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Regularity conditions on parabolic measures

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1. Introduction

We consider second-order parabolic differential operators of the following form:

$$L=L_t-\frac{\partial}{\partial t}=\frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(t,x)\frac{\partial^2}{\partial x_i\partial x_j}-\frac{\partial}{\partial t}.$$

Suppose that L is uniformly parabolic on $[s,\infty)\times\mathbf{R}^n$ for every $s>0$, i.e., we suppose that there exist positive constants $C(s)$ such that for each $(t,x)\in[s,\infty)\times\mathbf{R}^n$

$$C(s)|\xi|^2\le\sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j\le\frac{1}{C(s)}|\xi|^2 \quad \text{for every } \xi\in\mathbf{R}^n.$$

We assume that the matrix $(a_{ij}(t,x))_{i,j}$ is symmetric for every t,x.

Suppose that the coefficients of L are locally Hölder continuous on $(0,\infty)\times\mathbf{R}^n$. Then the initial value problem

$$Lu=0 \quad \text{for } t>s, \quad u(s,\cdot)=h$$

is uniquely solvable for every $s\ge 0$ and every bounded continuous function h.

The parabolic operator $L_t-\partial/\partial t$ generates a diffusion process with decreasing time parameter. That process can be characterized by the property that

$$t\mapsto v(t,X_t)$$

is a supermartingale (or a submartingale, resp.) for every function v which is continuously differentiable with respect to the time variable, twice continuously

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differentiable with respect to the space variables with bounded derivatives, and which satisfies the inequality

$$\left(L_t - \frac{\partial}{\partial t}\right)v \le 0 \quad \left(\text{or } \left(L_t - \frac{\partial}{\partial t}\right)v \ge 0, \text{ resp.}\right)$$

(see [9, Theorem 4.1]).

The solution u of the above initial value problem can be written by means of the transition probability measures $P_L(t,x;s,\cdot)$ of the diffusion process generated by the parabolic operator $L_t - \partial/\partial t$ as follows (cf. [9, Chapter 3]):

$$u(t,x) = \int_{\mathbf{R}^n} h(y)\, P_L(t,x;s,dy)$$

(notice that our notation differs from the notation in [9] since we suppose that $t>s$). The example of a singular parabolic measure given by Fabes and Kenig (see [1]) shows that the transition probability measure $P_L(t,x;0,\cdot)$ can be singular with respect to the Lebesgue measure even if the coefficients a_{ij} are uniformly continuous.

The aim of the present note is to show that $P_L(t,x;0,\cdot)$ satisfies nevertheless certain regularity conditions. More precisely, we will give polynomial upper and lower bounds for the probabilities $P_L(t,x;0,U_\varrho(y))$ for small ϱ and fixed t,x and y where $U_\varrho(y)$ denotes the open ball with radius ϱ and center y. The exponent of ϱ in the lower bound depends on the ratio of upper and lower bounds for the eigenvalues of the matrices $(a_{ij}(s,z))_{i,j}$ for every (s,z) from a neighbourhood of $(0,y)$.

The exponent of ϱ in the upper bound depends on the ratio of an upper bound for the eigenvalues of $(a_{ij}(s,z))$ and a lower bound for the traces of the restrictions of $(a_{ij}(s,z))_{i,j}$ to $(n-1)$-dimensional linear subspaces of $\mathbf{R}^n$ for every (s,z) from a neighbourhood of $(0,y)$. We remark that the assumptions which are needed for the proof of the upper bound are satisfied for a large class of parabolic operators which are degenerated at the boundary.

We show by appropriate counterexamples that the exponents which occur in the upper and lower bounds, resp., cannot be replaced by larger or smaller numbers, resp.

The upper bound for the case of space dimension n equal to 1 was first obtained by McNamara (see [7]). He also observed that an upper bound for the probabilities $P_L(t,x;0,U_\varrho(y))$ yields a lower bound for the Hausdorff dimension of every Borel set M with $P_L(t,x;0,M)>0$.

The lower bound is related to the value of the best constant in the Harnack inequality in a neighbourhood of $\{0\}\times\mathbf{R}^n$ (see [6] for the statement and a proof of the Harnack inequality for parabolic operators in nondivergence form).

For the sake of simplicity of the notation we restrict ourselves to the case $y=0$.

We obtain our estimates by comparison with appropriate solutions of the rotationally symmetric problem

$$\left(L^{\sigma_1,\sigma_2}_{r_*,s}-\frac{\partial}{\partial t}\right)u(t,x)=0 \quad \text{for all } t,x \text{ with } t>0 \text{ and } \frac{|x|}{\sqrt{t+s}}\neq r_* \tag{1}$$

where

$$L^{\sigma_1,\sigma_2}_{r_*,s}u(t,x)\equiv\begin{cases}\frac{1}{2}\sigma_1\triangle u(t,x) & \text{if } \dfrac{|x|}{\sqrt{t+s}}<r_*,\\ \frac{1}{2}(\sigma_1\triangle+(\sigma_2-\sigma_1)\partial^2_{x/|x|})u(t,x) & \text{if } \dfrac{|x|}{\sqrt{t+s}}>r_*;\end{cases}$$

here r_* and s are positive numbers and $\partial^2_{x/|x|}$ stands for the second directional derivative in the direction of $x/|x|$.

An elementary calculation yields that $(t,x)\mapsto(t+s)^{-\lambda/2}f(|x|/\sqrt{t+s})$, with $f\colon[0,\infty)\setminus\{r_*\}\to\mathbf{R}$ twice continuously differentiable, is a solution of (1) if and only if f is a solution of the following Sturm–Liouville equation with Neumann boundary condition at 0:

$$\sigma(r)f''+\left(r+\frac{(n-1)\sigma_1}{r}\right)f'+\lambda f=0 \quad \text{for every } r\in[0,\infty)\setminus\{r_*\} \tag{2}$$

where $\sigma(r)\equiv\sigma_1$ if $r<r_*$ and $\sigma(r)\equiv\sigma_2$ if $r>r_*$.

The optimal value of r_* depends on σ_1 and σ_2. It can, at least in principle, be determined by the solution of an eigenvalue problem for the Sturm–Liouville equation (2) with a free boundary at r_* (cf. [6, Section 2] and Section 2 below).

Theorem 1. *Suppose that we are given positive constants σ_1, σ_2, λ_0, r_0 and a positive solution $(t,x)\mapsto t^{-\lambda/2}f(|x|/\sqrt{t})$ of* (1) *for $s=0,\lambda=\lambda_0$, and $r_*=r_0$ (or equivalently a positive solution f of the Sturm–Liouville equation* (2)) *with the additional properties that f vanishes at least exponentially fast at infinity and that the second derivative of f at r_* exists and is equal to zero.*

1. *Assume that $(a_{ij}(t,x))_{i,j}\leq\sigma_2\mathbf{1}$ and $\operatorname{trace}(\Pi(a_{ij}(t,x))_{i,j})\geq(n-1)\sigma_1$ for every (t,x) from an appropriate neighbourhood of $(0,0)$ and every orthogonal projection Π on an $(n-1)$-dimensional linear subspace of $\mathbf{R}^n$.*

Then for every $t_0>0$ and $x_0\in\mathbf{R}^n$ the following holds:

$$\liminf_{\varrho\to0}\frac{\ln P_L(t_0,x_0;0,U_\varrho(0))}{\ln\varrho}\geq\lambda_0.$$

2. *Assume that $(a_{ij}(t,x))_{i,j}\geq\sigma_2\mathbf{1}$ and $(a_{ij}(t,x))_{i,j}\leq\sigma_1\mathbf{1}$ for every (t,x) from an appropriate neighbourhood of $(0,0)$.*

Then for every $t_0>0$ *and* $x_0\in\mathbf{R}^n$ *the following holds:*

$$\limsup_{\varrho\to 0} \frac{\ln P_L(t_0, x_0; 0, U_\varrho(0))}{\ln \varrho} \le \lambda_0.$$

Remark. We will show in Section 2, Proposition 1 that given an arbitrary pair σ_1, σ_2 of positive numbers with $\sigma_1\le\sigma_2$ (or $\sigma_1\ge\sigma_2$, resp.), there exist always a function f and positive numbers λ_0, r_0 such that the presuppositions of the first part of Theorem 1 are satisfied (or that the presuppositions of the second part of Theorem 1 are satisfied, resp.).

Estimates for that number λ_0 will be given in Section 2, Lemma 5.

Remark. A simple consequence of Theorem 1 and Section 2, Lemma 5 is that for coefficients $(a_{ij}(t,x))_{i,j}$ such that $\lim_{(t,x)\to(0,0)}(a_{ij}(t,x))_{i,j}$ exists and is nondegenerate the following holds: $\lim_{\varrho\to 0}(\ln P_L(t_0,x_0;0,U_\varrho(0))/\ln\varrho)=n$.

Remark. Suppose that $n\ge 3$. Then it is easy to see that the assumptions $(a_{ij}(t,x))_{i,j}\le\sigma_2\mathbf{1}$ and $\operatorname{trace}(\Pi(a_{ij}(t,x))_{i,j})\ge(n-1)\sigma_1$ for every (t,x) and every $(n-1)$-dimensional orthogonal projection Π are satisfied for a large class of parabolic operators which are degenerated at the boundary $\{0\}\times\mathbf{R}^n$ of the domain $[0,\infty)\times\mathbf{R}^n$. We notice that we can also generalize our results to processes of the type constructed in [5].

Remark. Under the assumptions $(a_{ij}(t,x))_{i,j}\le\sigma_2\mathbf{1}$, $(a_{ij}(t,x))_{i,j}\ge\sigma_1\mathbf{1}$ and the additional assumption that the coefficients $a_{ij}(t,x)$ of L_t are independent of the time variable t, the parabolic measures $P_L(t_0,x_0;0,\cdot)$ have a density with respect to the Lebesgue measure (see [9, Lemma 9.2.2]). Furthermore, that density belongs to $L^q(\mathbf{R}^n)$ for a constant q depending on σ_1, σ_2, and n with $q>n/(n-1)$ (see [2, Section 4]). Hence, the following estimate holds: $\liminf_{\varrho\to 0}(\ln P_L(t_0,x_0;0,U_\varrho(0))/\ln\varrho)\ge n-n/q$. We are not able to compare this estimate with our estimates since it seems to be difficult to give "good" bounds for q.

2. Existence and properties of a solution of an extremal problem for a Sturm–Liouville equation

We claim that there exists a positive number r_* such that the smallest nontrivial eigenvalue λ of the following boundary value problem attains its minimal value if $\sigma_1<\sigma_2$ or its maximal value if $\sigma_1>\sigma_2$, resp. We will denote that value of r_* by r_0. The corresponding minimal or maximal eigenvalue will be denoted by λ_0.

We consider solutions of the Sturm–Liouville equation (2) with Neumann boundary condition at 0, the condition that the solution vanishes at least exponentionally fast if the independent variable r tends to infinity, and the condition that the solution itself and its first derivative are continuous at r_*.

It is well known that the first nontrivial eigenvalue λ of that problem has multiplicity one and that the corresponding (real) eigenfunction does not change its sign.

The proof of the above claim is based on a compactness argument and explicit representations for the solutions of the following confluent hypergeometric equation which can be obtained from (2) by replacing the independent variable r by $z\equiv r^2/2\sigma_i$ for $i=1,2$ and by the substitution $f(r)=y(z)$:

$$zy''+\left(z+\frac{(n-1)\sigma_1+\sigma_i}{2\sigma_i}\right)y'+\frac{\lambda}{2}y=0. \tag{3}$$

We set $p\equiv\frac{1}{2}\lambda$ and $q_i\equiv(((n-1)\sigma_1+\sigma_i)/2\sigma_i)-\frac{1}{2}\lambda$ for $i=1,2$ and every $\lambda\geq 0$. An analytic solution y_1 on $[0,\infty)$ of the equation (3) with $i=1$ is given by the following power series representation:

$$y_1(z)\equiv 1+\sum_{k=1}^{\infty}\frac{p(p+1)\ldots(p+k-1)}{(p+q_1)(p+q_1+1)\ldots(p+q_1+k-1)}\frac{(-z)^k}{k!}. \tag{4}$$

Notice that y_1 is up to a scalar factor equal to $\int_0^1 t^{p-1}(1-t)^{q_1-1}\exp(-zt)\,dt$ if p and q_1 are greater than zero (cf. [8, Section 3.1]).

The integral on the right-hand side of the following equation

$$\widehat{y}_2(z)\equiv\oint t^{p-1}(t-1)^{q_2-1}\exp(-zt)\,dt \tag{5}$$

yields a solution of (3) on $(0,\infty)$ for $i=2$ if we consider the integral along the real axis from 1 to ∞ if $q_2>\frac{1}{4}$ and the integral round the contour consisting of the real axis from $+\infty$ to $\frac{3}{2}$, a circle with radius $\frac{1}{2}$ and center 1, and the real axis from $\frac{3}{2}$ to $+\infty$ if $q_2\leq\frac{1}{4}$. It is known that $\widehat{y}_2$ does not vanish identically (the proof for the case $q_2<0$ can be reduced to the case $q_2\geq 0$ by an iterated application of the transformation $\widehat{y}_2\mapsto-\widehat{y}_2'-\widehat{y}_2$). A simple consequence of the above integral representation for $\widehat{y}_2$ is that $|\widehat{y}_2(z)|<\exp(-\frac{1}{4}z)$ holds for every sufficiently large positive z.

Finally, we consider the function $\widetilde{y}_2$ which is defined by the right-hand side of (4) with q_2 in place of q_1. It is known that $\widetilde{y}_2(z)\sim z^{-p}$ for large positive z and $q_2\neq 0,-1,\ldots$ (see for instance [8, Section 4.1.1]). Hence, $\widehat{y}_2$ and $\widetilde{y}_2$ form a fundamental system of solutions of (3) if $q_2\neq 0,-1,\ldots$.

A standard application of Sturm's Comparison Theorem yields that for every λ_1 there exists an r_1 such that every solution of (3) has at most one zero in the interval $[r_1, \infty)$ if $\lambda < \lambda_1$. We aim to show that $\widehat{y}_2$ has no zero in the open interval (r_1, ∞) if $\lambda < \lambda_1$. Suppose that on the contrary $\widehat{y}_2(r) = 0$ for some $r > r_1$ and $\lambda < \lambda_1$. Uniqueness theorems for ordinary differential equations of second order imply that $\widehat{y}_2'(r) \neq 0$. We notice that $\widehat{y}_2$ and $\widehat{y}_2'$ depend continuously on the parameter q_2 if $q_2 \leq \frac{1}{4}$. Thus, we can assume without loss of generality that $q_2 \neq 0, -1, \dots$. If we take into account that $|\widehat{y}_2(z)| < \exp(-\frac{1}{4}z)$ and that $\widetilde{y}_2(z) \sim z^{-p}$ for z sufficiently large, we obtain that there exists a (small) real number δ such that $\widehat{y}_2 + \delta \widetilde{y}_2$ has at least two zeros in the interval (r_1, ∞). This is in contradiction to our assumption on $[r_1, \infty)$. We can conclude that there exist scalar multiples y_2 of $\widehat{y}_2$ with $|y_2| \leq |\widehat{y}_2|$ for all p and q such that every y_2 is nonnegative for z sufficiently large and such that y_2 and its derivatives of arbitrary order depend continuously on the parameter p.

We set

$$f_i^{(\lambda)}(r) \equiv y_i\left(\frac{r^2}{2\sigma_i}\right) \tag{6}$$

for every $\lambda > 0$, $i = 1, 2$, and every r with $r \geq 0$ if $i = 1$ and every r with $r > 0$ if $i = 2$. By (4), it is easy to see that $f_1^{(\lambda)}$ converges to 1 and its derivatives up to an arbitrary fixed order converge to zero uniformly on every bounded interval if the nonnegative number λ tends to zero. The above integral representation for $\widehat{y}_2$ yields that $f_2^{(\lambda)\prime}$ is negative on $[r_*, \infty)$ for every given $r_* > 0$ if λ is sufficiently small (take into account that $q_2 > \frac{1}{4}$ if $0 \leq \lambda < \frac{1}{2}$).

We define a real function F on $(0, \infty) \times (0, \infty)$ as follows:

$$F(\lambda, r_*) \equiv f_1^{(\lambda)\prime}(r_*) f_2^{(\lambda)}(r_*) - f_2^{(\lambda)\prime}(r_*) f_1^{(\lambda)}(r_*) \text{ for every } \lambda, r_* > 0. \tag{7}$$

It follows from the above considerations that $F(\lambda, r_*)$ is positive for every sufficiently small $\lambda > 0$.

It can be deduced from Sturm's Comparison Theorem that neither $f_1^{(\lambda)}$ nor $f_2^{(\lambda)}$ is nonnegative on $[0, r_*)$ or (r_*, ∞) for a given $r_* > 0$ if λ is sufficiently large. We consider the smallest positive number $\lambda_1(r_*)$ such that $f_1^{(\lambda_1(r_*))}(r_*) = 0$ or $f_2^{(\lambda_1(r_*))\prime}(r_*) = 0$. Notice that $f_1^{(\lambda)}(r) = f_1^{(\lambda)\prime}(r) = 0$ or $f_2^{(\lambda)\prime}(r) = f_2^{(\lambda)\prime\prime}(r) = 0$ for an arbitrary $r > 0$ are in contradiction to uniqueness theorems for ordinary differential equations of second order. Hence, elementary properties of zeros of differentiable functions imply that $f_1^{(\lambda)}(r) > 0$ for every r, λ with $r \leq r_*$ and $\lambda < \lambda_1(r_*)$. Moreover, we obtain that the condition $f_1^{(\lambda)\prime}(r) = 0$ and $f_1^{(\lambda)\prime\prime}(r) \geq 0$ for some r with $0 < r \leq r_*$ and $\lambda \leq \lambda_1(r_*)$ is in contradiction to (2) (take into account that $f_1^{(\lambda_1(r_*))\prime}(r) = 0$ implies that $f_1^{(\lambda_1(r_*))}(r)\neq 0)$. By (4), $f_1^{(\lambda)\prime}(0)=0$ and $f_1^{(\lambda)\prime\prime}(0)<0$. Hence, $f_1^{(\lambda)\prime}(r)<0$ for every r, λ with $0<r\leq r_*$ and $\lambda\leq\lambda_1(r_*)$.

The function $f_2^{(\lambda)\prime}$ is a solution of a Sturm–Liouville equation and we can deduce from Sturm's Comparison Theorem that there exists an r_1 such that $f_2^{(\lambda)\prime}$ has at most one zero in $[r_1,\infty)$ for every λ with $\lambda\leq\lambda_1(r_*)$. Recall that $f_2^{(\lambda)}(r)$ is nonnegative for r sufficiently large and that $\lim_{r\to\infty} f_2^{(\lambda)}(r)=0$ for every λ. Hence, $f_2^{(\lambda)\prime}(r)$ is negative for r sufficiently large and every λ with $\lambda<\lambda_1(r_*)$. Again, we can conclude that $f_2^{(\lambda)\prime}(r)<0$ for every $r\geq r_*$ and $\lambda<\lambda_1(r_*)$. In particular, we obtain that $f_2^{(\lambda_1(r_*))}(r_*)>0$.

Hence, $F(\lambda_1(r_*),r_*)=f_1^{(\lambda_1(r_*))\prime}(r_*)f_2^{(\lambda_1(r_*))}(r_*)<0$ for every positive r_*.

We can conclude that for every given positive number r_* there exists a $\lambda(r_*)\in(0,\lambda_1(r_*))$ such that $F(\lambda(r_*),r_*)=0$.

Moreover, we have $f_1^{(\lambda(r_*))}(r)>0$ for every $r\in[0,r_*]$, $f_1^{(\lambda(r_*))\prime}(r)<0$ for every $r\in(0,r_*]$, $f_2^{(\lambda(r_*))\prime}(r)<0$ and $f_2^{(\lambda(r_*))}(r)>0$ for every $r\in[r_*,\infty)$.

Lemma 1. *Suppose that $f^{(\lambda_1)}$ and $f^{(\lambda_2)}$ are nonnegative solutions of (2) on $[0,\infty)\setminus\{r_*\}$ for $\lambda=\lambda_1$ or $\lambda=\lambda_2$, resp., and a fixed positive number r_*. We assume that both functions $f^{(\lambda_i)}$ for $i=1,2$ satisfy the boundary conditions $f^{(\lambda_i)\prime}(0)=0$ and $f^{(\lambda_i)}$ vanishes at least exponentially fast at infinity. Suppose that $f^{(\lambda_1)}$ and $f^{(\lambda_2)}$ are continuous at r_* and that $f^{(\lambda_i)\prime\prime}(r_*)=(-1)^i c\delta_{r_*}$ for a positive c and $i=1,2$ in the distributional sense.*

Then $\lambda_1<\lambda(r_*)<\lambda_2$.

Proof. For the sake of simplicity of the notation we restrict ourselves to the proof of $\lambda_1<\lambda_2$. Suppose on the contrary that $\lambda_1\geq\lambda_2$. Without loss of generality we may and will assume that $f^{(\lambda_1)}(r_*)=f^{(\lambda_2)}(r_*)=1$.

We set $g^{(i)}\equiv f^{(\lambda_i)\prime}/f^{(\lambda_i)}$. By assumption, $g^{(1)}(0)=g^{(2)}(0)=0$. The Wronski determinant of every pair of solutions of (2) on $[0,\infty)$ is a scalar multiple of $r\mapsto(1/r^{n-1})\exp(-r^2/2\sigma_1)$. Hence, the solution $f^{(\lambda_i)}$ of (2) on $[0,r_*)$ with $f^{(\lambda_i)\prime}(0)=0$ is uniquely determined up to a scalar factor. An elementary calculation shows that $g^{(1)\prime}(0)<g^{(2)\prime}(0)$ (cf. (4)). Since $g^{(i)}$ is a solution of the Riccati equation

$$\sigma_1 g'+g^2+\left(r+\frac{(n-1)\sigma_1}{r}\right)g+\lambda_i=0 \tag{8}$$

on $[0,r_*)$ for the corresponding value of i, it can be deduced from comparison theorems for ordinary differential equations of first order that $g^{(1)}(r_*-0)\leq g^{(2)}(r_*-0)$.

Thus, the assumptions of the lemma imply that $g^{(1)}(r_*+0)<g^{(2)}(r_*+0)$. We aim to show that this leads to a contradiction. Since $f_2^{(\lambda)}$ and $f_2^{(\lambda)\prime}$ depend continuously on λ, we can restrict ourselves for the following considerations to the case that

$q_2^{(1)} \neq 0, -1, \ldots$ where $q_2^{(1)} \equiv (((n-1)\sigma_1+\sigma_2)/2\sigma_2) - \frac{1}{2}\lambda_1$. Let ε be a positive number such that

$$g^{(1)}(r_*+0)+\varepsilon < g^{(2)}(r_*+0) \tag{9}$$

and consider the unique solution h of the initial value problem $h(r_*)=0$ and $h'(r_*)=\varepsilon$ for the equation (2) with λ_1 in place of λ. If we apply Sturm's Comparison Theorem to the functions $f^{(\lambda_1)}$ and h we obtain that $h(r)>0$ for every r with $r>r_*$.

Recall that y_2 and $\widetilde{y}_2$ form a fundamental system of solutions of (3) and that $|y_2(z)|<\exp(-\frac{1}{4}z)$ and $\widetilde{y}_2(z) \sim z^{-p}$ for large z. We obtain in particular that a solution of (3) that vanishes at infinity faster than every power of z is uniquely determined up to a constant factor. The function h is not proportional to the function $f^{(\lambda_1)}$. The functions $f^{(\lambda_1)}$ and $f^{(\lambda_2)}$ vanish at infinity faster than every power of $r^2=z$. Hence, $(f^{(\lambda_1)}+h)(r) \geq f^{(\lambda_2)}(r)$ for every sufficiently large r.

In view of $(f^{(\lambda_1)}+h)(r_*)=f^{(\lambda_1)}(r_*)=f^{(\lambda_2)}(r_*)=1$ and

$$\frac{(f^{(\lambda_1)}+h)'}{f^{(\lambda_1)}+h}(r_*+0) = g^{(1)}(r_*+0)+\varepsilon < g^{(2)}(r_*+0) = \frac{f^{(\lambda_2)\prime}}{f^{(\lambda_2)}}(r_*+0)$$

(cf. (9)) we can deduce from $(f^{(\lambda_1)}+h)(r) \geq f^{(\lambda_2)}(r)$ for every sufficiently large r that there exists an $r_1>r_*$ with $(f^{(\lambda_1)}+h)(r_1)=f^{(\lambda_2)}(r_1)$ and $(f^{(\lambda_1)}+h)'(r_1) \geq f^{(\lambda_2)\prime}(r_1)$. Hence,

$$\frac{(f^{(\lambda_1)}+h)'}{f^{(\lambda_1)}+h}(r_1) \geq \frac{f^{(\lambda_2)\prime}}{f^{(\lambda_2)}}(r_1).$$

Since $(f^{(\lambda_1)}+h)'/(f^{(\lambda_1)}+h)$ and $f^{(\lambda_2)\prime}/f^{(\lambda_2)}$ are solutions of (8), comparison theorems for ordinary differential equations of first order lead to a contradiction.

Remark. A similar argument as in the proof of Lemma 1 shows that $\lambda(r_*)$ is the unique zero of $\lambda \mapsto F(\lambda, r_*)$ such that $f_1^{(\lambda)}(r) \geq 0$ for every $r \leq r_*$ and $f_2^{(\lambda)\prime}(r) \leq 0$ for every $r \geq r_*$. Hence, the continuity of F implies that the function $r_* \mapsto \lambda(r_*)$ is also continuous.

In order to apply a compactness argument we aim to show that there exists an $r_1>0$ such that for every r_* with $r_*>r_1$ the inequality $\lambda(r_2)>\lambda(r_*)$ holds for every r_2 with $r_*-\varepsilon<r_2<r_*$ and an appropriate positive ε if $\sigma_1<\sigma_2$ and similarly that $\lambda(r_2)<\lambda(r_*)$ holds if $\sigma_1>\sigma_2$.

The proof is based on the following lemma.

Lemma 2. *Let σ_1 be given. Then there exists a positive number r_1 such that every function $f_1^{(\lambda)\prime\prime}$ for $\lambda>0$ has at least one zero in the interval $[0, r_1)$ (cf. (6) for the definition of $f_1^{(\lambda)}$).*

Proof. Using an argument similar to the argument used at the beginning of the proof of Lemma 1, we can reduce the proof of the assertion of Lemma 2 to the case of all positive λ belonging to an arbitrary small neighbourhood of 0 (by differentiation of (2) we obtain a linear equation of second order for $f_1^{(\lambda)'}$; therefore the function $f_1^{(\lambda)''}/f_1^{(\lambda)'}$ satisfies a Riccati equation; that equation replaces (8)). In view of [7, Lemma 1], we can restrict ourselves to the case $n\geq 2$.

We have $f_1^{(\lambda)''}(r)=2zy_1''(z)+y_1'(z)$ for $z\equiv\frac{1}{2}r^2$ and $y_1(z)=f_1^{(\lambda)}(r)$. The power series representation of $-(n/\lambda)y_1'$ is given by the right-hand side of (4) for $p\equiv\frac{1}{2}\lambda+1$ and $q_1\equiv\frac{1}{2}n-\frac{1}{2}\lambda$. Moreover, the function $(1/\lambda)y_1'$ and its derivative with respect to z depend continuously on λ for $\lambda\geq 0$.

Therefore it is sufficient to prove that there exists an $z_1>0$ with $2z_1y_0'(z_1)+y_0(z_1)=0$ for the function y_0 defined by the right-hand side of (4) for $p\equiv 1$, $q_1\equiv\frac{1}{2}n$.

Since y_0 is up to a scalar factor equal to $\int_0^1(1-t)^{q_1-1}\exp(-zt)\,dt$ and since by definition $y_0(0)=1$, we obtain that $y_0(z)>0$ and $y_0'(z)<0$ for every $z>0$. Suppose that $2zy_0'(z)+y_0(z)>0$ for every $z>0$. Hence, by Gronwall's Lemma, $y_0(z)\geq(1/\sqrt{z})y_0(1)$ for every $z\geq 1$.

This is in contradiction to $\int_0^1(1-t)^{q_1-1}\exp(-zt)\,dt\leq\int_0^1\exp(-zt)\,dt<1/z$ (recall that $q_1=\frac{1}{2}n\geq 1$).

Lemma 3. *Assume that we are given λ_1 and $r_1>0$ with $F(\lambda_1,r_1)=0$ (cf. (7) for the definition of F). Suppose that $f_1^{(\lambda_1)''}(r_1)>0$.*

Then there exists an $\varepsilon>0$ with $\lambda(r_2)<\lambda_1$ $(=\lambda(r_1))$ for every $r_2\in[r_1-\varepsilon,r_1)$ if $\sigma_1<\sigma_2$ and $\lambda(r_2)>\lambda_1$ for every $r_2\in[r_1-\varepsilon,r_1)$ if $\sigma_1>\sigma_2$.

Proof. The definition of F yields that $f_1^{(\lambda_1)'}(r_1)f_2^{(\lambda_1)}(r_1)-f_2^{(\lambda_1)'}(r_1)f_1^{(\lambda_1)}(r_1)=0$. Moreover, we have $f_1^{(\lambda_1)}(r_1)>0$, $f_1^{(\lambda_1)'}(r_1)<0$, $f_2^{(\lambda_1)'}(r_1)<0$, and $f_2^{(\lambda_1)}(r_1)>0$.

Since $f_1^{(\lambda_1)}$ and $f_2^{(\lambda_1)}$ are both solutions of an equation of the form (2) with σ_1 or σ_2 in place of $\sigma(r)$, we can conclude from $f_1^{(\lambda_1)'}(r_1)/f_1^{(\lambda_1)}(r_1)=f_2^{(\lambda_1)'}(r_1)/f_2^{(\lambda_1)}(r_1)$ that

$$\sigma_1\frac{f_1^{(\lambda_1)''}(r_1)}{f_1^{(\lambda_1)}(r_1)}=\sigma_2\frac{f_2^{(\lambda_1)''}(r_1)}{f_2^{(\lambda_1)}(r_1)}. \tag{10}$$

Thus,

$$\frac{\partial}{\partial r_*}F(\lambda,r_*)|_{(\lambda_1,r_1)}=f_1^{(\lambda_1)''}(r_1)f_2^{(\lambda_1)}(r_1)-f_2^{(\lambda_1)''}(r_1)f_1^{(\lambda_1)}(r_1)\gtrless 0$$

if $\sigma_1\lessgtr\sigma_2$, resp. The assertion of Lemma 3 is now a consequence of Lemma 1.

Lemma 4. *Suppose that $f_1^{(\lambda)}$ is decreasing on $[0,r_*]$ (or that $f_2^{(\lambda)}$ is decreasing on $[r_*,\infty)$, resp.) for a positive number λ.*

Then $f_1^{(\lambda)''}$ has at most one zero in $[0,r_*]$ (or $f_2^{(\lambda)''}$ has at most two zeros in $[r_*,\infty)$, resp.).

Proof. It can be deduced from the power series representation for the corresponding solution of the equation (3) that $f_1^{(\lambda)''}(0)<0$. Hence the assertion of the lemma is established once we have shown that $f_i^{(\lambda)''}(r_1)=f_i^{(\lambda)''}(r_2)=0$ and $f_i^{(\lambda)'''}(r_1)\geq 0\geq f_i^{(\lambda)'''}(r_2)$ for r_1,r_2 with $0<r_1<r_2\leq r_*$ and $i=1$ or $r_*\leq r_1<r_2$ and $i=2$ leads to a contradiction. The derivative g of $f_i^{(\lambda)}$ satisfies the following Sturm–Liouville equation:

$$\sigma(r)g''(r)+\left(\frac{(n-1)\sigma_1}{r}+r\right)g'(r)+\left(\lambda+1-\frac{(n-1)\sigma_1}{r^2}\right)g(r)=0 \tag{11}$$

on $(0,r_*)$ for $i=1$ (or on (r_*,∞) for $i=2$) (recall that σ is by definition constant on each of the intervals $[0,r_*]$ and (r_*,∞)). By assumption, we have $g(r_k)\leq 0$ and $g'(r_k)=0$ for $k=1,2$. By uniqueness theorems for ordinary differential equations, $g(r_k)\neq 0$ for $k=1,2$. Since $g''(r_1)\geq 0\geq g''(r_2)$, we can deduce from (11) that $\lambda+1-((n-1)\sigma_1/r_1^2)\geq 0\geq \lambda+1-((n-1)\sigma_1/r_2^2)$. This is in contradiction to $r_1<r_2$.

Proposition 1. *Let σ_1 and σ_2 be positive numbers.*

Then there exist positive numbers λ_0 and r_0 such that the Sturm–Liouville equation (2) with $r_*=r_0$, with Neumann boundary condition at 0, and the condition that the solution vanishes at least exponentially fast at infinity has a positive solution f_0 which is twice continuously differentiable on $[0,\infty)$.

Moreover, we have $f_0''(r)\leq 0$ if $r\leq r_0$, $f_0''(r)\geq 0$ if $r\geq r_0$, and $f_0'(r)\leq 0$ for every r.

Proof. We restrict ourselves to the case $\sigma_1<\sigma_2$, the remaining cases $\sigma_1>\sigma_2$ and $\sigma_1=\sigma_2$ can be dealt with in a similar way.

By Lemma 2 and Lemma 4, there exists an r_1 such that $f_1^{(\lambda(r_*))''}(r_*)>0$ for every r_* with $r_*\geq r_1$. Thus, we can conclude from Lemma 3 that $r_*\mapsto\lambda(r_*)$ is increasing on $[r_1,\infty)$.

On the other hand, we can obtain (at least in principle) an upper bound for $\lambda(r_*)$ from Sturm's Comparison Theorem if we take into account that $f_1^{(\lambda(r_*))}$ is nonnegative on $[0,r_*]$ and that $f_2^{(\lambda(r_*))}$ is nonnegative on $[r_*,\infty)$. Thus, we can conclude from (4) that $f_1^{(\lambda(r_*))''}(r)<0$ for every $r\in[0,r_*]$ and every sufficiently small positive r_*. A similar argument as in the proof of Lemma 3 shows that $r_*\mapsto\lambda(r_*)$ is decreasing in an appropriate neighbourhood of 0.

Since F is continuous, there exists a nonnegative number r_0 such that $r_* \mapsto \lambda(r_*)$ attains its minimal value at $r_*=r_0$. We set $\lambda_0 \equiv \lambda(r_0)$.

By Lemma 3, we can deduce that the inequality $f_1^{(\lambda_0)''}(r_0) \leq 0$ holds. It can be shown in exactly the same way that $f_1^{(\lambda_0)''}(r_0) \geq 0$. Hence, $f_1^{(\lambda_0)''}(r_0)=0$. Notice, that we have in particular $r_0 \neq 0$.

We also have $f_2^{(\lambda_0)''}(r_0)=0$ (cf. (10)). We set $f_0(r) \equiv f_1^{(\lambda_0)}(r)$ if $r \leq r_0$ and $f_0(r) \equiv (f_1^{(\lambda_0)}(r_0)/f_2^{(\lambda_0)}(r_0)) f_2^{(\lambda_0)}(r)$ if $r \geq r_0$. This establishes the existence of a twice continuously differentiable positive solution f_0 of (2) on $[0,\infty)$.

The inequality $f_1^{(\lambda_0)''}(0)<0$ (cf. the first step of the proof of Lemma 4) and Lemma 4 imply that $f_1^{(\lambda_0)''}(r)<0$ if $r<r_0$. Hence, $f_1^{(\lambda_0)'''}(r_0) \geq 0$. If we take into account that $f_1^{(\lambda_0)'}$ and $f_2^{(\lambda_0)'}$ are solutions of (11) on $(0,r_*)$ or (r_*,∞), resp., and that $f_1^{(\lambda_0)'}(r_0)=f_2^{(\lambda_0)'}(r_0)$ and $f_1^{(\lambda_0)''}(r_0)=f_2^{(\lambda_0)''}(r_0)=0$, we can conclude that $f_2^{(\lambda_0)'''}(r_0) \geq 0$. Hence, by Lemma 4, $f_2^{(\lambda_0)''}(r) \geq 0$ if $r \geq r_0$.

The last part of the assertion of the proposition was already shown before Lemma 1.

Remark. An immediate consequence of Proposition 1, Lemma 4, and the uniqueness up to a scalar factor of solutions of (3) that are regular at the origin is that the positive number r_0 such that $r_* \mapsto \lambda(r_*)$ attains its minimal value if $\sigma_1<\sigma_2$ (or its maximal value if $\sigma_1>\sigma_2$) is uniquely determined.

Now we aim to give a (rough) estimate for λ_0 in terms of σ_1 and σ_2.

Lemma 5. *Suppose that λ_0 satisfies the conditions of Proposition* 1. *Then*

$$\frac{n\sigma_1}{\sigma_2} \leq \lambda_0 < \frac{(n-1)\sigma_1}{\sigma_2}+1 \quad \text{if } \sigma_1<\sigma_2$$

and

$$\frac{n\sigma_1}{\sigma_2} \geq \lambda_0 > \frac{(n-1)\sigma_1}{\sigma_2}+1 \quad \text{if } \sigma_1>\sigma_2.$$

Proof. Suppose that $\sigma_1<\sigma_2$.

We aim to show first that $\lambda_0<((n-1)\sigma_1/\sigma_2)+1$. Our proof is based on Lemma 1. We set $\lambda_2 \equiv ((n-1)\sigma_1/\sigma_2)+1$. Now, we define a function $f^{(\lambda_2)}$ on $[0,\infty)$ by $f^{(\lambda_2)}(r) \equiv \exp(-r^2/2\sigma_2)$ if $r \geq \varepsilon$ and

$$f^{(\lambda_2)}(r) \equiv y_1\left(\frac{r^2}{2\sigma_1}\right)\frac{\exp(-\varepsilon^2/2\sigma_2)}{y_1(\varepsilon^2/2\sigma_1)}$$

if $r \leq \varepsilon$; the function y_1 is defined by the right-hand side of (4) for $p \equiv \frac{1}{2}\lambda_2$ and $q \equiv \frac{1}{2}n-\frac{1}{2}\lambda_2$ and the positive number ε will be chosen later.

Thus, $f^{(\lambda_2)}$ is a solution of (2) on $[0,\infty)$ with $f^{(\lambda_2)\prime}(0)=0$ and such that $f^{(\lambda_2)}$ vanishes at least exponentially at infinity. Once we have shown that $f^{(\lambda_2)\prime\prime}(\varepsilon)=+\infty$ in the distributional sense for every sufficiently small positive ε, the assertion follows from $\lambda_0=\inf_{r_*>0}\lambda(r_*)\le\lambda(\varepsilon)<\lambda_2$ (the last step is a consequence of Lemma 1). The proof can be completed by the observation that for ε small the following holds:

$$\begin{aligned} f^{(\lambda_2)\prime}(\varepsilon-0) &= \frac{\varepsilon}{\sigma_1} y_1'\left(\frac{\varepsilon^2}{2\sigma_1}\right)\frac{\exp(-\varepsilon^2/2\sigma_2)}{y_1(\varepsilon^2/2\sigma_1)} = -\frac{\varepsilon}{\sigma_1}\frac{\lambda_2}{n}+O(\varepsilon) \\ &< -\frac{\varepsilon}{\sigma_2} = f^{(\lambda_2)\prime}(\varepsilon+0)+O(\varepsilon). \end{aligned}$$

(We remark that the number $((n-1)\sigma_1/\sigma_2)+1$ can be obtained as the smallest nontrivial eigenvalue of an eigenvalue problem for an operator whose coefficients do not depend on the time variable; see [2, Section 4].)

Next, we show that under the above assumption the second part of the assertion holds, i.e. we show that $\lambda_0\ge n\sigma_1/\sigma_2$ if $\sigma_1<\sigma_2$. Let ε be an arbitrary positive number. We set

$$v(t,x)\equiv t^{-n\sigma_1/2(\sigma_2+\varepsilon)}\exp\left(-\frac{|x|^2}{2(\sigma_2+\varepsilon)t}\right) \quad \text{for } t>0,\ x\in\mathbf{R}^n.$$

Then (we write r for $x/\sqrt{t}$; the number r_0 was defined in Proposition 1):

$$\begin{aligned} \frac{t}{v(t,x)}\left(L_{r_0}^{\sigma_1,\sigma_2}-\frac{\partial}{\partial t}\right)v(t,x) &= \frac{\sigma(r)}{2(\sigma_2+\varepsilon)^2}r^2-\frac{1}{2(\sigma_2+\varepsilon)}r^2 \\ &\quad -\frac{(n-1)\sigma_1+\sigma(r)}{2(\sigma_2+\varepsilon)}+\frac{n\sigma_1}{2(\sigma_2+\varepsilon)}\le 0 \end{aligned}$$

(recall that $\sigma(r)=\sigma_1$ if $r\le r_*$ and $\sigma(r)=\sigma_2$ if $r>r_*$). On the other hand, for

$$\tilde{u}(t,x)\equiv t^{-\lambda_0/2}f_0\left(\frac{|x|}{\sqrt{t}}\right) \quad \text{for every } t>0,\ x\in\mathbf{R}^n$$

we obtain that

$$L_{r_0}^{\sigma_1,\sigma_2}\tilde{u}(t,x)=0 \quad \text{for every } t,x.$$

Using a similar integral representation for $\hat{y}_2$ as in (5) for a contour in the complex plane that contains a circle with center 1 and sufficiently small positive radius, we can show that given an arbitrary positive number δ the function y_2 can be estimated from above by $\exp(-(1-\delta)z)$ for every sufficiently large z. Hence, there exists a positive constant C such that $Cv(1,x)-\tilde{u}(1,x)\ge 0$ for every $x\in\mathbf{R}^n$. The maximum principle yields that $Cv(t,x)-\tilde{u}(t,x)\ge 0$ for every $t>1$ and $x\in\mathbf{R}^n$. Thus, we can conclude that $\lambda_0 \geq n\sigma_1/(\sigma_2+\varepsilon)$. Since ε was an arbitrary positive number, $\lambda_0 \geq n\sigma_1/\sigma_2$.

The inequality $\lambda_0 \leq n\sigma_1/\sigma_2$ for $\sigma_1 > \sigma_2$ can be proven in a similar way if we take into account that for every positive number δ there exists an z_0 such that $y_2(z) \geq \exp(-(1+\delta)z)$ for every z with $z \geq z_0$. Because the author could not find a good reference, a sketch of a proof is given.

We will restrict ourselves to the case when $\delta < \frac{1}{2}$. The estimate $0 \leq y_2(z) \leq \exp(-(1-\delta)z)$ for large z and the fact that $(\exp((1-2\delta)z)y_2(z))'$ satisfies a Sturm–Liouville equation for which every solution only has a finite number of zeros yield that $z \mapsto \exp((1-2\delta)z)y_2(z)$ is decreasing on $[z_1, \infty)$ for a sufficiently large z_1. Hence, $0 \leq y_2(z) \leq -(1/(1-2\delta))y_2'(z)$ for sufficiently large z. We can conclude from (3) that $-(1-\frac{1}{2}\delta)y_2'(z) \leq y_2''(z) \leq -(1+\frac{1}{2}\delta)y_2'(z)$ for sufficiently large z. The proof can be completed by an application of Gronwall's Lemma.

The inequality $\lambda_0 > ((n-1)\sigma_1/\sigma_2)+1$ for $\sigma_1 > \sigma_2$ can be established in exactly the same way as the corresponding inequality for $\sigma_1 < \sigma_2$.

Lemma 6. *Suppose that $f_0 \in C^2[0,\infty)$ satisfies the conditions of Proposition* 1. *Assume in addition that $\sigma_1 \leq \sigma_2$.*

Then $f_0''(r) \geq (1/r) f_0'(r)$ for every $r > 0$.

Proof. The assertion is obvious if $r \geq r_0$ or if $\sigma_1 = \sigma_2$. Therefore we suppose that $r < r_0$ and that $\sigma_1 < \sigma_2$.

We set $z \equiv \frac{1}{2}r^2$ and $f_0(r) = y(z)$. Thus, we obtain that $f_0''(r) - (1/r)f_0'(r) = y''(z)$.

By Lemma 5, we have $\lambda_0 < n$ and therefore $q_1 \equiv \frac{1}{2}n - \frac{1}{2}\lambda_0 > 0$. Hence, $y_1(z) = C\int_0^1 t^{p-1}(1-t)^{q_1-1}\exp(-zt)\,dt$ for a positive constant C (cf. [8, Section 3.1]). This completes the proof.

3. Proof of the main result and applications

The objective of this section is to prove Theorem 1 and to give several related results which can be shown by similar methods.

Proof of the first part of Theorem 1. In view of well-known localization arguments (see [9, Section 6.6]), we may and will restrict ourselves to the case where the assumptions of the first part of the theorem are satisfied for every $(t,x) \in [0,\infty) \times \mathbf{R}^n$. We will furthermore suppose that $\sigma_1 < \sigma_2$ since the assertion is trivial if $\sigma_1 = \sigma_2$. We claim that

$$t \mapsto (t+s)^{-\lambda_0/2} f_0\left(\frac{|X_t|}{\sqrt{t+s}}\right)$$

is a supermartingale with decreasing time parameter t for the diffusion process $t\mapsto X_t$ with differential generator $L_t-\partial/\partial t$ and for every positive s.

By Section 2, Proposition 1,

$$\text{(12)}\quad \operatorname{trace}\left((a_{ij}(t,x))_{i,j}\left(\frac{\partial^2}{\partial x_i\partial x_j}f_0\left(\frac{|x|}{\sqrt{t+s}}\right)\right)_{i,j}\right) \le \frac{1}{t+s}\left(\sigma_2 f_0''\left(\frac{|x|}{\sqrt{t+s}}\right)+\sigma_1(n-1)\frac{\sqrt{t+s}}{|x|}f_0'\left(\frac{|x|}{\sqrt{t+s}}\right)\right)$$

for every s,t,x with $|x|/\sqrt{t+s}\ge r_0$.

The estimate $\operatorname{trace}(\Pi(a_{ij}(t,x))_{i,j})\ge(n-1)\sigma_1$ for every $(n-1)$-dimensional orthogonal projection Π implies in particular that $\operatorname{trace}((a_{ij}(t,x))_{i,j})\ge n\sigma_1$.

By Section 2, Proposition 1 and Lemma 6, $0\ge f_0''(r)\ge(1/r)f_0'(r)$ for every r with $r\le r_0$. Thus, we can conclude that

$$\text{(13)}\quad \operatorname{trace}\left((a_{ij}(t,x))_{i,j}\left(\frac{\partial^2}{\partial x_i\partial x_j}f_0\left(\frac{|x|}{\sqrt{t+s}}\right)\right)_{i,j}\right) \le \frac{1}{t+s}\left(\sigma_1 f_0''\left(\frac{|x|}{\sqrt{t+s}}\right)+\sigma_1(n-1)\frac{\sqrt{t+s}}{|x|}f_0'\left(\frac{|x|}{\sqrt{t+s}}\right)\right)$$

for every s,t,x with $|x|/\sqrt{t+s}\le r_0$.

By (12) and (13),

$$\left(L_t-\frac{\partial}{\partial t}\right)\left((t+s)^{-\lambda_0/2}f_0\left(\frac{|x|}{\sqrt{t+s}}\right)\right) \le\left(L^{\sigma_1,\sigma_2}_{r_0,s}-\frac{\partial}{\partial t}\right)\left((t+s)^{-\lambda_0/2}f_0\left(\frac{|x|}{\sqrt{t+s}}\right)\right)\equiv 0$$

for every s,t,x.

This establishes the claim. Hence,

$$E_{t_0,x_0}\left[s^{-\lambda_0/2}f_0\left(\frac{|X_0|}{\sqrt{s}}\right)\right]\le(t_0+s)^{-\lambda_0/2}f_0\left(\frac{x_0}{\sqrt{t_0+s}}\right);$$

E_{t_0,x_0} denotes the expectation operator with respect to the diffusion with differential generator $L_t-\partial/\partial t$ starting at (t_0,x_0). The corresponding probabibility measure will be denoted by P_{t_0,x_0}.

We can estimate from above the right-hand side of the last inequality by a constant C. Since f_0 is decreasing, we obtain from Chebychev's inequality that

$$f_0(1)P_{t_0,x_0}[|X_0|\le\sqrt{s}]\le Cs^{\lambda_0/2}\quad\text{for every positive number } s.$$

This completes the proof of the first part of the theorem.

Proof of the second part of Theorem 1. Now, suppose that the assumptions of the second part of the theorem are satisfied and that $\sigma_1>\sigma_2$.

It can be shown in a similar way as above that

$$t\mapsto (t+s)^{-\lambda_0/2} f_0\left(\frac{|X_t|}{\sqrt{t+s}}\right) \tag{14}$$

is a submartingale with decreasing time parameter t for every positive s.

We notice that the analogue of Section 2, Lemma 6 does not hold if $\sigma_1>\sigma_2$. The analogue of (13) is nevertheless correct since the presupposition $\operatorname{trace}(\Pi(a_{ij}(t,x))_{i,j}) \geq (n-1)\sigma_1$ for the first part of the theorem was replaced by the "stronger" presupposition $(a_{ij}(t,x))_{i,j}\leq\sigma_1\mathbf{1}$.

Since (14) is a submartingale,

$$E_{t_0,x_0}\left[s^{-\lambda_0/2} f_0\left(\frac{|X_0|}{\sqrt{s}}\right)\right] \geq (t_0+s)^{-\lambda_0/2} f_0\left(\frac{|x_0|}{\sqrt{t_0+s}}\right)$$

for every positive s. Since f_0 is decreasing on $[0,\infty)$, we obtain that

$$E_{t_0,x_0}\left[s^{-\lambda_0/2} f_0\left(\frac{|X_0|}{\sqrt{s}}\right)\right] \geq \frac{f_0(|x_0|/\sqrt{t_0})}{(2t_0)^{\lambda_0/2}}$$

for every s with $0<s\leq t_0$.

On the other hand,

$$E_{t_0,x_0}\left[f_0\left(\frac{|X_0|}{\sqrt{s}}\right)\right] \leq f_0(0) P_L(t_0,x_0;0,U_\varrho(0)) + f_0\left(\frac{\varrho}{\sqrt{s}}\right)$$

for every positive ϱ and s.

We can conclude that

$$P_L(t_0,x_0;0,U_\varrho(0)) \geq \frac{1}{f_0(0)}\left\{s^{\lambda_0/2}\frac{f_0(|x_0|/\sqrt{t_0})}{(2t_0)^{\lambda_0/2}} - f_0\left(\frac{\varrho}{\sqrt{s}}\right)\right\}$$

for every positive ϱ and every s with $0<s\leq t_0$.

Let a positive number ε with $\varepsilon<1$ be given. We set $s\equiv\varrho^{2(1+\varepsilon)}$ for every ϱ with $\max\{\varrho^2;\varrho^4\}\leq t_0$. Thus,

$$P_L(t_0,x_0;0,U_\varrho(0)) \geq \frac{1}{f_0(0)}\left\{\varrho^{(1+\varepsilon)\lambda_0}\frac{f_0(|x_0|/\sqrt{t_0})}{(2t_0)^{\lambda_0/2}} - f_0(\varrho^{-\varepsilon})\right\}$$

for every sufficiently small positive ϱ. Taking into account that f_0 vanishes at least exponentionally fast at infinity, it is easy to see that

$$\lim_{\varrho\downarrow 0}\frac{f_0(\varrho^{-\varepsilon})}{\varrho^{(1+\varepsilon)\lambda_0}}=0.$$

Hence,

$$P_L(t_0,x_0;0,U_\varrho(0))\geq\frac{1}{2f_0(0)}\frac{f_0(|x_0|/\sqrt{t_0}\,)}{(2t_0)^{\lambda_0/2}}\varrho^{(1+\varepsilon)\lambda_0}$$

for every sufficiently small positive ϱ. The assertion is herewith established because ε was an arbitrary positive number with $\varepsilon<1$.

Corollary 1. *Suppose that the assumptions of the first part of Theorem* 1 *are satisfied for a differential operator* $L_t-\partial/\partial t$, *constants* σ_1, σ_2, λ_0, r_0 *with* $\sigma_1<\sigma_2$, *and every* $(t,x)\in(0,\infty)\times\mathbf{R}^n$ *(i.e., we suppose that the assumptions on* $(a_{ij}(t,x))_{i,j}$ *are satisfied uniformly on* $(0,\infty)\times\mathbf{R}^n$*).*

Then $P_L(t,x;0,M)=0$ *for every Borel set* M *with Hausdorff* λ_0*-measure equal to zero.*

Proof. The corollary can be proved in exactly the same way as [7, Theorem 2] if we take into account the following remark. The same argument as in the proof of Theorem 1 yields that given $(t_0,x_0)\in(0,\infty)\times\mathbf{R}^n$ and a positive constant R, there exists a constant C with $P_{t_0,x_0}[|X_0-y|\leq\sqrt{s}\,]\leq Cs^{\lambda_0/2}$ for every y with $|y|\leq R$ and every positive s.

Corollary 2. *Suppose that the assumptions of the second part of Theorem* 1 *are satisfied in an appropriate neighbourhood of* $(0,0)$ *for a differential operator* $L_t-\partial/\partial t$ *and constants* σ_1, σ_2, λ_0, r_0 *with* $\sigma_1>\sigma_2$. *Let* $(t_0,x_0)\in(0,\infty)\times\mathbf{R}^n$ *and a positive number* ε *be given.*

Then there exists a positive constant C *such that for every positive solution* u *of the equation* $(L_t-\partial/\partial t)u=0$ *on* $(0,\infty)\times\mathbf{R}^n$ *the following holds*:

$$u(t_0,x_0)\geq Ct^{(1+\varepsilon)\lambda_0/2}u(t,x)\quad\text{for every } t,x \text{ with } 0<t\leq\tfrac{1}{2}t_0,\ |x|^2\leq\frac{t}{\varepsilon}.$$

Proof. By Harnack's inequality (see [6]), there exists a positive constant $C_1(\varepsilon)$ such that for every positive solution u of $(L_t-\partial/\partial t)u=0$ on $(0,\frac{3}{2}t)\times U_{\sqrt{2t/\varepsilon}}(0)$ and every t with $0<t\leq\frac{1}{2}t_0$ the following holds:

$$u(\tfrac{5}{4}t,x_1)\geq C_1(\varepsilon)u(t,x_2)\quad\text{for every } x_1,x_2\in U_{\sqrt{t/\varepsilon}}(0). \tag{15}$$

We notice that we can restrict ourselves to the case where (t,x) belongs to an arbitrary small neighbourhood of $(0,0)$. Therefore we may and will assume that the assumptions on $(a_{ij}(t,x))_{i,j}$ are satisfied uniformly on $(0,\infty)\times\mathbf{R}^n$.

A similar argument as in the proof of Theorem 1 shows that there exists a positive constant $C_2(\varepsilon)$ such that

$$P_L(t_0,x_0;\tfrac{5}{4}t,U_{\sqrt{t/\varepsilon}}(0))\geq C_2(\varepsilon)\left(\frac{t}{\varepsilon}\right)^{(1+\varepsilon)\lambda_0/2}$$

for every t with $0<t\leq\frac{1}{2}t_0$. Hence,

$$u(t_0,x_0)\geq C_2(\varepsilon)\left(\frac{t}{\varepsilon}\right)^{(1+\varepsilon)\lambda_0/2}\inf_{\{5t/4\}\times U_{\sqrt{t/\varepsilon}}(0)}u \tag{16}$$

for every positive solution u of $(L_t-\partial/\partial t)u=0$ on $(0,\infty)\times\mathbf{R}^n$. The assertion of the corollary is now an immediate consequence of (15) and (16).

Examples. We consider a positive solution f_0 of the Sturm–Liouville equation (2) for real numbers λ_0 and r_0 such that the conditions of Proposition 1 are satisfied. We claim that there exists a continuous function $\widetilde{\sigma}$ on $[0,\infty)$ with $\sigma_1\leq\widetilde{\sigma}(r)\leq\sigma_2$ for every $r\geq0$ or with $\sigma_1\geq\widetilde{\sigma}(r)\geq\sigma_2$ for every $r\geq0$ and a solution $\widetilde{f}$ of a Sturm–Liouville equation of type (2) with $\widetilde{\sigma}$ in place of σ and with similar boundary conditions as in Theorem 1 such that the difference of the corresponding eigenvalues λ_0 and $\tilde{\lambda}$ is arbitrary small.

Once this claim is established, we obtain by similar arguments as at the ends of the proofs of the first and second part of Theorem 1 that for the diffusion process with differential generator

$$\frac{1}{2}\left(\sigma_1\triangle+\left(\widetilde{\sigma}\left(\frac{|x|}{\sqrt{t}}\right)-\sigma_1\right)\partial^2_{x/|x|}\right)$$

the following equality holds:

$$\lim_{\varrho\downarrow0}\frac{\ln P_L(t_0,x_0;0,U_\varrho(0))}{\ln\varrho}=\tilde{\lambda}$$

(cf. [9, Chapter 7] for existence and uniqueness results for diffusion processes with differential generators with continuous coefficients).

We can conclude that the "exponent" λ_0 in the assertion of Theorem 1 cannot be improved.

In order to prove the claim, we consider for every r with $r\leq r_0$ the function $\sigma^{(r)}$ on $[0,\infty)$ with $\sigma^{(r)}(s)\equiv\sigma_1$ for $s\leq r$, $\sigma^{(r)}(s)\equiv\sigma_2$ for $s\geq r_0$, and such that $\sigma^{(r)}$ is affine linear on $[r, r_0]$. Given a number r, the eigenvalue $\lambda(r)$ of a Sturm–Liouville equation similar to (2) with $\sigma^{(r)}$ in place of σ, with the same boundary conditions as in Theorem 1, and the condition that the solution is continuously differentiable on $[0,\infty)$ can be defined as the zero of a function $\lambda \mapsto F^{(r)}(\lambda, r_0)$. That function can be defined in a similar way as the function $F(\cdot, r_0)$ from Section 2. As in the remark after Lemma 1, we can conclude that $\lambda(r)$ depends continuously on r. This establishes the claim.

Finally, we consider a simple degenerated situation. Suppose that we are given natural numbers n and k with $2 \le k < n$. We set $a_{ij}(t,x) \equiv 1$ if $i = j \le k$ and $a_{ij}(t,x) \equiv 0$ otherwise for every t, x. A similar argument as in the proof of Part 1 of Theorem 1 (cf. also the last remark in Section 2) shows that $\lim_{\varrho \downarrow 0} \inf \ln P_L(t_0, x_0; 0, U_\varrho(0))/\ln \varrho \ge \lambda_0$ where λ_0 is given by Proposition 1 for $\sigma_1 \equiv (k-1)/(n-1)$ and $\sigma_2 \equiv 1$ (notice that the presupposition $(a_{ij}(t,x)) \ge C(t)\mathbf{1}$ for a positive number $C(t)$ is not satisfied in this situation). On the other hand, it is obvious that

$$\lim_{\varrho \downarrow 0} \inf \ln P_L(t_0, x_0; 0, U_\varrho(0))/\ln \varrho = k.$$

Hence, $k \ge \lambda_0$.

Notice that Lemma 5 yields the estimate $(n(k-1))/(n-1) \le \lambda_0 < k$.

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