ω be a differential which is analytic in $\overline{W}$ except for a pole of residue 1 at σ. Let the points of Z be denoted by ϱ_i ($i = 1$ to m). For each $k = 1$ to m there is a single-valued analytic function which has zeros of the second order at all the points ϱ_j with $j \neq k$ and a zero of only the first order at ϱ_k. A differential with the required properties may be constructed by adding to ω a suitable linear combination of the differentials of these m functions.

(ii) Let the points of P be denoted by σ_j ($j = 1$ to n). For each $m = 1$ to n there is a single-valued analytic function g_m which has zeros at all the points σ_j with $j \neq m$ but which does not vanish at σ_m. Let ω be a differential which is analytic in $\overline{W}$ with vanishing γ-periods and whose only non-vanishing α-period is its α_i-period. A function with the required properties may be constructed by adding to an integral of this differential a suitable linear combination of the functions g_m.

2.7. THEOREM. *If* g *is a function of L-character on* γ *such that* $\int_\gamma g\,\omega = 0$ *for all* $\omega \in \mathfrak{D}(P, Z, S, T)$ *which are analytic on* γ, *then* $g \in \mathfrak{J}(P, Z, S, T)$.

Proof. We select the functionals $f_i(\omega)$ of Theorem 2.2 to be the following: the γ_j-periods of ω when $j \in T$, the α_j-periods of ω when $j \in S$, and the functionals $L_\omega(0)$ where $L_\omega(z)\,dz$ is the form taken by ω in a particular local coordinate system in which $z = 0$ corresponds to a point in Z. All the functions $f_i(\omega_\tau)$ are then analytic functions of τ except on the curves γ_j ($j \in T$) and α_j ($j \in S$), and at the points of Z where there can be simple poles: while the differentials $d f_i(\omega_\tau)$ are analytic in W except at the points of Z. If $j \in T$ then $\int_{\gamma_j} \omega_\tau$ has a constant discontinuity across γ_j but is otherwise analytic on γ; while the remaining functions $f_i(\omega_\tau)$ are analytic on the whole of γ. It follows that $H(\tau)$ defined by (2.2.4) has properties (i), (iii), (iv) and (v) of the class $\mathfrak{J}(P, Z, S, T)$.

To show that it also has property (ii) we choose ω in $\mathfrak{D}(P, Z, S, T)$ to be analytic in $\overline{W}$ except for a simple pole of residue 1 at $\sigma \in P$ (Lemma 2.6). The sum of the residues of $H\omega$ in W is equal to $\int_\gamma H\omega$: but $\int_\gamma H\omega = \int_\gamma g\,\omega$, which vanishes by hypothesis. Consequently H has a zero at σ. Thus $g \in \mathfrak{J}(P, Z, S, T)$.

2.8. THEOREM. *If the differential* ω *is integrable on* γ *and if* $\int_\gamma f\omega = 0$ *for all* $f \in \mathfrak{J}(P, Z, S, T)$ *which are analytic on* γ, *then* $\omega \in \mathfrak{D}(P, Z, S, T)$.

Proof. Let ϕ be a differential analytic and non-zero in $\overline{W}$ except for simple poles at the points of P and simple zeros at the points of Z. Such a differential exists as the product of an arbitrary analytic differential with a suitable single-valued function (Lemma 2.4). Then for every differential ψ analytic in $\overline{W}$

$$\int_\gamma \left(\frac{\omega}{\phi}\right)\psi = \int_\gamma \left(\frac{\psi}{\phi}\right)\omega = 0$$

since ψ/ϕ is a single-valued function, analytic in $\overline{W}$ except for simple poles at the points of Z and vanishing at the points of P, and hence belongs to $\mathfrak{I}(P, Z, S, T)$. Thus, by Theorem 2.7, $\omega/\phi \in H_1(\gamma)$ and ω/ϕ is a single-valued analytic function on W. Therefore $\omega \in K_1(\gamma)$, ω is analytic on W with the possible exception of simple poles at the points of P, and ω has zeros at the points of Z.

Suppose now that $f \in \mathfrak{I}(P, Z, S, T)$. Let F be the corresponding analytic function from which f differs by a constant on certain of the γ_j. Suppose $j \in T$. We form a function f_1 by adding a non-zero constant c to f on γ_j. Then $f_1 \in \mathfrak{I}(P, Z, S, T)$, and $\int_\gamma f_1 \omega = 0$ as well as $\int_\gamma f\omega = 0$. Hence $\int_{\gamma_j} c\,\omega = 0$, and so the γ_j-period of ω is zero. Next, there exists a function f vanishing at the points of P, whose differential is analytic in $\overline{W}$ and has all its periods zero except for the period around the curve α_k which is conjugate to α_i (Lemma 2.6). If $i \in S$ this function f belongs to $\mathfrak{I}(P, Z, S, T)$, and so

$$0 = \int_\gamma f\omega = \sum_{j=1}^{p}\left[\int_{\alpha_j} df \int_{\alpha_{j+p}} \omega - \int_{\alpha_j} \omega \int_{\alpha_{j+p}} df\right],$$

by applying Cauchy's Theorem to W cut along the α_j,

$$= -\int_{\alpha_i} \omega \int_{\alpha_{i+p}} df.$$

It follows that the α_i-period of ω is zero, and hence $\omega \in \mathfrak{D}(P, Z, S, T)$.

3. Existence of solutions to extremal problems

3.1. The methods of Rogosinski and Shapiro [13] can be applied, in conjunction with the boundary-value theorems of the preceding section, to extremal problems on Riemann surfaces. We require a number of properties of normed linear spaces, and in particular of L and L^∞. A rather fuller summary than that which follows is to be found in [13]: the basic results are proved in [3].

(i) Let F be a normed linear space over the field of complex numbers, and let G be a linear sub-space of F. Let $I(g)$ be a bounded linear functional on G. Then a bounded linear functional B on F is said to be an extension of I if $B(g)=I(g)$ for all $g \in G$. Functionals which are extensions of the same functional on G will be called G-equivalent.

(ii) The norm of a bounded linear functional B on F, denoted by $\|B\|_F$, is

$$\sup_{\|f\|\leqslant 1} |B(f)|, \quad f \in F.$$

(iii) Consider the set of bounded linear functionals on a normed linear space F which are extensions of a functional I on the linear subspace G. By the Hahn-Banach Theorem there exists in this set a functional with minimal norm on F, and this minimal norm is equal to the norm of I on G.

(iv) Let (a, b) be a finite interval. We denote by L the class of complex-valued functions integrable over (a, b): they form a normed vector space with norm defined by $\|g(t)\| = \int_a^b |g(t)|\,dt$. And we denote by L^∞ the class of all essentially bounded measurable complex-valued functions in (a, b): they form a normed vector space with norm defined by $\|g\|_\infty = \text{ess. sup}\,|g|$.

(v) Given any sequence $\{f_n\}$ in $L^\infty(a, b)$ with $\|f_n\| \leqslant 1$, there exist a subsequence $\{f_{n_k}\}$ and a function $f \in L$ with $\|f\| \leqslant 1$ such that

$$\lim_{n\to\infty} \int_a^b f_n g\,dt = \int_c^b f g\,dt$$

for every $g \in L$.

(vi) The general linear functional B on $L(a, b)$ is of the form

$$B(f) = \int_a^b f(t)\,g(t)\,dt$$

where $g \in L^\infty$. Moreover $\|B\| = \text{ess. sup}\,|g|$.

(vii) If C is the subspace of $L^\infty(a, b)$ consisting of continuous functions $c(t)$, then the general functional B on C is of the form

$$B(c) = \int_a^b c(t)\,dm(t)$$

where the complex-valued function $m(t)$ is of bounded variation in (a, b). Moreover

$$\|B\|_C = \int_a^b |dm(t)|$$

(compare [3] pp. 61–65).

(viii) Let S_1, S_2 be the spaces of continuous functions on the intervals I_1, I_2 respectively with the L^∞ norm, and let $S_1 + S_2$ denote their direct sum. Let a norm on $S_1 + S_2$ be defined by

$$\|(s_1, s_2)\| = \max(\sup|s_1|, \ \sup|s_2|).$$

Then every bounded linear functional B on $S_1 + S_2$ is of the form

$$B(s_1, s_2) = L(s_1) + M(s_2)$$

where L is a bounded linear functional on S_1 and M is a bounded linear functional on S_2. Moreover

$$\|B(s_1, s_2)\| = \|L(s_1)\| + \|M(s_2)\|.$$

These results can be extended in the natural way to direct sums of more than two spaces.

3.2. LEMMA. *Let dg_1 be a differential integrable on γ. Then*

$$\sup\left|\int_\gamma f\,dg_1\right| \leqslant \inf \int_\gamma |dg| \tag{3.2.1}$$

where in taking the supremum we consider functions $f \in \mathfrak{J}(P, Z, S, T)$ with ess. sup. $|f| \leqslant 1$ *on γ, and in taking the infimum we consider differentials dg with $dg - dg_1 \in \mathfrak{D}(P, Z, S, T)$.*

This lemma is an immediate consequence of the fact (Lemma 2.4) that

$$\left|\int_\gamma f\,dg_1\right| = \left|\int_\gamma f\,dg\right| \leqslant \int_\gamma |dg|$$

for every function and differential in the classes considered. Later we shall be able to assert strict equality in (3.2.1).

3.3. THEOREM. *Let dg_1 be a differential integrable on γ. Then among the functions $f \in \mathfrak{J}(P, Z, S, T)$ which have* ess. sup $|f| \leqslant 1$ *on γ there is one which maximizes $\left|\int_\gamma f\,dg_1\right|$.*

Proof. The functions f which are of L-character and are essentially bounded on γ form a normed vector space with norm defined by $\|f\|_\infty = \text{ess. sup}\,|f|$, and it is easy to justify the use of properties analogous to those of $L^\infty(a, b)$. Let I be the linear functional on this vector space defined by

$$I(f)=\int_\gamma f\,d g_1.$$

Then there exists a sequence $\{f_n\}$ in the subspace G of essentially bounded functions belonging to $\mathfrak{J}(P, Z, S, T)$ such that $\|f_n\|_\infty=1$ and

$$\lim_{n\to\infty} I(f_n)=\|I\|_G,$$

because $\|I\|_G$ is $\sup_{\|f\|=1}|I|$. Hence, by the property analogous to 3.1 (v), there exists a subsequence $\{f_{n_k}\}$ and a function F with ess. sup $|F|\leqslant 1$ such that

$$\int_\gamma f_{n_k}\,dg \to \int_\gamma F\,dg$$

for every dg integrable on γ. In particular this holds for all $dg\in\mathfrak{D}(P, Z, S, T)$ analytic on γ. But in this case the left-hand side is zero for each n_k by Lemma 2.4, and it follows that $\int_\gamma F\,dg=0$ for every $dg\in\mathfrak{D}(P, Z, S, T)$ analytic on γ. Hence, by Theorem 2.7, $F\in\mathfrak{J}(P, Z, S, T)$. Also

$$\sup\left|\int_\gamma f\,dg_1\right|=\|I\|_G=\lim_{n\to\infty}\int_\gamma f_{n_k}\,dg_1=\int_\gamma F\,dg_1,$$

the supremum being taken over all $f\in G$ with $\|f\|=1$. This completes the proof.

3.4. THEOREM. *Let dg_1 be a differential integrable on γ. Then among the differentials dg such that $dg-dg_1\in\mathfrak{D}(P, Z, S, T)$ there is one which minimizes $\int_\gamma|dg|$. Moreover the minimizing differential dg has the property*

$$\int_\gamma|dg|=\sup\left|\int_\gamma f\,dg_1\right|,\quad |f|\leqslant 1 \text{ on } \gamma,\quad f\in\mathfrak{J}(P, Z, S, T).$$

Proof. Let C be the space of continuous functions on $0\leqslant\theta\leqslant 2\pi$. Let E be the space of q-tuples with elements drawn from C, and let the norm of $(c_1(\theta), c_2(\theta), \ldots, c_q(\theta))$ be defined as the greatest of the q quantities

$$\sup_{0\leqslant\theta\leqslant 2\pi}|c_j(\theta)|,\quad j=1 \text{ to } q.$$

Let Δ be the class of functions which belong to $\mathfrak{J}(P, Z, S, T)$ and are continuous on each of the γ_j. Corresponding to each member f of Δ we can form a member $(f_1, f_2, \ldots, f_q)$ of E by setting $f_j(\theta)=f[\lambda_j(e^{i\theta})]$. The q-tuples of this type form a linear subspace Γ of E.

In terms of the jth annulus parameter $z = r e^{i\theta}$ let $d g_1 = h_j(\theta)\, d\theta$ on γ_j ($j = 1$ to q). Then $d g_1$ determines a bounded linear functional I on the space E, defined by

$$I[c_1(\theta), \ldots, c_q(\theta)] = \sum_{j=1}^{q} \int_0^{2\pi} c_j(\theta)\, h_j(\theta)\, d\theta.$$

Let B be any bounded linear functional on E which is Γ-equivalent to I. Then, being a functional on E, it is of the form

$$B[(c_1, c_2, \ldots, c_q)] = \sum_{j=1}^{q} \int_0^{2\pi} c_j(\theta)\, d\mu_j(\theta)$$

where $\mu_j(\theta)$ is of bounded variation: and its norm on E is

$$\sum_{j=1}^{q} \int_0^{2\pi} |d\mu_j(\theta)|.$$

We have here used 3.1 (vii) and 3.1 (viii).

Let f be a function analytic in $\overline{W}$ which vanishes at the points of P and at the point σ in W. Then $f \in \Delta$, and since B is Γ-equivalent to I,

$$\sum_{j=1}^{q} \int_0^{2\pi} f_j(\theta)\, d\mu_j(\theta) = \sum_{j=1}^{q} \int_0^{2\pi} f_j(\theta)\, h_j(\theta)\, d\theta,$$

that is,

$$\sum_{j=1}^{q} \int_0^{2\pi} f_j(\theta)\, d M_j(\theta) = 0$$

where $M_j(\theta) = \int_0^{\theta} h_j(\theta)\, d\theta - \mu_j(\theta)$. Integrating by parts we obtain

$$\sum_{j=1}^{q} \int_0^{2\pi} f_j'(\theta)\, M_j(\theta)\, d\theta = \sum_{j=1}^{q} f_j(0)\,[M_j(2\pi) - M_j(0)].$$

Now let $d\phi$ be a differential analytic in $\overline{W}$ with the possible exception of a simple pole at σ, and with γ_j-period equal to $M_j(2\pi) - M_j(0)$ for $j = 1$ to q. In terms of the jth annular coordinate system let $d\phi = k_j(z)\, dz$. Then

$$\sum_{j=1}^{q} \int_0^{2\pi} f_j(\theta)\, k_j(e^{i\theta})\, i e^{i\theta}\, d\theta = \int_{\gamma} f\, d\phi = 0$$

by Cauchy's Theorem. Let $K_j(\theta)=\int_0^\theta k_j(e^{i\theta})\,i\,e^{i\theta}\,d\theta$. Then integrating by parts,

$$\sum_{j=1}^{q}\int_0^{2\pi} f_j'(\theta)\,K_j(\theta)\,d\theta=\sum_{j=1}^{q} f_j(0)\,[K_j(2\pi)-K_j(0)].$$

Thus
$$\sum_{j=1}^{q}\int_0^{2\pi} f_j'(\theta)\,[M_j(\theta)-K_j(\theta)]\,d\theta=0,$$

and so
$$\int_\gamma V\,df=0$$

where V is the function which takes the value $i\,e^{-i\theta}[K_j(\theta)-M_j(\theta)]$ at the point of γ_j with local parameter $e^{i\theta}$. We have shown this to be true under the condition that df is analytic in $\overline{W}$, has zero periods and satisfies $\int_{\beta_k} df=0$ for a set of arcs β_k linking up σ and the points of P. Therefore, by Theorem 2.2, $V\in H_1(\gamma)$. But $i\,e^{-i\theta}[K_j(\theta)-M_j(\theta)]$ is of bounded variation in $(0,\,2\pi)$, and so, by 1.3 (ix), V is absolutely continuous on γ. Hence $\mu_j(\theta)$ is absolutely continuous in $(0,\,2\pi)$ for each $j=1$ to q; and so, setting $\mu_j'(\theta)=F_j(\theta)$, we have $F_j(\theta)\in L$ and

$$B[c_1(\theta),\,c_2(\theta),\,\dots,\,c_q(\theta)]=\sum_{j=1}^{q}\int_0^{2\pi} c_j(\theta)\,F_j(\theta)\,d\theta. \tag{3.4.1}$$

Moreover the norm of B on E is

$$\sum_{j=1}^{q}\int_0^{2\pi} |F_j(\theta)|\,d\theta.$$

But we know that among the bounded linear functionals on E that are Γ-equivalent to I there is one with minimum norm, and that this minimum norm is equal to the norm of I on Γ. Now not only is it true, as we have proved, that every such functional is of the form (3.4.1), but every functional of the form (3.4.1) is a bounded linear functional on E. Thus we have proved that among the sets of functions $F_j(\theta)\in L(0,\,2\pi)$ such that

$$\sum_{j=1}^{q}\int_0^{2\pi} f_j(\theta)\,F_j(\theta)\,d\theta=\sum_{j=1}^{q}\int_0^{2\pi} f_j(\theta)\,h_j(\theta)\,d\theta$$

for all $(f_1(\theta),\,\dots,\,f_q(\theta))\in\Gamma$ there is one set which minimizes

$$\sum_{j=1}^{q} \int_0^{2\pi} |F_j(\theta)|\, d\theta,$$

and that the minimal value is

$$\sup \left| \sum_{j=1}^{q} \int_0^{2\pi} f_j(\theta)\, h_j(\theta)\, d\theta \right|, \quad f \in \Delta, \ |f| \leqslant 1 \text{ on } \gamma.$$

In other words, among the differentials dg such that

$$\int_\gamma f\, dg = \int_\gamma f\, dg_1$$

for all $f \in \Delta$ there is one which minimizes $\int_\gamma |dg|$: moreover the minimizing differential dg has the property

$$\int_\gamma |dg| = \sup \left| \int_\gamma f\, dg_1 \right|, \quad |f| \leqslant 1 \text{ on } \gamma, \ f \in \Delta.$$

In view of Theorem 2.8 and Lemma 2.4 this is precisely what we seek to prove, except that the above supremum is taken over a more restricted class of functions. In reality we have proved a rather stronger form of the theorem than was stated. The stated result follows immediately from the inequalities (Lemma 3.2)

$$\sup_{f \in \Delta} \left| \int_\gamma f\, dg_1 \right| \leqslant \sup_{f \in \mathcal{J}} \left| \int_\gamma f\, dg_1 \right| \leqslant \inf \int_\gamma |dg|.$$

4. Boundary behaviour of the extremals

4.1. In the theorems of § 3 the functions and differentials were not restricted to be analytic on γ itself. But in certain circumstances we can show that the extremal functions and differentials are in fact analytic *on* γ and consequently that they solve extremal problems in more restricted classes. We first prove three lemmas on the classical H_p classes.

4.2. LEMMA. *If $f(z) \in H_p$ and $g(z)$ is analytic on the unit circumference U then $f(z)\, g(z) = F(z) + G(z)$ where $F(z) \in H_p$ and $G(z)$ is analytic on U.*

Proof. The function $f(z)\, g(z)$ is analytic in some annulus $a < |z| < 1$ and is consequently of the form $F(z) + G(z)$ where $F(z)$ is analytic in $|z| < 1$ and $G(z)$ is analytic in $|z| > a$. We now deduce from Minkowski's and related inequalities that

$$\int_0^{2\pi} |F(re^{i\theta})|^p\, d\theta = O(1)$$

as $r \to 1-$, and $F(z)$ accordingly belongs to H_p.

4.3. LEMMA. *If $F(z) \in H_1$, $G(z)$ is analytic on U, and $F(z)+G(z)$ has real boundary values on U, then $F(z)$ can be continued as an analytic function on U.*

Proof. By Lemma 4.2

$$\frac{F(z)+G(z)}{z} = F_1(z) + G_1(z)$$

where $F_1(z) \in H_1$ and $G_1(z)$ is analytic on U. Therefore by 1.3 (v) the function

$$\int^{z} [F(z)+G(z)]\frac{dz}{iz},$$

with suitable precautions to ensure single-valuedness, is continuous on any arc of U. But also this function is real on U. Thus it can be continued analytically across U, and the same is consequently true of $F(z)$.

4.4. LEMMA. *Suppose that*

(i) *$F(z) = G(z) + K(z)$ where $G(z) \in H_\infty$, $K(z)$ is analytic on U;*

(ii) *$h(z) = g(z) + k(z)$ where $g(z) \in H_1$, $k(z)$ is analytic on U;*

(iii) *almost everywhere on the part of U where $h(z) \neq 0$, $|F(z)|$ and* arg $F(z)\,h(z)\,dz$ *are constant.*

Then both $F(z)$ and $h(z)$ can be continued analytically across U.

Proof. We may clearly suppose that $|F(z)| = 1$ and $F(z)\,h(z)\,dz \geqslant 0$. Then p.p. on U

$$|F(z)\,h(z)\,dz| = F(z)\,h(z)\,dz,$$

and consequently

$$|h(z)| = iz\,F(z)\,h(z).$$

We deduce that the function

$$h(z) - z^2[F(z)]^2 h(z)$$

has real boundary values, and the function

$$h(z) + z^2[F(z)]^2 h(z)$$

has imaginary boundary values on U. It follows from Lemmas 4.2 and 4.3 that both these functions can be continued as analytic functions across U. Therefore $h(z)$ and $[F(z)]^2 h(z)$ are analytic on U, and $[F(z)]^2$ is analytic on U except possibly for poles at zeros of $h(z)$. But the zeros of $h(z)$ being finite in number, $|F(z)|^2 = 1$ p.p. and $[F(z)]^2$ is accordingly analytic and non-zero on U. Hence $F(z)$, as well as $h(z)$, is analytic on U.

4.5. Theorem. *If F is an extremal function in Theorem 3.3 and df is an extremal differential in Theorem 3.4, and if $dg_1 \in K_1(\gamma)$, then both F and df are analytic on γ.*

Proof. By Theorem 3.4 and Lemma 2.4

$$\int_\gamma |df| = \left|\int_\gamma F\,dg_1\right| = \left|\int_\gamma F\,df\right|.$$

But since ess. sup $|F| \leqslant 1$ on γ equality is only possible if $|F| = 1$ and $\arg F\,df$ is constant p.p. on that part of γ where $df \neq 0$. We now express F and df in terms of the jth annulus parameter as $F(z)$ and $h(z)\,dz$. Since $dg_1 \in K_1(\gamma)$, therefore also $df \in K_1(\gamma)$, and it follows that $F(z)$ and $h(z)$ satisfy all the conditions of Lemma 4.4. Consequently F and df are analytic on γ.

The reasoning of this proof also shows that the extremal function in Theorem 3.3 is unique to the extent of an arbitrary constant multiple of absolute value one: for it can be expressed in the form

$$A\,\frac{|df|}{df}$$

where A is constant. The uniqueness is independent of the condition $dg_1 \in K_1(\gamma)$.

4.6. Corollary. *Let $dg_1 \in K_1(\gamma)$. Then*

(i) *Among the differentials dg analytic on γ such that $dg - dg_1 \in \mathcal{D}(P, Z, S, T)$ there is one which minimizes $\int_\gamma |dg|$.*

(ii) *Among the functions $f \in \mathcal{J}(P, Z, S, T)$ which are analytic and of absolute value not exceeding one on γ there is one which maximizes $\left|\int_\gamma f\,dg_1\right|$.*

(iii) *The extrema in these two problems are equal.*

4.7. The conjugate classes $\mathcal{D}(P, Z, S, T)$, $\mathcal{J}(P, Z, S, T)$ can clearly be generalized in a variety of ways to yield further conjugate extremal problems. For instance we can allow the differentials of $\mathcal{D}$ to have poles of order higher than the first if at the same time we restrict the functions of $\mathcal{J}$ to have zeros of correspondingly higher order. Or, if β is an arc of W which does not meet the curves α_i, we can restrict the differentials df of $\mathcal{D}$ by insisting that $\int_\beta df$ should vanish. In the corresponding $\mathcal{J}$-class we then have to allow the functions to have a discontinuity across β and their differentials to have simple poles at the end-points of β. In these cases the preceding theory applies with little modification.

5. Applications

5.1. In Corollary 4.6 take dg_1 to be a differential analytic in $\overline{W}$ with α_i-period equal to a_i ($i=1$ to $2p$). Take P, Z, T to be empty, and let S be the set of all integers 1 to $2p$. Then the differentials of $\mathfrak{D}(P, Z, S, T)$ which are analytic on γ are precisely those which can be extended as differentials analytic in $\overline{W}$ with vanishing α-periods. Corollary 4.6 therefore takes the following form.

(i) Among the differentials dg analytic in $\overline{W}$ with α_i-period equal to a_i ($i=1$ to $2p$) there is one which minimizes $\int\limits_\gamma |dg|$.

(ii) Among the functions F, which are integrals of differentials analytic in $\overline{W}$ with vanishing γ_i-periods, and whose absolute value on γ does not exceed 1, there is one which maximizes

$$\left|\sum_{i=1}^{p}(a_i P_{i+p} - a_{i+p} P_i)\right|$$

where P_i is the period of dF around α_i.

(iii) The extrema in these two problems are equal.

These are the problems studied in [10].

5.2. In Corollary 4.6 take dg_1 to be a differential analytic in $\overline{W}$ except for simple poles at the fixed points σ, σ_1 in W. Take Z, S, T to be empty sets; and let P contain just one point, namely σ_1. Then the differentials of $\mathfrak{D}(P, Z, S, T)$ which are analytic on γ are precisely those which can be extended as differentials analytic in $\overline{W}$ with the possible exception of a simple pole at σ_1: and the functions of $\mathfrak{J}(P, Z, S, T)$ which are analytic on γ are precisely those which can be extended as single-valued analytic functions in $\overline{W}$ vanishing at σ_1. Corollary 4.6 therefore takes the following form.

(i) Among the differentials dg which are analytic in $\overline{W}$ apart from a simple pole of residue $1/2\pi$ at σ and a simple pole at σ_1 there is one which minimizes $\int\limits_\gamma |dg|$.

(ii) Among the analytic functions on $\overline{W}$ whose absolute value does not exceed 1 and which vanish at σ_1 there is one whose absolute value at σ is a maximum.

(iii) The extrema in these two problems are equal.

These problems have been studied by Ahlfors [1].

5.3. In Corollary 4.6 take dg_1 to be a differential analytic in $\overline{W}$. Take P, S and T to be empty sets; and let Z contain just one point τ_0. Then the differentials of $\mathcal{D}(P, Z, S, T)$ which are analytic on γ are precisely those which are analytic in $\overline{W}$ with zeros at τ_0: and the functions of $\mathcal{J}(P, Z, S, T)$ which are analytic on γ are precisely those which are analytic in $\overline{W}$ with the possible exception of a simple pole at τ_0. Corollary 4.6 therefore takes the following form.

(i) Among the differentials dg analytic in $\overline{W}$, such that $dg - dg_1$ vanishes at τ_0, there exists one which minimizes $\int_\gamma |dg|$.

(ii) Among the functions f, analytic in $\overline{W}$ with the possible exception of a simple pole at τ_0 and with absolute value not exceeding 1 on γ, there is one which maximizes $\left|\int_\gamma f\, dg_1\right|$.

(iii) The extrema in these two problems are equal.

In the case when W is a plane region this statement can be further simplified as follows.

(i) Among the functions $h(z)$ analytic in $\overline{W}$ which take the value 1 at z_0 there is one which minimizes $\frac{1}{2\pi}\int_\gamma |h(z)|\, ds$.

(ii) Among the functions $f(z)$ analytic in $\overline{W}$ with the possible exception of a simple pole at z_0 and with absolute value not exceeding 1 on γ, there is one whose residue at z_0 has maximum absolute value.

(iii) The extrema in these two problems are equal.

These problems should be compared with those studied in [6].

5.4. In Corollary 4.6 take dg_1 to be a differential analytic in $\overline{W}$ with a pole of the second order at σ. Take Z, S and T to be empty sets: and let P be the set whose only member is σ. We restrict ourselves to the case in which W is a plane region: we then require only a single parameter for the whole of W and we write $h(z)\,dz$ for dg, $h_1(z)\,dz$ for dg_1 and z_0 for σ. Corresponding to differentials of $\mathcal{D}(P, Z, S, T)$ we have functions with a simple pole at z_0, and the functions of $\mathcal{J}(P, Z, S, T)$ have a zero at z_0. If we suppose $h_1(z)$ to have the form

$$h_1(z) = \frac{1}{(z - z_0)^2}$$

then Corollary 4.6 yields the following result.

(i) Among the functions $h(z)$ analytic in $\overline{W}$ except at z_0 where $h(z)$ has an expansion of the form

$$h(z) = \frac{1}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \cdots$$

there is one which minimizes $\frac{1}{2\pi}\int\limits_\gamma |h(z)|\,ds$.

(ii) Among the functions $F(z)$ analytic in $\overline{W}$, satisfying $|F(z)| \leqslant 1$ and with the expansions

$$F(z) = \beta_F(z-z_0) + b_1(z-z_0)^2 + \cdots$$

about z_0, there is one which maximizes $|\beta_F|$.

(iii) The extrema in these two problems are equal.

These problems have been studied in [5].

5.5. In Corollary 4.6 take W to be a plane region, and let P and T be empty sets. Let $dg_1 = h_1(z)\,dz$ where $h_1(z)$ vanishes at the points of Z with the single exception of the point t where it is to take the value 1. Corollary 4.6 then takes the following form.

(i) Among the functions $h(z)$, analytic in $\overline{W}$, which vanish at the points of Z, with the single exception of the point t where they take the value 1, there is one which minimizes $\frac{1}{2\pi}\int\limits_\gamma |h(z)\,dz|$.

(ii) Among the functions which are analytic in $\overline{W}$ except for simple poles at the points of Z, and whose absolute value on γ does not exceed one, there is one whose residue at t has maximum absolute value.

(iii) The extrema in these two problems are equal.

An equivalent statement of part (ii) of the above result is as follows.

(ii)′ Among the functions analytic in $\overline{W}$ except for simple poles at the points of Z, with residue 1 at t, there is one whose maximum absolute value on γ is least.

Part (i) can also be expressed in various equivalent ways. Let $G(z)$ be a function which is analytic and non-zero in $\overline{W}$ except for a simple pole of residue -1 at t and zeros elsewhere in Z. Then by setting $h(z) = f(z)\,G(z)$ we obtain:

(i)′ Among the functions $f(z)$, analytic in $\overline{W}$, which vanish at t and have

$$\frac{1}{2\pi}\int\limits_\gamma |f(z)\,G(z)\,dz| \leqslant 1$$

there is one which maximizes $|f'(t)|$.

Naturally we have equality between the extrema in (i)′ and (ii)′. In [12] Rudin treats this problem in the special case when $G(z)$ is $P'(z)$ where Re $P(z)$ is the Green's function of W with pole at t.

References

[1]. L. V. Ahlfors, Open Riemann surfaces and extremal problems on compact subregions. *Comment. Math. Helv.*, 24 (1950).
[2]. ——, Variational methods in function theory. (*Lectures at Harvard University*, 1953, transcribed by E. C. Schlesinger.)
[3]. S. Banach, *Théorie des opérations linéaires*. Warsaw, 1932.
[4]. C. Caratheodory, *Theory of Functions of a Complex Variable*. New York, 1954.
[5]. P. Garabedian, Schwarz's Lemma and the Szegö kernel function. *Trans. Amer. Math. Soc.*, 67 (1949).
[6]. ——, The classes L^p and conformal mapping. *Trans. Amer. Math. Soc.*, 69 (1950).
[7]. P. D. Lax, Reciprocal extremal problems in function theory. *Comm. Pure Appl. Math.*, 8 (1955).
[8]. R. Nevanlinna, *Uniformisierung*. Berlin, 1953.
[9]. W. F. Osgood, *Lehrbuch der Funktiontheorie*, Band II, 2. Leipzig, 1932.
[10]. A. H. Read, Harvard Thesis (not in print).
[11]. ——, Conjugate extremal problems of class $p=1$. *Ann. Ac. Sci. Fenn.* To appear.
[12]. W. Rudin, Analytic functions of class H_p. *Trans. Amer. Math. Soc.*, 78 (1955).
[13]. W. W. Rogosinski & H. S. Shapiro, On certain extremum problems for analytic functions. *Acta Math.*, 90 (1953).
[14]. V. Smirnoff, Sur les valeurs limites des fonctions analytiques. *C. R.*, 188 (1929), pp. 131–133.
[15]. A. Zygmund, *Trigonometrical Series*. New York, 1952.
[16]. J. L. Doob, A minimum problem in the theory of analytic functions. *Duke Math. J.*, 8 (1941).