ON EMBEDDINGS OF SPHERES

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Introduction

If we embed an $(n-1)$-sphere in an n-sphere, the complement consists of two components. Our problem is to describe the components more exactly.

For $n=2$, there is a classical theorem of Schöenflies which says that an arbitrary simple closed curve in the two-dimensional sphere S^2 separates S^2 into two components whose closures are both topologically equivalent to a disk. The Riemann mapping theorem yields, moreover, a conformal equivalence between the interior of a simple closed curve and the open disk.

The reasonable conjecture to make would be that some analogous result holds for all dimensions; more precisely, that the complementary components of an $(n-1)$-sphere embedded in n-space are topologically equivalent to n-cells.

A classical counter-example (in dimension $n=3$) to this unrestricted analogue of the two-dimensional Schöenflies theorem is a wild embedding of S^2 in S^3 known as the Alexander Horned Sphere [1]. One of the complementary components of this embedding is not homeomorphic with the n-cell, and, in fact, not simply connected.

One's intuition shrugs at this counter-example, attributes its existence to the 'pathology of the non-differentiable', or whatever, and persists in believing the statement true—at least for nice imbeddings. In particular, the Alexander Horned Sphere embedding can be made neither differentiable nor polyhedral.

Under the assumption that the two-sphere S^2 is embedded polyhedrally in S^3, Alexander [1], and later, Moise [4], proved that the closures of the complementary components of S^2 were topological 3-cells.

(1) A research announcement has already appeared in *Bull. Amer. Math. Soc.* 65, 1959.

My aim is to prove that if S^{n-1} is embedded nicely in S^n, the closure of the complementary components are n-cells. The word "nicely" is defined in section 1, and it includes the class of differentiable embeddings as a special case. However, it is yet unknown whether the class of polyhedral embeddings is also included as a subclass of nice embeddings.

The main theorem is proved in section 2. A corollary, the Open Star Theorem, is given in the last section. It states that the open star of a vertex in a triangulated manifold is homeomorphic with Euclidean n-space. This is a weak partial result towards what is known as the Sphere Problem. The Sphere Problem asks whether or not the closed star of a vertex in a triangulated manifold is combinatorially equivalent to a closed n-cell, I^n. It is not known yet whether the closed star is even topologically equivalent to I^n.

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1. Section of terminology

The Euclidean n-space, or the Cartesian product of n copies of the real line R, will be denoted R^n. A point $x \in R^n$ is thus an n-tuple of real numbers $(x_1, \ldots, x_n)$ and

$$\|x\| = \max_{i=1,\ldots,n} |x_i|.$$

We use the following notations.

The *standard n-cube* $I^n = \{x \in R^n \mid \|x\| \leqslant 1\}$, $I^1 = I$.

The *standard n-sphere* $S^n = \{x \in R^{n+1} \mid \|x\| = 1\}$.

The *standard n-annulus* $A^n = \{x \in R^n \mid 1 \leqslant \|x\| \leqslant 2\}$.

The *standard n-stock* St^n is obtained by attaching two copies $(A^n)_1$ $(A^n)_2$ of A^n via the identification:

$$\{x \in (A^n)_1 \mid x_1 = 2\} \leftrightarrow \{x \in (A^n)_2 \mid x_1 = -2\}.$$

The n-annulus has two boundary components, each homeomorphic with S^n. I shall refer to $\{x \in R^n \mid \|x\| = 1\}$ as the internal boundary component, denoted in ∂A^n, and $\{x \in R^n \mid \|x\| = 2\}$ as the external boundary component, denoted ex ∂A^n. I shall also need names for standard homeomorphisms of S^{n-1} onto each boundary component. Denote by i: $S^{n-1} \to$ in ∂A^n the identity homeomorphism, and by i^*: $S^{n-1} \to$ ex ∂A^n the radial projection homeomorphism.

Similarly, I shall need a name for the external boundary component of St^n. Call it ex ∂St^n. Call the two internal boundary components w_1 and w_2. Let τ_1, τ_2 be standard identity homeomorphisms of S^{n-1} into each internal boundary component.

Let τ be some fixed homeomorphism of $\operatorname{ex} \partial St^n$ onto S^{n-1}, for instance:

$$\tau(x) = \begin{cases} \left(\dfrac{x_1 - 2}{4}, \dfrac{x_2}{2}, \ldots, \dfrac{x_n}{2}\right) & \text{for } x \in (A^n)_1 \cap \operatorname{ex} \partial St^n, \quad x = (x_1, \ldots, x_n) \\ \left(\dfrac{x_1 + 2}{4}, \dfrac{x_2}{2}, \ldots, \dfrac{x_n}{2}\right) & \text{for } x \in (A^n)_2 \cap \operatorname{ex} \partial St^n, \quad x = (x_1, \ldots, x_n). \end{cases}$$

If X and Y are topological spaces, $X \approx Y$ will signify that there is a homeomorphism between them.

If X is a manifold, ∂X will refer to its boundary, and $\operatorname{int} X$ will be the space $X - \partial X$.

If X and Y are topological spaces, and $f: A \to Y$ a continuous map, $X \cup_f Y$ or $Y \cup_f X$ will refer to the topological space $X \cup Y$ with the equivalence relation $x \sim f(x)$, equipped with the identification topology. There is no ambiguity arising from reversing the order of X and Y; where it is absolutely clear which attaching map f is meant, $X \cup_f Y$ may be referred to as $X \cup Y$. And further, in the course of the proof, iterated identifications $X_1 \cup_{f_1} X_2 \cup_{f_2} \ldots \cup_{f_n} X_n$ will be used. Where no confusion can arise I shall dispense with the parentheses necessary to indicate the precise order of identifications.

By CX, the *cone* over a space X, is meant the space $X \times [0, 1]$, with $X \times 0$ identified to a point. By a *subcone* $C_t X$ is meant the image in CX of $X \times [0, t]$ for $0 \leqslant t \leqslant 1$.

A *similarity transformation* S: $CX \to CX$ is a map S of the form $S(X \times t) = X \times \bar{S}(t)$, where $\bar{S}$: $[0, 1] \to [0, 1]$ is a monotonic continuous function such that $\bar{S}(0) = 0$.

Finally, a *Euclidean similarity transformation* is a mapping T of $R^n \to R^n$ which is of the form:

T: $x \to \lambda x + b$, where $b \in R^n$ and λ is a positive number, for $x \in R^n$.

DEFINITION: An embedding π: $S^{n-1} \to R^n$ is *nice* if one can extend π to a homeomorphism π^*: $I \times S^{n-1} \to R^n$ (i.e. $\pi^*(0 \times S^{n-1}) = \pi(S^{n-1})$) such that

(i) $\pi^*((-1) \times S^{n-1})$ is contained in the bounded complementary component of $\pi^*(0 \times S^{n-1})$ [this requirement is made for convenience only] and

(ii) π^* is linear in the neighborhood of some point of $(-\frac{1}{2}, 0) \times S^{n-1} \approx A^n \subset R^n$. (This requirement, also, is phrased in a manner which saves words in a future application. Manifestly, there is no loss of generality incurred by assuming that the 'linear point' lie in that restricted territory.) Linear is meant in the sense of a map of a subset of the vector space R^n into itself.

2. The main theorem

The main theorem to be proved is the following:

THEOREM. *Let S^{n-1} be nicely imbedded in S^n. Then the closures of the complementary components of S^{n-1} are homeomorphic to the n-cell.*

Outline of the proof.

(A) In the class of n-manifolds which bound $(n-1)$-spheres (in a nice way), a multiplication is defined. Intuitively, one takes two such manifolds M, N and attaches them to the interior boundaries of an n-stock, forming a new manifold, $M \cdot N$, which again has an $(n-1)$-sphere boundary. Complications exist in the definition since we must prove that the final space $M \cdot N$ is well defined. For, a priori, $M \cdot N$ depends strongly on the homeomorphisms used to attach M and N to the interior boundaries.

(B) (Lemma 2) $M \cdot N = N \cdot M$.

(C) If $M \subset S^n$, one can construct a manifold N for which $M \cdot N \approx I^n$. (N is roughly the complementary manifold $S^n - \operatorname{int} M$.)

(D) Let $B^\infty = M \cdot N \cdot M \cdot N \ldots \cup \infty$ where ∞ is the one-point compactification of the rest, then (Lemma 3) there are two ways of viewing B^∞.

(a) $$B^\infty = (M \cdot N) \cdot (M \cdot N) \ldots \cup \infty$$

and since $M \cdot N \approx I^n$, $$B^\infty = I^n \cdot I^n \ldots \cup \infty,$$

from which one easily deduces that $B^\infty \approx I^n$.

(b) $$B^\infty = M \cdot (N \cdot M) \cdot (N \cdot M) \ldots \cup \infty$$

and since $N \cdot M = M \cdot N$ by (B),

$$B^\infty = M \cdot I^n \cdot I^n \ldots \cup \infty$$

from which one easily finds that $B^\infty \approx M$.

(E) $M \approx I^n$ for by (b) $M \approx B^\infty$, and by (a), $B^\infty \approx I^n$.

The proof.

The Semi-Group X. Let $\check{\mathrm{X}}$ be the collection of couples $\mathfrak{M} = (M, \Phi)$, where

(i) M is a compact n-manifold with boundary embeddable in Euclidean n-space in such a manner that it has an $(n-1)$-sphere boundary ∂M nicely embedded in R^n;

(ii) $\Phi\colon \partial M \to S^{n-1}$ is a homeomorphism.

If $\mathcal{M} \in \mathrm{X}$, $\mathcal{M} = (M, \Phi)$, I will denote M by $|\mathcal{M}|$ and Φ by $\Phi_{\mathcal{M}}$ when it is helpful to do so. Call $|\mathrm{X}|$ the set of manifolds M satisfying condition (i) above. Two elements $\mathcal{M}$, $\mathcal{M}' \in \check{\mathrm{X}}$ will be called *equivalent* (denoted: $\mathcal{M} \sim \mathcal{M}'$) if there is a homeomorphism $h\colon |\mathcal{M}| \to |\mathcal{M}'|$ and a commutative diagram:

$$\begin{array}{ccc} \partial|\mathcal{M}| & \xrightarrow{h} & \partial|\mathcal{M}'| \\ \downarrow{\scriptstyle \Phi_{\mathcal{M}}} & & \downarrow{\scriptstyle \Phi_{\mathcal{M}'}} \\ S^{n-1} & \xleftrightarrow{id} & S^{n-1} \end{array}$$

The object of real interest is the set of equivalence classes of $\check{\mathrm{X}}$, under the relation defined above. Denote this set by X. A multiplication is defined in X as follows: If $\mathcal{M}$, $\mathcal{N}$ are representatives in $\check{\mathrm{X}}$ of equivalence classes of X,

$$\mathcal{M} \cdot \mathcal{N} = (|\mathcal{M}| \cup_{\tau_1 \Phi_{\mathcal{M}}} St^n \cup_{\tau_2 \Phi_{\mathcal{N}}} |\mathcal{N}|, \tau).$$

Notice that this is just a definition of what is commonly known as "addition of manifolds". The reason for carefulness is that at present we cannot prove that the naïve definition of "manifold addition" is independent of the attaching homeomorphism. This is, in fact, the only reason that the elements of $\check{\mathrm{X}}$ were chosen to be couples (M, Φ) rather than just topological spaces.

In order to justify the definition of $\mathcal{M} \cdot \mathcal{N}$ we must prove the following two lemmas.

LEMMA. X *is closed under this multiplication.*

We have to show that $|\mathcal{M} \cdot \mathcal{N}|$ satisfies (i). The proof is however simple and can be omitted.

I might also include the remark that it is not strictly necessary for the proof of the main theorem.

LEMMA. *The above multiplication is well-defined on the equivalence classes of* $\check{\mathrm{X}}$, *and hence yields a multiplication in* X.

All one need show to prove the lemma is that if $\mathcal{M}$ is replaced by an element $\mathcal{M}' \in \check{\mathrm{X}}$ equivalent to it, $\mathcal{M} \cdot \mathcal{N}$ and $\mathcal{M}' \cdot \mathcal{N}$ are again equivalent.

By the definition of equivalence, $\mathfrak{M}$ and $\mathfrak{M}'$ satisfy a commutative diagram:

$$\begin{array}{ccc} \partial|\mathfrak{M}| & \xleftarrow{h} & \partial|\mathfrak{M}'| \\ \downarrow{\scriptstyle \Phi_{\mathfrak{M}}} & & \downarrow{\scriptstyle \Phi_{\mathfrak{M}'}=\Phi_{\mathfrak{M}}\circ h} \\ S^{n-1} & & S^{n-1} \end{array}$$

where h is a homeomorphism $h\colon |\mathfrak{M}'| \to |\mathfrak{M}|$ and

$$\begin{aligned} \mathfrak{M}'\cdot\mathfrak{N} &= (|\mathfrak{M}'| \cup_{\tau_1\circ\Phi_{\mathfrak{M}'}} St^n \cup_{\tau_2\circ\Phi_{\mathfrak{N}}} |\mathfrak{N}|, \tau) \\ &= (|\mathfrak{M}| \cup_{\tau_1\circ\Phi_{\mathfrak{M}}\circ h} St^n \cup_{\tau_2\circ\Phi_{\mathfrak{N}}} |\mathfrak{N}|, \tau). \end{aligned}$$

The homeomorphism H: $|\mathfrak{M}'\cdot\mathfrak{N}| \to |\mathfrak{M}\cdot\mathfrak{N}|$ defined as follows:

$$\begin{cases} H/St^n \cup |\mathfrak{N}| = \text{identity} \\ H/|\mathfrak{M}'| = h \end{cases}$$

yields the equivalence. H is thus defined on $|\mathfrak{M}'\cdot\mathfrak{N}|$; clearly it is defined compatibly with attaching maps and is a homeomorphism. Finally, $H/\partial|\mathfrak{M}'\cdot\mathfrak{N}|$ is the identity homeomorphism yielding the commutative diagram:

$$\begin{array}{ccc} \partial|\mathfrak{M}'\cdot\mathfrak{N}| & \xrightarrow{H=\mathrm{id}} & \partial|\mathfrak{M}\cdot\mathfrak{N}| \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau} \\ S^{n-1} & & S^{n-1} \end{array}$$

We stated that X is a semi-group, or, in other words, the multiplication defined above is associative. This could easily be proved from the definition. However, it is not needed for the main theorem, and so I shall not prove it.

Let $\mathfrak{I} \in \mathrm{X}$ be the couple (I^n, ι) where ι: $\partial I^n \to S^{n-1}$ is the identity map.

LEMMA 1. *If $M \in |\mathrm{X}|$ and f is a homeomorphism of ∂M onto in ∂A^n, then $M \cup_f A^n \approx M$.*

LEMMA 1′. *If $\mathfrak{M} \in \mathrm{X}$, then $\mathfrak{M}\cdot\mathfrak{I} = \mathfrak{M}$.*

Proof of Lemma 1. Intuitively the situation is clear. $M \in |\mathrm{X}|$ is so defined that ∂M has an annulus neighborhood, A^n. Since ∂M is an $(n-1)$-sphere nicely embedded in R^n, $\partial M = \varrho(S^{n-1}\times 0)$ where ϱ is a homeomorphism of, say, $S^{n-1}\times[0, 1]$ into M. Let A^* be the annulus $\varrho(S^{n-1}\times[0, 1])$ in M, and denote by M^* the closure of the complement of A^* in M. Then $M = M^* \cup A^*$, and $M \cup_f A^n = M^* \cup A^* \cup_f A^n$.

A homeomorphism from M to $M \cup_f A^n$ can be obtained by leaving the complement of A^* fixed and stretching it over itself and A^n. To define such a homeomorphism μ: $M \to M \cup_f A^n$ formally, let μ be the identity on M^* and define μ from

$$A^* \text{ to } A^* \cup_f A^n \text{ by } \mu(\varrho(s,t)) = \begin{cases} \varrho(s, 2t) \in A^*, & 0 \leqslant t \leqslant \frac{1}{2}, \\ (f\varrho(s,0),\ 4t-1) \in A^n, & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Lemma 1′ follows immediately from Lemma 1, since

$$|\mathfrak{M}\cdot\mathfrak{I}| = |\mathfrak{M}| \cup_{\tau_1\Phi_{\mathfrak{M}}} (St^n \cup_{\tau_2\iota} I^n) \quad \text{and } \Phi_{\mathfrak{M}\cdot\mathfrak{I}} = \tau.$$

There is a homeomorphism k: $St^n \cup_{\tau_2\iota} I^n \to A^n$, sending τ_1 to ι and τ to ι^{*-1}. Thus

$$|\mathfrak{M}\cdot\mathfrak{I}| = |\mathfrak{M}| \cup_{\tau_1\Phi_{\mathfrak{M}}} A^n$$

and $\Phi_{\mathfrak{M}\cdot\mathfrak{I}} = i^{*-1}$. Furthermore, μ of Lemma 1 yields a commutative diagram:

$$\begin{array}{ccc} \partial|\mathfrak{M}| & \xrightarrow{\mu} & \partial|\mathfrak{M}\cdot\mathfrak{I}| \\ \downarrow{\scriptstyle\Phi_{\mathfrak{M}}} & & \downarrow{\scriptstyle\Phi_{\mathfrak{M}\cdot\mathfrak{I}}} \\ S^{n-1} & & S^{n-1} \end{array}$$

LEMMA 2. $\mathfrak{M}\cdot\mathfrak{N} \sim \mathfrak{N}\cdot\mathfrak{M}$.

The essential point of the proof is contained in a statement concerning St^n.

LEMMA 2′. *There is a homeomorphism R^* of the standard n-stock which interchanges the interior boundaries by a rigid Euclidean motion, and leaves the external boundary pointwise fixed. Further,*

$$T/w_1 = \tau_2\tau_1^{-1};\ T/w_2 = \tau_1\tau_2^{-1}.$$

Lemma 2′ is obvious, and Lemma 2 follows immediately, for there is a homeomorphism T: $|\mathfrak{M}\cdot\mathfrak{N}| \to |\mathfrak{N}\cdot\mathfrak{M}|$ i.e.

$$T: |\mathfrak{M}| \cup_{\tau_1\circ\Phi_{\mathfrak{M}}} St^n \cup_{\tau_2\circ\Phi_{\mathfrak{N}}} |\mathfrak{N}| \to |\mathfrak{M}| \cup_{\tau_2\circ\Phi_{\mathfrak{M}}} St^n \cup_{\tau_1\circ\Phi_{\mathfrak{N}}} |\mathfrak{N}|,$$

such that

$$\begin{cases} T/St^n = R^* \\ T/|\mathfrak{M}| = \text{identity} \\ T/|\mathfrak{N}| = \text{identity}. \end{cases}$$

It is merely a verification to show that T is well defined, and yields an equivalence between $\mathfrak{M}\cdot\mathfrak{N}$ and $\mathfrak{N}\cdot\mathfrak{M}$.

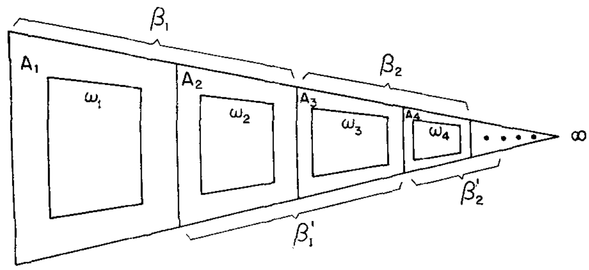

Fig. 1.

LEMMA 3. *If* $|\mathfrak{M}\cdot\mathfrak{N}|\approx I^n$, *then* $|\mathfrak{M}|\approx I^n$, $|\mathfrak{N}|\approx I^n$.

Let A^∞ be a topological space composed of an infinite number of annuli A_i attached side by side, compactified by a single point ∞. A^∞ is to be thought of as embedded in the Euclidean space. (See Fig. 1.)

Any two adjacent annuli $A_i \cup A_{i+1}$ form an n-stock. Denote, as above (Fig. 1), the n-stocks $A_{2i-1}\cup A_{2i}$ by β_i and $A_{2i}\cup A_{2i+1}$ by β_i'. Let ω_i be the internal boundary of A_i. There are Euclidean similarity transformations σ_i, σ_i' bringing $\beta_1 \xrightarrow[\sigma_i]{} \beta_i$ and $\beta_1 \xrightarrow[\sigma_i']{} \beta_i'$. Let ν_i be the euclidean similarity transformation bringing ω_1 to ω_i. Further, β_1 is to be identified with the standard n-stock, and ω_1 with S^{n-1}.

We attach a copy M_i of $M=|\mathfrak{M}|$ in each ω_{2i-1} by a homomorphism

$$\partial M \xrightarrow[\Phi_{\mathfrak{M}}]{} S^{n-1} \xrightarrow[\nu_{2i-1}]{} \omega_{2i-1}$$

and similarly a copy N_i of $N=|\mathfrak{N}|$ in each ω_{2i} by a homeomorphism

$$\partial N \xrightarrow[\Phi_{\mathfrak{N}}]{} S^{n-1} \xrightarrow[\nu_{2i}]{} \omega_{2i}.$$

Let B^∞ be the "filled-in" space $A^\infty \bigcup_i N_i \bigcup_j M_j$ and $\bar\beta_i=\beta_i\cup M_i\cup N_i$; $\bar\beta_i'=\beta_i'\cup N_i\cup M_{i+1}$. From the definition of $\bar\beta_i$, $\bar\beta_i'$, it is clear that

$$\bar\beta_i\approx|\mathfrak{M}\cdot\mathfrak{N}|\approx I^n,$$

and, using Lemma 2, $\qquad \bar\beta_i'\approx|\mathfrak{N}\cdot\mathfrak{M}|\approx I^n.$

(I) $B^{\infty} \approx I^n$.

Let K be the union of the external boundaries of β_i, compactified by ∞,

$$K = \bigcup_{i=1}^{\infty} \operatorname{ext} \partial \beta^i \cup \infty.$$

K can be considered as a subset of A^{∞} and hence of R^n. As a subset of R^n, $\operatorname{ext} \partial \beta^i$ bounds a cell, c_i, and it is clear that

$$V = \bigcup_{i=1}^{\infty} c_i \cup \infty$$

is homeomorphic to a cell, $V \approx I^n$.

There is a homeomorphism Π from V onto B^{∞}. Π is defined to be the identity mapping from $K \subset V$ to $K \subset B^{\infty}$. This defines Π on ∂c_i for each i

$$\Pi: \partial c_i \to \partial \bar{\beta}_i.$$

Since both c_i and $\bar{\beta}_i$ are topological n-cells, one need only prove a simple lemma.

Lemma (a). *If* Π *is a homeomorphism of* $\partial I^n \to \partial I^n$, *then* Π *can be extended to a homeomorphism*

$$\Pi^*: I^n \to I^n.$$

Proof. Consider each I^n to be the unit ball in R^n, and extend Π radially. Therefore, one can extend Π to each c_i. This yields a homeomorphism $\Pi^*: V \to B^{\infty}$. Thus $B^{\infty} \approx V \approx I^n$.

(II) $B^{\infty} \approx M$.

There is a second way to decompose B^{∞}:

$$B^{\infty} = (A_1 \cup M_1) \cup (\bigcup_{\substack{i>1 \\ j>1}} A_i \cup M_j \cup N_j) \cup \infty,$$

or

$$B^{\infty} = (A_1 \cup M) \cup (\bigcup_i \bar{\beta}_i' \cup \infty).$$

Lemma (b). $A_1 \cup (\bigcup_i \bar{\beta}_i' \bigcup_i \infty) \approx A^n$.

Proof. Let

$$K' = A_1 \cup \operatorname{ext} \partial \beta_i' \cup \infty,$$

considered as a subset of A^{∞} and hence of R^n. Each $\operatorname{ext} \partial \beta_i'$ bounds a cell c_i' in R^n. Let

$$V' = K' \cup (\bigcup_i c_i'), \quad V' \subset R^n.$$

It is clear that $V' \approx A^n$.

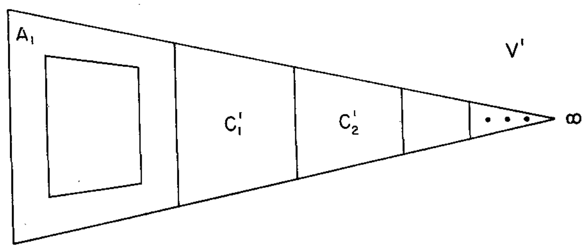

Fig. 2.

Define a homeomorphism Π' of V' onto $A_1 \cup (\cup_i \bar{\beta}_i') \cup \infty$ as follows. Π' maps $K' \subseteq V'$ to $K' \subseteq A_1 \cup (\bigcup_i \bar{\beta}_i') \cup \infty$ by the identity map. This defines Π' on each $\partial c_i'$.

$$\Pi' \colon \partial c_i' \to \partial \bar{\beta}_i'.$$

Since we know that $c_i' \approx I^n$ and $\bar{\beta}_i' \approx I^n$, the above Lemma (a) allows us to extend Π' to a homeomorphism of each c_i' with each $\bar{\beta}_i'$. This yields the homeomorphism

$$\Pi^{*\prime} \colon V' \to A_1 \cup (\cup_i \bar{\beta}_i') \cup \infty,$$

proving Lemma (b).

We have obtained $B^\infty \approx M \cup A^n$. Lemma 1 applies, yielding

$$B^\infty \approx M \cup A^n \approx M.$$

(I) and (II) yield a proof of Lemma 3, for

$$I^n \approx B^\infty \approx M.$$

The argument is symmetrical in $|\mathfrak{M}|$ and $|\mathfrak{N}|$, and so both $|\mathfrak{M}|$ and $|\mathfrak{N}|$ are topological cells.

We can now complete the proof of the main theorem.

Let M and N be the two complementary components. Let $\varrho\colon S^{n-1} \to S^n$ be the nice embedding. Then one can assume that there is a map $\varrho\colon I \times S^{n-1} \to S^n$, and, moreover,

$$\varrho([-1, 0] \times S^{n-1}) \subset M, \qquad \varrho([0, +1] \times S^{n-1}) \subset N.$$

Let
$$M^* = M - \varrho([-\tfrac{1}{2}, 0] \times S^{n-1}).$$

Then $M^* \approx M$. To see this apply Lemma 1, after noticing that

$$M = M^* \cup \varrho([-\tfrac{1}{2}, 0] \cup S^{n-1})$$

and that $\varrho([-\tfrac{1}{2}, 0] \times S^{n-1}) = A$ is topologically an annulus. Then

$$S^n = M^* \cup A \cup N \approx M \cup A \cup N.$$

Since ϱ is linear in the neighborhood U of a point in $[-\tfrac{1}{2}, 0] \times S^{n-1}$, let Δ be a standard simplex in U. Then $A - \operatorname{int} \Delta$ is homeomorphic with the standard n-stock by a homeomorphism μ:

$$St^n \xrightarrow[\mu]{\approx} A - \operatorname{int} \Delta.$$

Also, $\varrho(\Delta)$ is a standard simplex in S^n, and so $S^n - \operatorname{int} \varrho(\Delta) \approx I^n$. Therefore,

$$I^n \approx M \cup \varrho \circ \mu (St^n) \cup N.$$

Let

$$f = \mu^{-1} \circ \varrho^{-1} / \partial M, \quad g = \mu^{-1} \circ \varrho^{-1} / \partial N.$$

Then $I^n = M \cup_f St \cup_g N$. Or, if one sets

$$\mathfrak{M} = (M, f), \quad \mathfrak{N} = (N, g),$$

then

$$|\mathfrak{M} \cdot \mathfrak{N}| \approx I^n.$$

But Lemma 3 applies and

$$N = |\mathfrak{N}| \approx I^n, \quad M = |\mathfrak{M}| \approx I^n,$$

proving the theorem.

The differentiable case

The main theorem merely states that a topological equivalence between the closure of the interior component of a nicely embedded sphere and the standard cell can be obtained. This raises the question whether or not a diffeomorphism between X and the unit cell can be obtained when the embedding $S^{n-1} \to S^n$ is assumed to be a diffeomorphism. That any differentiable embedding $S^{n-1} \to S^n$ is nice in the above sense is a standard lemma (See Thom [3]). The methods of the proof can be refined, in this case, to yield a homeomorphism between X and I^n which is actually an equivalence of differential structures except at the point ∞. More precisely, one can extend the given embedding

$$\Phi \colon \partial I^n = S^{n-1} \to S^n$$

to a homeomorphism $\Phi^*: I^n \to S^n$

that is a diffeomorphism except at ∞ [4].

In fact, this is the best one can hope for. Milnor [5] has exhibited a diffeomorphism $\Phi: S^6 \to S^6 \subset S^7$ which cannot be extended to a diffeomorphism $\Phi^*: I^7 \to S^7$.

The simplicial case

If one assumes that Φ is a simplicial embedding it is unknown whether Φ is nice in the above sense. Assume, then, that Φ is a simplicial, nice embedding $S^{n-1} \xrightarrow[\Phi]{} S^n$. Then it is an open question whether or not Φ can be extended to a simplicial homeomorphism $\Phi^*: I^n \to S^n$. Unlike the differentiable analogue there is no counter-example to this.

Moreover, it is a simple matter to refine the above proof to yield an extension Φ^* which is simplicial except at ∞. This means that there is an infinite triangulation T_X of $X - \infty$ and an infinite triangulation T_{I_n} of $I^n - \Phi^{-1}(\infty)$ such that with respect to each of these triangulations Φ^* is simplicial, and T_X is compatible with the triangulation of $X - \infty$ inherited from X, and T_{I_n} is compatible with the triangulation of $I^n - \Phi(\infty)$ inherited from I^n.

Some further generalizations

Let $\mathcal{K}$ be the semi-group of knot types. $\mathcal{K}$ is the set of equivalence classes of combinatorial imbeddings of S^1 in S^3. Two imbeddings are equivalent if one can be brought to the other by a combinatorial automorphism of S^3 onto itself.

The definition of addition of knots is standard and the set $\mathcal{K}$ forms a semi-group with respect to this operation. It was an observation of Fox that the construction used in the main theorem could be also used to prove that $\mathcal{K}$ has no inverses, a theorem due originally to Shubert.

This remark can be generalized:

Define an imbedding $\Phi: S^k \to E^n$ to be *invertible* if it satifies (1) and (2) below.

(1) Φ is linear on some open set of S^k, which may be chosen to be the lower hemisphere, by obvious shifting. If two embeddings $\Phi, \Psi: S^k \to E^n$ satisfy (1), then there is a natural way to "add" them, obtaining an imbedding denoted $\Phi + \Psi: S^k \to E^n$.

(2) There is a Ψ, satisfying (1), such that $f = \Phi + \Psi: S^k \to E^n$ extends to a homeomorphism

$$f^*: D^{k+1} \to E^n,$$

where D^{k+1} is the $(k+1)$-cell, and S^k is considered as its boundary.

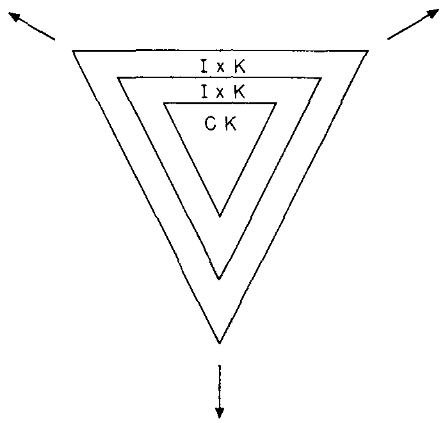

Fig. 3.

The interior boundary of the $(j+1)$st copy of $I \times K$ is attached to the exterior boundary of the j-th.

THEOREM. *If Φ: $S^k \to E^n$ is invertible, then Φ can be extended to a homeomorphism*

$$\Phi^*: D^{k+1} \to E^n.$$

A proof and applications of this fact will be given elsewhere.

3. The open star theorem

By a triangulated manifold will be meant a simplicial complex that is topologically locally Euclidean. The *closed star* of a vertex v in a triangulated manifold M will be the subcomplex of M consisting of all simplices containing v. The *open star* of v is just the interior of the closed star of v.

THEOREM. *The open star of a vertex in a triangulated n-manifold is topologically equivalent to R^n.*

Proof. Let K denote the boundary of the closed star of a vertex v. Then the open star of v can be considered to be just $CK \cup K \times I \cup K \times I \cup \ldots$ where the attaching maps are as in Fig. 3.

Choose σ, an n-dimensional simplex with v as vertex. Let σ' be a simplex contained in σ, and similar in shape to σ. Then $\partial(\sigma')$ is an $(n-1)$-sphere nicely embedded in σ.

Let U be a neighborhood of v homeomorphic with R^n, disjoint from γ, γ', the opposite faces of v in σ and σ'. I assume that σ' has been chosen sufficiently near σ so that σ' intersects U and that $U \subset CK$.

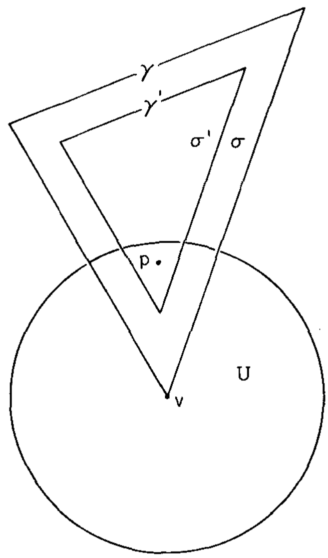

Fig. 4.

Choose a point $p \in \operatorname{int} \sigma' \cap U$. Since U is Euclidean space, there is a homeomorphism λ: $U \to U$, such that λ is the identity homeomorphism outside of a sufficiently large sphere $S \subset U$, and $\lambda(p) = v$. The homeomorphism λ can be extended to a homeomorphism of $CK \to CK$ by defining π to be the identity outside S. $\lambda(\partial(\sigma'))$ is a sphere Σ, which contains v in its interior. If γ is the face opposite v, $\lambda(\gamma) = \gamma$.

Choose a sufficiently small subcone $C'K$ of CK so that $C'K$ lies entirely in the interior of Σ. The region between CK and int $C'K$ is topologically $K \times I$. Therefore $\Sigma \subset K \times I$, and there is a copy, Σ_j, of Σ embedded in each $K \times I$ contained in the open star. See Fig. 5.

Lemma 4. *The closed region* Λ *between* Σ_j *and* Σ_{j-1} *is topologically an annulus.*

One could obtain Lemma 4 by first proving the simple but technical corollary to the main theorem:

Let Φ, Ψ *be two nice embeddings of* S^{n-1} *into* S^n *such that* $\Phi(S^{n-1}) \cap \Psi(S^{n-1})$ *is empty. Then the closure of the complementary component of* $\Phi(S^{n-1}) \cup \Psi(S^{n-1})$ *which has* $\Phi(S^{n-1}) \cup \Psi(S^{n-1})$ *as boundary is topologically an annulus.*

I content myself, however, with a direct application of the main theorem.

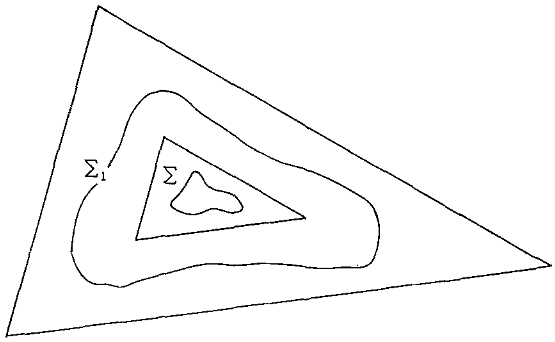

Fig. 5.

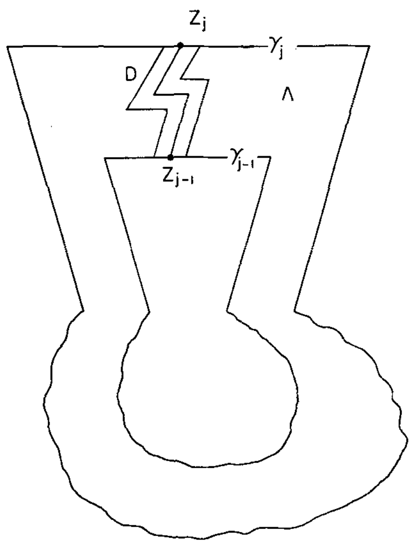

Fig. 6.

Each $\Sigma_j \subset CK \bigcup_{k=1}^{j} (K\times I)_k$ is the image of $\Sigma = \lambda(\partial\sigma')$ under a similarity transformation τ_j. Both Σ_j and Σ_{j-1} are contained in $\tau_j\lambda(\sigma)$ and

$$\Sigma_j = \tau_j\lambda(\partial\sigma'),$$
$$\Sigma_{j-1} = \tau_{j-1}\lambda(\partial\sigma').$$

Since λ is the identity transformation on γ, $\gamma\subset\partial\sigma'$ is mapped linearly onto $\gamma_j = \tau_j\lambda(\gamma)$ in Σ_j, and also onto $\gamma_{j-1} = \tau_{j-1}\lambda(\gamma)$ in Σ_{j-1}. Thus γ_j, γ_{j-1} are linear simplices.

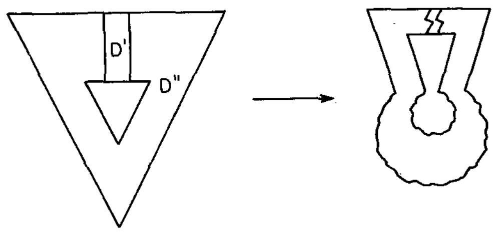

Fig. 7.

LEMMA 5. *There is an n-cell (pictorially it is a tube) D contained in Λ and intersecting γ_j and γ_{j-1} on $(n-1)$-cells Δ_j and Δ_{j-1} respectively.*

The construction of D is straightforward, and I shall merely sketch the method by which one may obtain such a D.

First a point $z_j \in \gamma_j$ can be joined to a point $z_{j-1} \in \gamma_{j-1}$ by an arc C in Λ. Take an open neighborhood of C, and in it, replace C by a polygonal arc $\check{C}$. Finally take a closed regular neighborhood of $\check{C}$ in Λ, $\check{D}$. Since γ_j and γ_{j-1} are hyperplanes (locally) one can modify $\check{D}$ near z_j and z_{j-1} to yield D, a regular neighborhood which intersects γ_j and γ_{j-1} at $(n-1)$-cells $\Delta_j \ni z_j$, $\Delta_{j-1} \ni z_{j-1}$.

LEMMA 6. *$\partial(\Lambda - D)$ is an $(n-1)$-sphere nicely embedded in $\tau_j \lambda(\sigma)$, where $\tau_j \lambda(\sigma)$ is considered to be a simplicial n-cell with simplicial structure transported from σ by $\tau_j \lambda$.*

The proof of Lemma 6 is straightforward and tedious, and so it is omitted.

We now prove Lemma 4.

Consider A^n to be $\partial\sigma' \times I$. Define a mapping δ bringing $\partial\sigma' \times 0$ to Σ_j by $\tau_j \lambda$, and $\partial\sigma' \times 1$ to Σ_{j-1} by $\tau_{j-1}\lambda$. Construct a tube

$$D' \subset \partial\sigma' \times I$$

from

$$\tau_j^{-1} \Delta_j \subset \partial\sigma' \times 0$$

to

$$\tau_{j-1}^{-1} \Delta_{j-1} \subset \partial\sigma' \times 1,$$

in such a way that the closure of

$$\partial\sigma' \times I - D'$$

is topologically an n-cell, D''.

δ is already defined on

$$\partial\tau_j^{-1} \Delta_j \cup \partial\tau_{j-1}^{-1} \Delta_{j-1},$$

but undefined on

$$L' = \partial D' - \tau_j^{-1} \Delta_j \cup \tau_{j-1}^{-1} \Delta_{j-1}.$$

Our problem is to extend δ to a homeomorphism δ of

$$\partial\sigma' \times \{0\} \cup \{1\} \subset L'$$

to

$$\Sigma_j \cup \Sigma_{j-1} \cup L,$$

where

$$L = \partial D - \Delta_j \cup \Delta_{j-1}.$$

Since L and L' are both homeomorphic with A^{n-1}, and $\delta/\partial L$ is a combinatorial homeomorphism of $\partial L'$ onto ∂L, the problem is merely to extend a given combinatorial homeomorphism δ from ∂A^{n-1} onto itself to a combinatorial homeomorphism of A^{n-1} onto itself. This requires checking that the two homeomorphisms obtained by restricting δ to each of the components of ∂A^{n-1} behave compatibly with respect to orientation, and then applying Theorem 1 of [6]. In this way δ is extended to L.

By Lemma 6, $\partial(\Lambda - D)$ is a nicely embedded sphere, hence the closure of the interior of $\partial(\Lambda - D)$ is topologically an n-cell. Now $\delta/\partial D''$ maps the boundary of an n-cell onto the boundary of an n-cell and so can be extended to a homeomorphism of D''. Similarly $\delta/\partial D'$ can be extended to a homeomorphism of D' onto D. δ is therefore a homeomorphism of A^n onto Λ.

Proof of the open star theorem.

$$CK \bigcup_i (I \times K)_i$$

can be regarded as

$$\lambda(\sigma') \bigcup_i \Lambda_i$$

which is topologically

$$I^n \cup A_1^n \cup A_2^n \cup \dots$$

when the interior boundary of A_{j+1}^n is attached to the exterior boundary of A_j^n, and the interior boundary of A_1^n is attached to ∂I^n. Hence

$$CK \bigcup_i (K \times I)_i \approx R^n.$$

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