A simplified proof for a moving boundary problem for Hele-Shaw flows in the plane

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0. Introduction

In [7] Richardson derived a mathematical model for describing Hele-Shaw flows with a free boundary produced by the injection of fluid into a narrow channel. This model can be represented in the following form (see also [3]): Given $f_0(z)$, $f_0(0)=0$, analytic and univalent in a neighbourhood of $|z|\leq 1$, find $f(z,t)$, analytic and univalent as a function of z in a neighbourhood of $|z|\leq 1$, continuously differentiable with respect to t in a right-sided neighbourhood of $t=0$, satisfying

$$\text{(1)} \qquad \operatorname{Re}\left(\frac{1}{z}\frac{\partial f}{\partial t}(z,t)\overline{\frac{\partial f}{\partial z}(z,t)}\right)=1 \qquad \text{for } |z|=1;$$

$$\text{(2)} \qquad f(z,0)=f_0(z) \qquad \text{for } |z|\leq 1;$$

$$\text{(3)} \qquad f(0,t)=0.$$

With the results of Vinogradov–Kufarev [9] one gets the existence and uniqueness of solutions which depend analytically on z and t under the additional assumption $f_z(0,t)>0$. But the proofs in [9] are fairly complicated.

For this reason Gustafsson gave in [3] a more elementary proof of existence and uniqueness of solutions of (1)–(3) in the case that $f_0(z)$ is a polynomial or a rational function. In both cases the solution is of the same sort with regard to z as the initial value $f_0(z)$. The restriction to rational initial values seems to be indispensable for the used reduction of (1) to a finite system of ordinary differential equations in t.

The goal of the present paper is to give a simplified proof for a generalized Hele-Shaw problem containing as a special case the above formulated problem (1)–(3). This proof is based on the application of the non-linear abstract Cauchy–Kovalevsky theorem which was proved by Nishida in [5]. Moreover, this theorem gives uniqueness for solutions depending continuously differentiably on t.

Theorem 1 ([5]). *Let us consider the abstract Cauchy–Kovalevsky problem*

$$\frac{dw}{dt} = \mathcal{L}(t,w), \quad w(0) = 0 \tag{4}$$

satisfying the following conditions in a scale of Banach spaces $\{B_s, \|\cdot\|_s\}_{0<s\le 1}$ *(A family of continuously embedded Banach spaces* $\{B_s, \|\cdot\|_s\}_{0<s\le 1}$ *is called a Banach space scale if for all* $0<s'\le s\le 1$ *the norm of the canonical embedding operator* $\|I_{s\to s'}\|\le 1$.) *(C, K, R and T are certain positive constants independent of* s', s, t*):*

(i) *the right-hand side* $\mathcal{L}(t,w)$ *is a continuous, in t, mapping of*

$$[0,T]\times\{w\in B_s : \|w\|_s < R\} \quad \text{into } B_{s'} \quad \text{for all } 0 < s' < s \le 1; \tag{5}$$

(ii) *the continuous function* $\mathcal{L}(t,0)$ *satisfies*

$$\|\mathcal{L}(t,0)\|_s \le K/(1-s) \quad \text{for all } 0 < s < 1; \tag{6}$$

(iii) *for all* $0<s'<s\le 1$, $t\in[0,T]$ *and* w_1, w_2 *belonging to* $\{\|w\|_s<R\}$ *we have*

$$\|\mathcal{L}(t,w_1) - \mathcal{L}(t,w_2)\|_{s'} \le \frac{C}{s-s'}\|w_1 - w_2\|. \tag{7}$$

Under these assumptions there exists one and only one solution

$$w \in C^1([0, a_0(1-s)), B_s)_{0<s<1}, \quad \|w(t)\|_s < R,$$

where a_0 *is a suitable positive constant.*

This theorem represents an essential tool for solving non-linear time-dependent mixed problems for harmonic or holomorphic functions in the mathematical literature ([1, 2, 4, 6]). Our problem (1)–(3) is of such a type. We shall show that after the reduction of the generalized Hele-Shaw problem to an equivalent problem for $w = (\partial f/\partial z)^{-1}$, which fulfills all the conditions (5)–(7) in suitable scales of Banach spaces, the abstract theorem is applicable and yields immediately the main result of [9] as a special case.

The result of Gustafsson [3] can be interpreted as a regularity result concerning the corresponding structures of the initial value and the solution. A result of the same type is derived at the end of this paper for $(\partial f/\partial z)^{-1}$ or $(\partial f_0/\partial z)^{-1}$ belonging to special classes of entire functions.

1. Heuristic considerations and the derivation of a scale-type problem

Let us start with a generalization of (1) to

$$\mathrm{Re}\left(\frac{1}{h(z,t)}\frac{\partial f}{\partial t}(z,t)\overline{\frac{\partial f}{\partial z}(z,t)}\right)=g(z,\bar z,t) \tag{8}$$

for all $|z|=1$ and $t>0$, where

(i) the real-valued function $g=g(z,\bar z,t)$ is continuous on $\{|z|=1\}\times[0,T]$ and possesses a holomorphic extension from $|z|=1$ into a circular ring

$$K_b=\{1/b<|z|<b\},\quad b>1,\qquad \text{for all } t\in[0,T]; \tag{9}$$

(ii) the function $h=h(z,t)$ is continuous in $t\in[0,T]$ and for each such t analytic in a neighbourhood of

$$|z|\le 1,\quad h(0,t)=0,\quad h_z(0,t)\neq 0\quad \text{for all } t\in[0,T] \tag{10}$$

and

$$h(z,t)\neq 0\quad \text{for all } (z,t)\in\{0<|z|\le 1\}\times[0,T].$$

Setting $h(z,t)=z$ and $g(z,\bar z,t)=1$ in (8) we have the condition (1). The condition (8) is equivalent to

$$\mathrm{Re}\left(\frac{1}{h(z,t)}\frac{\partial f}{\partial t}(z,t)\left(\frac{\partial f}{\partial z}\right)^{-1}(z,t)\right)=\left|\frac{\partial f}{\partial z}(z,t)\right|^{-2}g(z,\bar z,t).$$

From the assumptions (3), (9), (10) and the univalence of $f(z,t)$ in a neighbourhood of $\{|z|\le 1\}$ for all $t\in[0,T]$ we get the holomorphy of

$$\frac{\partial f}{\partial t}(z,t)\left(\frac{\partial f}{\partial z}\right)^{-1}(z,t)/h(z,t)$$

in $\{|z|<1\}$. Using (8) and the fact that every holomorphic function in $\{|z|<1\}$ with prescribed real part on $\{|z|=1\}$ is uniquely determined by the value for the imaginary part in $z=0$ we are able to formulate the additional condition

$$\mathrm{Im}\left(\frac{1}{h(z,t)}\frac{\partial f}{\partial t}(z,t)\left(\frac{\partial f}{\partial z}\right)^{-1}(z,t)\right)(0,t)=0. \tag{11}$$

The application of the Schwarz formula leads to

$$\frac{\partial f}{\partial t}(z,t)-h(z,t)\frac{\partial f}{\partial z}(z,t)\frac{1}{2\pi i}\int_{|z|=1}\left|\frac{\partial f}{\partial \varrho}\right|^{-2}g(\varrho,\bar\varrho,t)\frac{\varrho+z}{\varrho-z}\,\frac{d\varrho}{\varrho}=0 \tag{12}$$

for $|z|<1$. For our further investigations we need the space $\mathcal{H}(G_r)\cap C(\overline{G}_r)$, that is the space of all complex-valued functions defined and continuous in $\overline{G}_r$ and holomorphic in $G_r=\{|z|<r\}$. In the same manner we introduce the spaces $\mathcal{H}(G_r)\cap C^\alpha(\overline{G}_r)$, $\mathcal{H}(G_r)\cap C^1(\overline{G}_r)$ and $\mathcal{H}(G_r)\cap C^{1,\alpha}(\overline{G}_r)$.

Lemma 1. *Let us suppose that* $f(z,t)\in C^1([0,a_0),\mathcal{H}(G_1)\cap C^1(\overline{G}_1))$ *is for each* $t\in[0,a_0)$ *a univalent function in* $|z|\leq 1$ *and in* $G_1\times(0,a_0)$ *a solution of the problem* (8), (11), (2) *and* (3), *and equivalently, of the problem* (12), (2) *and* (3). *Then* $v(z,t)=(\partial f/\partial z)^{-1}\in C^1([0,a_0),\mathcal{H}(G_1)\cap C(\overline{G}_1))$ *is a solution of*

$$\frac{\partial v}{\partial t}-hT_t(v)\frac{\partial v}{\partial z}+v\frac{\partial}{\partial z}(hT_t(v))=0 \quad \textit{for } (z,t)\in G_1\times(0,a_0), \tag{13}$$

$$v(z,0)=v_0(z)=(\partial f_0/\partial z)^{-1} \quad \textit{for } z\in\overline{G}_1, \tag{14}$$

where $v(z,t)\neq 0$.

Here $T_t(v)$ *denotes the non-linear operator*

$$T_t(v):=\frac{1}{2\pi i}\int_{\partial G_1}|v(\varrho)|^2 g(\varrho,\bar{\varrho},t)\frac{\varrho+z}{\varrho-z}\frac{d\varrho}{\varrho}. \tag{15}$$

Conversely, let us suppose that $v(z,t)\in C^1([0,a_1),\mathcal{H}(G_1)\cap C(\overline{G}_1))$ *is a solution of* (13) *and* (14) *with* $v(z,t)\neq 0$ *in* $\overline{G}_1\times[0,a_0)$. *Then* $f(z,t)=\int_0^z (d\varrho)/(v(\varrho,t))$ *belonging to* $C^1([0,a_0),\mathcal{H}(G_1)\cap C^1(\overline{G}_1))$ *represents a locally univalent solution of* (12), (2), *and* (3) *and, equivalently, of* (8), (11), (2) *and* (3) *in* $\overline{G}_1\times[0,a_0)$.

Proof. Let $f=f(z,t)$ as a univalent solution of (12), (2) and (3) satisfy the conditions of this lemma. Then $v=(\partial f/\partial z)^{-1}$ belongs to $C^1([0,a_0),\mathcal{H}(G_1)\cap C(\overline{G}_1))$. Differentiating (12) with respect to z, one obtains with $v=(\partial f/\partial z)^{-1}$

$$\frac{\partial(1/v)}{\partial t}-hT_t(v)\frac{\partial(1/v)}{\partial z}-\frac{1}{v}\frac{\partial}{\partial z}(hT_t(v))=0,$$

and hence,

$$\frac{\partial v}{\partial t}-hT_t(v)\frac{\partial v}{\partial z}+v\frac{\partial}{\partial z}(hT_t(v))=0 \quad \text{with } v(z,0)=(\partial f_0/\partial z)^{-1}.$$

Conversely, if $v\in C^1([0,a_0),\mathcal{H}(G_1)\cap C(\overline{G}_1))$ solves (13) and (14) with $v(z,t)\neq 0$ in $\overline{G}_1\times[0,a_0)$, then $1/v$ belongs to $C^1([0,a_0),\mathcal{H}(G_1)\cap C(\overline{G}_1))$ and f belongs to $C^1([0,a_0),\mathcal{H}(G_1)\cap C^1(\overline{G}_1))$, where $\partial_z f(z,t)\neq 0$. Hence, f is locally univalent. The definition of f implies $f(0,t)=0$ for $t\in[0,a_0)$. Furthermore,

$$f(z,0)=\int_0^z\frac{d\varrho}{v(\varrho,0)}=\int_0^z\frac{\partial f_0}{\partial\varrho}\,d\varrho=f_0(z)-f_0(0)=f_0(z).$$

Thus the conditions (2) and (3) are fulfilled.

If v solves (13), then the same reasoning as above gives

$$\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial t}-hT_t\left(\left(\frac{\partial f}{\partial \varrho}\right)^{-1}\right)\frac{\partial f}{\partial z}\right)=0.$$

For $t\in(0,a_0)$ the term in the brackets is holomorphic in G_1, hence,

$$\frac{\partial f}{\partial t}-h\frac{\partial f}{\partial z}T_t\left(\left(\frac{\partial f}{\partial \varrho}\right)^{-1}\right)=k(t),$$

a constant depending on t. Inserting $z=0$, this shows that $k(t)=0$, hence (12) is satisfied.

Finally from the holomorphy of $(1/h)(\partial f/\partial t)(\partial f/\partial z)^{-1}$ we obtain (8) and (11).

Remark 1. An analogous statement is valid for $f\in C^1([0,a_0),\mathcal{H}(G_1)\cap C^{1,\alpha}(\overline{G}_1))$ and $v\in C^1([0,a_0),\mathcal{H}(G_1)\cap C^{\alpha}(\overline{G}_1))$ instead of $f\in C^1([0,a_0),\mathcal{H}(G_1)\cap C^1(\overline{G}_1))$ and $v\in C^1([0,a_0),\mathcal{H}(G_1)\cap C(\overline{G}_1))$.

The lemma of equivalence just proved makes it possible to restrict ourselves to the problem (13) and (14). This is a scale-type problem. Thus it remains to show how we can interpret the problem (13) and (14) as a special case of (4) (see Section 3).

There is a gap between Richardson's mathematical model and Lemma 1. In Lemma 1 we obtain in the converse direction merely the local univalence of $f(z,t)$. But the following statement holds:

Suppose, that

(i) the initial value $f_0(z)$ from (2) is an analytic and univalent function in $\overline{G}_r, r>1$;

(ii) the family $\{f_t(z)\}$ of analytic functions belongs to $C([0,T],\mathcal{H}(G_{r'})\cap C(\overline{G}_{r'}))$, $r'<r$.

Then there exists a positive constant $T_0(r')$ such that $f_t(z)$ is univalent in $\overline{G}_{r'}$ for all $t\in[0,T_0(r'))$.

Using this statement the conditions

(i) univalence of the analytic function $f_0(z)$ in $\overline{G}_r$;

(ii) $v\in C^1([0,a_0),\mathcal{H}(G_{r'})\cap C(\overline{G}_{r'}))$ with $v(z,t)\neq 0$;

imply the univalence of $f(z,t)$ for small t in a neighbourhood of $\{|z|\leq 1\}$.

In Chapter 3 we shall prove the existence of such functions $v=v(z,t)$ as solutions of a modified problem to (13) and (14).

2. About the action of an operator $\widetilde{T}_t$ representing a continuation of T_t in some Banach spaces

Let v be in $C([0,T],H(G_r)\cap C(\overline{G}_r))$ with $r>1$. Then $T_t(v)$ belongs to $\mathcal{H}(G_1)$

for each $t\in[0,T]$. But moreover $T_t(v)$ possesses an analytic continuation in a larger domain depending on G_r and K_b from (9).

Lemma 2. *For an arbitrary $v\in\mathcal{H}(G_r)\cap C(\overline{G}_r)$ the image $T_t(v)$ of the nonlinear operator T_t applied to v can be analytically continued into G_{r_0} with $r_0=\min(b,r)$.*

Proof. From (15) we get

$$T_t(v)=\frac{1}{2\pi i}\int_{\partial G_1}|v(\varrho)|^2 g(\varrho,\bar\varrho,t)\frac{\varrho+z}{\varrho-z}\frac{d\varrho}{\varrho}$$
$$=\frac{1}{2\pi i}\int_{\partial G_1}v(\varrho)\overline{v(1/\bar\varrho)}g(\varrho,1/\varrho,t)\,\frac{\varrho+z}{\varrho-z}\frac{d\varrho}{\varrho}\quad\text{for all } z\in G_1.$$

The assumption $v\in\mathcal{H}(G_r)\cap C(\overline{G}_r)$ and (9) guarantee that the kernel of the integral is holomorphic in the set $\{1/r_0<|\varrho|<r_0\}\setminus\{z\}$ for all $t\in[0,T]$ and $z\in G_1$. Consequently,

$$T_t(v)=\frac{1}{2\pi i}\int_{\partial G_a}v(\varrho)\overline{v(1/\bar\varrho)}\,g(\varrho,1/\varrho,t)\,\frac{\varrho+z}{\varrho-z}\frac{d\varrho}{\varrho}$$

for all $z\in G_1$ and $1<a<r_0$. Obviously, the right-hand-side can be defined for all $z\in G_a$, and $T_t(v)$ possesses an analytic continuation

$$\widetilde{T}_t(v)=\frac{1}{2\pi i}\int_{\partial G_a}v(\varrho)\overline{v(1/\bar\varrho)}\,g(\varrho,1/\varrho,t)\,\frac{\varrho+z}{\varrho-z}\frac{d\varrho}{\varrho}\tag{16}$$

belonging to $\mathcal{H}(G_a)$. Since $G_{r_0}=\bigcup_{1<a<t_0}G_a$ the operator $\widetilde{T}_t$ maps $\mathcal{H}(G_r)$ into $\mathcal{H}(G_{r_0})$. For all $z\in G_1$ we conclude $\widetilde{T}_t(v)(z)=T_t(v)(z)$. Hence $\widetilde{T}_t(v)$ represents an analytic continuation of $T_t(v)$ for $v\in\mathcal{H}(G_r)\cap C(\overline{G}_r)$ into G_{r_0}.

There arises the question whether it is possible to estimate the action of $\widetilde{T}_t$ as a mapping of a Banach space B into itself. In the next lemma we shall give a positive answer for the case $B=\mathcal{H}(G_p)\cap C(\overline{G}_p), 1<p<r_0$.

Lemma 3. (a) *For every function v from $\mathcal{H}(G_p)\cap C(\overline{G}_p)$ the following estimate connecting the norms $\|v\|_p=\sup_{G_p}|v|$ and $\|\widetilde{T}_t(v)\|_p=\sup_{G_p}|\widetilde{T}_t(v)|$ holds:*

$$\|\widetilde{T}_t(v)\|_p\le C(p,g)\|v\|_p^2,$$

where the constant C is independent of $v\in\mathcal{H}(G_p)\cap C(\overline{G}_p)$ and $t\in[0,T]$. Moreover, we obtain for all $v_1,v_2\in B$ with $\|v_1\|_p,\|v_2\|_p<R$ the Lipschitz condition

$$\|\widetilde{T}_t(v_1)-\widetilde{T}_t(v_2)\|_p\le 2C(p,g)R\|v_1-v_2\|_p.$$

(b) *The family of operators* $\{\widetilde{T}_t(v)\}_{t\in[0,T]}$ *depends continuously on* $t\in[0,T]$. *This means*

$$\lim_{t_1\to t_2} \|\widetilde{T}_{t_1}(v)-\widetilde{T}_{t_2}(v)\|_p=0 \quad \textit{for all } v\in\mathcal{H}(G_p)\cap C(\overline{G}_p).$$

Proof. (a) Let us remember that

$$\widetilde{T}_t(v)=\frac{1}{2\pi i}\int_{\partial G_p} v(\varrho)\overline{v(1/\bar{\varrho})}\, g(\varrho,1/\varrho,t)\,\frac{\varrho+z}{\varrho-z}\,\frac{d\varrho}{\varrho}.$$

Using the holomorphy of $v(\varrho)\overline{v(1/\bar{\varrho})}g(\varrho,1/\varrho,t)(\varrho+z)/\varrho$ in $\{1/p<|\varrho|<p\}$, we obtain for all $z\in\partial G_{p'}, p'\to p$, and $t\in[0,T]$

$$\begin{aligned}\widetilde{T}_t(v)(z)=&\frac{1}{2\pi i}\int_{\partial G_{1/p}} v(\varrho)\overline{v(1/\bar{\varrho})}\, g(\varrho,1/\varrho,t)\frac{\varrho+z}{\varrho-z}\,\frac{d\varrho}{\varrho}\\&+\frac{1}{2\pi i}\int_{\partial\mathcal{U}_a(z)} v(\varrho)\overline{v(1/\bar{\varrho})}\, g(\varrho,1/\varrho,t)\frac{\varrho+z}{\varrho-z}\,\frac{d\varrho}{\varrho},\end{aligned}$$

where $\mathcal{U}_a(z)$ is a sufficiently small neighbourhood of z contained in G_p. From Cauchy's integral formula and a simple estimation it follows that

$$\begin{aligned}\widetilde{T}_t(v)(z)|\le&\left|\frac{1}{2\pi}\int_0^{2\pi} v\Big(\frac{1}{p}e^{i\varphi}\Big)\overline{v(pe^{i\varphi})}g\Big(\frac{1}{p}e^{i\varphi},pe^{-i\varphi},t\Big)\frac{e^{i\varphi}/p+z}{e^{i\varphi}/p-z}\,d\varphi\right|\\&+2\big|v(z)v(1/\bar{z})g(z,1/z,t)\big|\\&\le\|v\|_p^2\sup_{(z,t)\in\{1/p<|z|<p\}\times[0,T]}\big|g(z,1/z,t)\big|\left(2+\frac{|z|+1/p}{|z|-1/p}\right)\end{aligned}$$

for all $z\in\partial G_{p'}$. But the continuity of v in $\overline{G}_p$ guarantees that the last inequality remains valid for all $z\in\partial G_p$. Hence, by the maximum principle

$$\|\widetilde{T}_t(v)\|_p=\sup_{z\in G_p}|\widetilde{T}_t(v)(z)|\le C(p,g)\|v\|_p^2$$

with

$$C(p,g)=\sup_{(z,t)\in\{1/p<|z|<p\}\times[0,T]}\big|g(z,1/z,t)\big|\left(2+\frac{p^2+1}{p^2-1}\right).$$

By (9) and $1<p<r_0\le b$ the constant $C(p,g)$ is finite. The same reasoning leads to the Lipschitz condition.

(b) As in the proof of (a) one deduces

$$\|\widetilde{T}_{t_1}(v)(z)-\widetilde{T}_{t_2}(v)(z)\|_p \leq \left(2+\frac{p^2+1}{p^2-1}\right) \sup_{z\in\{1/p<|z|<p\}} \left|g(z,1/z,t_1)-g(z,1/z,t_2)\right| \leq \varepsilon$$

for $|t_1-t_2|$ sufficiently small and all $1<p<r_0$, taking into consideration the uniform continuity of g in $\{1/p\leq|z|\leq p\}\times[0,T]$.

Remark 2. It is possible to prove a corresponding inequality between $\|v\|_{p,\alpha}$ and $\|\widetilde{T}_t(v)\|_{p,\alpha}, 0<\alpha<1$, where $\|v\|_{p,\alpha}$ denotes the Hölder-norm of $v\in\mathcal{H}(G_p)\cap C^\alpha(\overline{G}_p)$. The proof of $\|\widetilde{T}_t(v)\|_{p,\alpha}\leq C(p,\alpha,g)\|v\|^2_{p,\alpha}$ is omitted.

For proving a regularity result for $(\partial f/\partial z)^{-1}$ in the sense of the results in [3] the next lemma represents an essential tool. For the formulation of this lemma we choose the following family $\{E_r\}_{r>0}$ of Banach spaces of entire functions:

$$\{E_r\}_{r>0} = \left\{v\in\mathcal{H}(\mathbf{C}): \sup_{z\in\mathbf{C}}\left|v(z)e^{-r|z|}\right| = \|v\|_r<\infty\right\}_{r>0}.$$

Now we are choosing $g=1$ in (16).

Lemma 4. *The operator*

$$\widetilde{T}(v)(z)=\frac{1}{2\pi i}\int_{\partial G_a} v(\varrho)\overline{v(1/\bar{\varrho})}\,\frac{\varrho+z}{\varrho-z}\,\frac{d\varrho}{\varrho},$$

$z\in G_a$, $a>1$ arbitrary, maps E_r into itself, where $\|\widetilde{T}(v)\|_r\leq\frac{11}{3}\exp(5r/2)\|v\|^2_r$.

Moreover, we obtain for all $v_1,v_2\in E_r$ with $\|v_1\|_r,\|v_2\|_r<R$ the Lipschitz condition $\|\widetilde{T}_t(v_1)-\widetilde{T}_t(v_2)\|_r\leq\frac{22}{3}Re^{5r/2}\|v_1-v_2\|_r$.

Proof. Supposing $v\in E_r$ the above-defined function $\widetilde{T}(v)(z)$ makes sense for all $z\in\mathbf{C}$. This follows from the fact that $v(\varrho)\overline{v(1/\bar{\varrho})}(\varrho+z)$ is holomorphic in $\mathbf{C}\setminus\{0\}$. Hence $\widetilde{T}(v)$ is an entire function.

Now let us fix $z_0\in\mathbf{C}$ with $|z_0|\geq2$. Then as in the proof of Lemma 3(a) we arrive at

$$\widetilde{T}(v)(z_0)=\frac{1}{2\pi i}\int_{\partial G_{1/b}} v(\varrho)\overline{v(1/\bar{\varrho})}\,\frac{\varrho+z_0}{\varrho-z_0}\,\frac{d\varrho}{\varrho}+2v(z_0)\overline{v(1/\bar{z}_0)}$$

for an arbitrary $b>|z_0|$, and

$$\widetilde{T}(v)(z_0)\exp(-r|z_0|)=\frac{1}{2\pi i}\int_{\partial G_{1/b}} v(\varrho)\overline{v(1/\bar{\varrho})}e^{-r/|\varrho|}e^{r(1/|\varrho|-|z_0|)}\,\frac{\varrho+z_0}{\varrho-z_0}\,\frac{d\varrho}{\varrho}$$
$$+2v(z_0)e^{-r|z_0|}\overline{v(1/\bar{z}_0)}.$$

But this leads immediately to

$$\begin{aligned}\left|\widetilde{T}(v)(z_0)e^{-r|z_0|}\right| &\leq \tfrac{5}{3}\max_{|z|=1/b}|v(z)|\max_{|z|=b}\left|v(z)e^{-r|z|}\right|e^{-r(b-|z_0|)}\\ &\quad +2\max_{|z|=1/2}|v(z)|\,|v(z_0)|e^{-r|z_0|}\\ &\leq \tfrac{11}{3}\max_{|z|=1/2}|v(z)|\,\|v\|_r\end{aligned}$$

if one takes into account that

$$\frac{|z_0|+1/b}{|z_0|-1/b}\leq\frac{|z_0|+\frac{1}{2}}{|z_0|-\frac{1}{2}}\leq\frac{5}{3}\quad \text{for } |z_0|\geq 2,\ b>0$$

and $e^{r(b-|z_0|)}\to 1$ for $|z_0|\to b$.

From the definition of $\|v\|_r$ we obtain

$$\max_{|z|=1/2}|v(z)|\leq\|v\|_r e^{r/2}\quad\text{and}\quad\max_{|z|=2}|v(z)|\leq\|v\|_r e^{2r}.$$

Thus it is possible to draw the following two conclusions:

$$\left|\widetilde{T}(v)(z_0)e^{-r|z_0|}\right|\leq\tfrac{11}{3}e^{r/2}\|v\|_r^2\quad\text{for each } z_0\in C \text{ with } |z_0|\geq 2,$$

and

$$\left|\widetilde{T}(v)(z_0)e^{-r|z_0|}\right|\leq\max_{|z|=2}\left|\widetilde{T}(v)(z)e^{-2r}e^{2r}\right|\leq\tfrac{11}{3}e^{5r/2}\|v\|_r^2,$$

for each $z_0\in C$ with $|z_0|<2$.

But these conclusions yield $\|\widetilde{T}(v)\|_r\leq\frac{11}{3}e^{5r/2}\|v\|_r^2$.

The same reasoning gives the Lipschitz condition.

In this section we introduced the operator $\widetilde{T}_t(v)$ and studied some of its properties as for example the relation between T_t and $\widetilde{T}_t$. The results obtained are useful in examining the problem

$$\text{(17)}\qquad \frac{\partial v}{\partial t}-h\widetilde{T}_t(v)\frac{\partial v}{\partial z}+v\frac{\partial}{\partial z}(h\widetilde{T}_t(v))=0,\quad v(z,0)=v_0(z)=(\partial f_0/\partial z)^{-1}.$$

The restriction of a solution $v\in C^1([0,a_0),\mathcal{H}(G_r)\cap C(\overline{G}_r))$ of this problem to $(z,t)\in\overline{G}_1\times[0,a_0)$ represents a solution $v\in C^1([0,a_0),\mathcal{H}(G_1)\cap C(\overline{G}_1))$ of (13) and (14).

3. The problem (17) and (14) as a special case of (4)

To apply Theorem 1 to the problem (17) and (14), we only have to show that the conditions (5)–(7) are fulfilled. The assumptions concerning f_0 and h guarantee the existence of constants $1<r_2<b$ and $R>0$ such that

$$R \le |v_0(z)| = |(\partial f_0/\partial z)^{-1}| \quad \text{in } \overline{G}_{r_2},$$

and $h \in C([0,T], \mathcal{H}(G_{r_2}) \cap C(\overline{G}_{r_2}))$. For a fixed $1<r_1<r_2$ let us choose the Banach space scale

$$\{B_s, \|\cdot\|_s\}_{0<s\le 1} = \{\mathcal{H}(G_{r_1+s(r_2-r_1)}) \cap C(\overline{G}_{r_1+s(r_2-r_1)}), \sup_{G_{r_1+s(r_2-r_1)}} |\cdot|\}_{0<s\le 1}.$$

Following Lemma 1 ($v(z,t)\neq 0$) it is necessary to choose the sphere

$$\{w \in B_s : \|w\|_s < R\}.$$

Introducing $w(z,t)=v(z,t)-v_0(z)$, this implies a homogeneous initial condition. Thus the problem (17) and (14) can be transformed to

$$\frac{\partial w}{\partial t} = \mathcal{L}_0(t,w) = -(w+v_0)\frac{\partial}{\partial z}(h\widetilde{T}_t(w+v_0)) + h\widetilde{T}_t(w+v_0)\frac{\partial}{\partial z}(w+v_0), \tag{18}$$

$$w(z,0)=0. \tag{19}$$

Lemma 5. *The operator $\mathcal{L}_0$ satisfies in the above-introduced Banach space scale $\{B_s, \|\cdot\|_s\}_{0<s\le 1}$ the conditions* (5)–(7) *of Theorem* 1.

Proof. Every space B_s forms a Banach algebra. Consequently, from Lemma 3(a), $v_0 \in B_1$ and $h \in C([0,T], B_1)$ we conclude that $h\widetilde{T}_t(w+v_0) \in B_s$ for all $0< s\le 1$ and all $w\in B_s$. Using the result of Tutschke [8] that $\partial/\partial z$ is a bounded operator as the mapping of B_s into B_s' with $\|\partial/\partial z\|_{s\to s'} \le ((r_2-r_1)(s-s'))^{-1}$ one obtains $\mathcal{L}_0(t,w) \in B_{s'}$ for every $(t,w) \in [0,T] \times \{w \in B_s : \|w\|_s < R\}$. From Lemma 3(b) it follows that for a given $w \in B_s$ the term $\widetilde{T}_t(w+v_0)$ depends continuously on t. But this leads to $\lim_{t_1\to t_2} \|\mathcal{L}_0(t_1,w)-\mathcal{L}_0(t_2,w)\|_{s'}=0$ for all $t_1,t_2\in[0,T]$ and all $w\in B_s$. This proves (5).

Let us further consider the difference

$$\begin{aligned}\mathcal{L}_0(t,w_1)-\mathcal{L}_0(t,w_2) = &-(w_1-w_2)\frac{\partial}{\partial z}(h\widetilde{T}_t(w_1+v_0)) - (w_2+v_0)\frac{\partial}{\partial z}\big(h(\widetilde{T}_t(w_1+v_0)\\ &-\widetilde{T}_t(w_2+v_0))\big) + h(\widetilde{T}_t(w_1+v_0)-\widetilde{T}_t(w_2+v_0))\frac{\partial}{\partial z}(w_1+v_0)\\ &+h\widetilde{T}_t(w_2+v_0)\frac{\partial}{\partial z}(w_1-w_2).\end{aligned}$$

Using

$$\left\|\frac{\partial}{\partial z}(w_1-w_2)\right\|_p \le 2C(p,g)(R+\|v_0\|_1)\|w_1-w_2\|_p$$

for all $w_1,w_2\in\mathcal{H}(G_p)\cap C(\overline{G}_p)$ with $\|w_1\|_p,\|w_2\|_p<R$ and all $t\in[0,T]$ the following estimates are valid in $\{B_s,\|\cdot\|_s\}_{0<s\le 1}$:

$$\begin{aligned}\|\mathcal{L}_0(t,w_1)-\mathcal{L}_0(t,w_2)\|_{s'} &\le \|w_1-w_2\|_s\|h\|_1\frac{\|\widetilde{T}_t(w_1+v_0)\|_s}{(s-s')(r_2-r_1)}\\ &\quad+\frac{\|h\|_1\|w_2+v_0\|_s}{(s-s')(r_2-r_1)}\|\widetilde{T}_t(w_1+v_0)-\widetilde{T}_t(w_2+v_0)\|_s\\ &\quad+\|h\|_1\|\widetilde{T}_t(w_1+v_0)-\widetilde{T}_t(w_2+v_0)\|_s\frac{\|w_1+v_0\|_s}{(s-s')(r_2-r_1)}\\ &\quad+\|h\|_1\|\widetilde{T}_t(w_2+v_0)\|_s\frac{\|w_1-w_2\|_s}{(r_2-r_1)(s-s')}\\ &\le\frac{\|w_1-w_s\|_2}{(s-s')(r_2-r_1)}\|h\|_1(R+\|v_0\|_1)^2 6C(r_2,r_1,g)\end{aligned}$$

with

$$C(r_2,r_1,g)=\sup_{(z,t)\in\{1/r_2<|z|<r_2\}\times[0,T]}\left|g(z,1/z,t)\right|\left(2+\frac{r_2^2+1}{r_1^2-1}\right).$$

So, also (7) is proved.

Finally, in the same manner it can be verified that

$$\|\mathcal{L}_0(t,0)\|_s=\left\|v_0\frac{\partial}{\partial z}(h\widetilde{T}_t(v_0))-h\widetilde{T}_t(v_0)\frac{\partial}{\partial z}v_0\right\|_s\le K/(1-s)$$

with a certain constant K independent of s and t. Hence also (6) is true, which completes the proof of this lemma.

Now the application of Theorem 1 to the problem (18) and (19) yields one and only one solution

$$w\in C^1\big([0,a_0(1-s)),\mathcal{H}(G_{r_1+s(r_2-r_1)})\cap C(\overline{G}_{r_1+s(r_2-r_1)})\big)_{0<s<1}$$

with $\sup_{G_{r_1+s(r_2-r_1)}}|w(z,t)|<R$ for all $t\in[0,a_0(1-s))$.

But then $v(z,t)=w(z,t)+v_0(z)$ represents a solution

$$v\in C^1\big([0,a_0(1-s)),\mathcal{H}(G_{r_1+s(r_2-r_1)})\cap C(\overline{G}_{r_1+s(r_2-r_1)})\big)_{0<s<1}$$

of the problem (17) and (14) with $\sup_{G_{r_1+s(r_2-r_1)}}|v(z,t)|>0$ for all $t\in[0,a_0(1-s))$.

The coincidence of the operators $\widetilde{T}_t$ and T_t for all $v\in\mathcal{H}(G_1)\cap C(\overline{G}_1)$ guarantees that the restriction of $v(z,t)$ to $C^1([0,a_0),\mathcal{H}(G_{r_1})\cap C(\overline{G}_{r_1}))$ is a solution of (13) and (14) with $\sup_{G_{r_1}}|v(z,t)|>0$ for all $t\in[0,a_0)$. From this result together with Lemma 1, the end of Chapter 1 and the equivalence of (12) with (8) and (11) we get the following theorem concerning problem (8), (2) and (3).

Theorem 2. *Suppose that*

(i) *the real-valued function* $g=g(z,\bar{z},t)$ *is continuous in* $\{|z|=1\}\times[0,T]$ *and possesses a holomorphic extension into a circular ring* $K_b=\{1/b<|z|<b\}$ *for all* $t\in[0,T]$;

(ii) *the function* $h=h(z,t)$ *belongs to the space* $C([0,T],\mathcal{H}(G_{r_2})\cap C(\overline{G}_{r_2}))$, $1<r_2<b$, $G_{r_2}=\{|z|<r_2\}$, *where* $h(0,t)=0$, $h_z(0,t)\neq 0$ *and* $h(z,t)\neq 0$ *for all* $(z,t)\in\{0<|z|\leq 1\}\times[0,T]$;

(iii) *the function* $f_0(z)$, $f_0(0)=0$, *is holomorphic and univalent in* $\overline{G}_{r_2}$.

Then for every $1<r_1<r_2$ *there exist a positive constant* $a_0(r_1)$ *and a uniquely determined function* $f=f(z,t)$, *holomorphic and univalent in* $\overline{G}_{r_1}$, *belonging to* $C^1([0,a_0(r_1)),\mathcal{H}(G_{r_1})\cap C^1(\overline{G}_{r_1}))$ *and satisfying the conditions*

$$\mathrm{Re}\left(\frac{1}{h(z,t)}\frac{\partial f}{\partial t}(z,t)\overline{\frac{\partial f}{\partial z}(z,t)}\right)=g(z,\bar{z},t)\quad \textit{for all } (z,t)\in\{|z|=1\}\times(0,a_0(r_1));$$

$$\mathrm{Im}\left(\frac{1}{h(z,t)}\frac{\partial f}{\partial t}(z,t)\overline{\frac{\partial f}{\partial z}(z,t)}\right)(0,t)=0\quad \textit{for } t\in(0,a_0(r_1));$$

$$f(z,0)=f_0(z)\quad \textit{for } z\in\overline{G}_{r_1};$$

$$f(0,t)=0\quad \textit{for } t\in[0,a_0(r_1)).$$

As a conclusion from Theorem 2 we immediately get a statement concerning the classical Hele-Shaw problem in the plane ($h(z,t)=z$, $g(z,\bar{z},t)=1$).

Corollary 1. *Under the assumption that the function* $f_0(z)$, $f_0(0)=0$, *is holomorphic and univalent in* $\overline{G}_{r_2}$, *for every* $1<r_1<r_2$ *there exist a positive constant* $a_0(r_1)$ *and one and only one holomorphic and univalent in* $\overline{G}_{r_1}$ *function* $f=f(z,t)\in C^1([0,a_0(r_1)),\mathcal{H}(G_{r_1})\cap C^1(\overline{G}_{r_1}))$ *satisfying*

$$\mathrm{Re}\left(\frac{1}{z}\frac{\partial f}{\partial t}(z,t)\overline{\frac{\partial f}{\partial z}(z,t)}\right)=1\quad \textit{for } (z,t)\in\{|z|=1\}\times(0,a_0(r_1));$$

$$\mathrm{Im}\left(\frac{1}{z}\frac{\partial f}{\partial t}(z,t)\overline{\frac{\partial f}{\partial z}(z,t)}\right)=0\quad \textit{for } t\in(0,a_0(r_1));$$

$$f(z,0)=f_0(z)\quad \textit{for } z\in\overline{G}_{r_1};$$

$$f(0,t)=0\quad \textit{for } t\in[0,a_0(r_1)).$$

Remark 3. In connection with the moment problem for holomorphic functions Gustafson [3] studied the conditions

$$\mathrm{Re}\left(\frac{1}{z}\frac{\partial f}{\partial t}(z,t)\overline{\frac{\partial f}{\partial z}(z,t)}\right)=\begin{cases}\cos n\varphi=(z^n+\bar{z}^n)/2\\ \sin n\varphi=(z^n-\bar{z}^n)/(2i)\end{cases}\text{on } |z|=1$$

instead of (1).

These conditions are special cases of (8), (9) and (10). The conditions (8)–(10) represent the most general conditions for a successful application of the non-linear abstract Cauchy–Kovalevsky Theorem due to Nishida [5].

Remark 4. Comparing (11) ($h(z,t)=z$) with Gustafsson's condition $f_z(0,t)>0$, it is easy to see that this assumption leads to (11). Hence the solutions of Theorem 2 for the classical Hele-Shaw problem coincide with the solutions constructed by Gustafsson in [3]. On the other hand, since $h(z,t)\sim h_z(0,t)z$ as $z\to 0$, (11) is equivalent to the representation $f_z(0,t)=\exp(i\alpha)\exp(g(t))$ if we additionally suppose that $h_z(0,t)$ is real-valued (α is a real constant, $g=g(t)$ a real-valued continuous function). Thus, (11) really generalizes the condition $f_z(0,t)>0$.

Remark 5. From Theorem 1 applied to problem (18) and (19) one obtains the estimate $\sup_{\overline{G}_{r_1}} |(\partial f(z,t)/\partial z)^{-1}| \le \|(\partial f_0/\partial z)^{-1}\|_{r_2}+R$, where $f=f(z,t)$ is the solution from Theorem 2 and R fulfills $\|(\partial f_0/\partial z)^{-1}\|_{r_2}\ge R$ for all $z\in\overline{G}_{r_2}$.

Taking account of Remarks 1 and 2 and the result of [8] that the operator $\partial/\partial z$ is bounded as a mapping of $\mathcal{H}(G_p)\cap C^\alpha(\overline{G}_p)$ into $\mathcal{H}(G_{p'})\cap C^\alpha(\overline{G}_{p'})$; $\big(p'<p$, $0<\alpha<1$ and $\|\partial/\partial z\|_{p\to p'}\le C/(p-p')\big)$, we are able to prove a result corresponding to Theorem 2 based on the scale of Banach spaces

$$\{B_s,\|\cdot\|_s\}_{0<s\le 1}=\big\{\mathcal{H}\big(G_{r_1+s(r_2-r_1)}\big)\cap C^\alpha\big(\overline{G}_{r_1+s(r_2-r_1)}\big),\|\cdot\|_{s,\alpha}\big\}.$$

For in general a smaller interval $t\in[0,b_0)$ an upper bound for the Hölder-norm of $(\partial f(z,t)/\partial z)^{-1}$ in $\overline{G}_{r_1}$ can be obtained by $\|(\partial f_0/\partial z(z))^{-1}\|_{r_2,\alpha}+R$ with the same R as in the case of the supremum-norms.

4. About the coincidence of the structures of $(\partial f_0/\partial z)^{-1}$ and $(\partial f(z,t)/\partial z)^{-1}$

Gustafsson proved in [3] that, if the initial value $f_0(z)$ is a univalent polynomial or a univalent rational function in a neighbourhood of $|z|\le 1$, then the solution of (1)–(3) is as a function of z of the same structure as $f_0(z)$, which means a univalent polynomial or a univalent rational function. In the polynomial case this coincidence of the structures can be expressed by the aid of the derivatives in the following form:

> If $\partial f_0/\partial z$ is a polynomial which has no zeros in a neighbourhood of $|z|\le 1$ then also $\partial f(z,t)/\partial z$ is a polynomial which has no zeros in a neighbourhood of $|z|\le 1$ for t from a suitable right-sided neighbourhood of $t=0$.

Such a formulation cannot be deduced for the rational case from the results of [3].

Using $(\partial f_0/\partial z)^{-1}$ and $(\partial f(z,t)/\partial z)^{-1}$ the last statement concerning the derivatives $\partial f_0/\partial z$ and $\partial f(z,t)/\partial z$ gets a new formulation.

If $(\partial f_0/\partial z)^{-1} = 1/P(z)$, where $P(z)$ is a polynomial without zeros in a neighbourhood of $|z| \le 1$, then $(\partial f(z,t)/\partial z)^{-1} = 1/Q(z,t)$, where $Q(z,t)$ is a polynomial in z without zeros in a neighbourhood of $|z| \le 1$ for every t from a right-sided neighbourhood of $t=0$.

In the following we are interested in the proof of a result of the same type. For this purpose, let us choose with arbitrary $0<s_1<s_2$ the Banach space scale of entire functions

$$\{B_s, \|\cdot\|_s\}_{0<s\le 1} = \{E_{s_1+(s_2-s_1)(1-s)}, \|\cdot\|_{s_1+(s_2-s_1)(1-s)}\}_{0<s\le 1},$$

where the spaces E_r were introduced in Section 2.

Theorem 3. *In addition to the assumptions of Corollary* 1 *suppose that* $(\partial f_0/\partial z)^{-1}$ *is an entire function belonging to* E_{s_1}. *Then it is known that besides the statement of Corollary* 1, *there holds* $(\partial f(z,t)/\partial z)^{-1} \in C^1([0,a_0(s_2)), B_{s_2})$ *with* $s_2>s_1$ *and a certain positive constant* $a_0(s_2)$. *In particular this means that* $(\partial f(z,t)/\partial z)^{-1}$ *is an entire function for all* $t\in[0,a_0(s_2))$.

(In $[0,a_0(r_1,s_2)), a_0(r_1,s_2)=\min(a_0(s_2),a_0(r_1))$, both properties of $f(z,t)$ are fulfilled.)

Proof. It remains to prove the statement for $(\partial f(z,t)/\partial z)^{-1}$, which follows from the application of Theorem 1 to the problem (18) and (19), equivalently, (17) and (14) with the scale $\{B_s, \|\cdot\|_s\}_{0<s\le 1}$.

From Lemma 4 the continuity of $\widetilde{T}_t(v)$ as a mapping of B_s into itself is clear. Hence we only have to study the behaviour of the differential operator $\partial/\partial z$ and the multiplication operator $z\cdot$ in the scale $\{B_s, \|\cdot\|_s\}_{0<s\le 1}$.

For the first let v be a function from E_r. Applying Cauchy's integral formula in a small neighbourhood $U_a(z_0)$ of a fixed point z_0 we obtain

$$\frac{\partial v}{\partial z}(z_0)e^{-rz_0} = \frac{1}{2\pi}\int_0^{2\pi} \frac{v(\varrho)e^{-r|\varrho|}e^{r(|\varrho|-|z_0|)}}{\varrho - z_0}\, d\varphi$$

with $\varrho=z_0+a\exp(i\varphi)$. Using $||\varrho|-|z_0|| \le |\varrho - z_0|$ this relation leads to

$$\left|\frac{\partial v}{\partial z}(z_0)e^{-r|z_0|}\right| \le \|v\|_r e^{ar}/a.$$

With $a=1/r$ we have $\|\partial v/\partial z\|_r \le er\|v\|_r$. Hence $\partial/\partial z$ is a bounded operator from E_r, respectively, from B_s into itself. In the second place let v be a function from

E_r. Then it cannot be expected, that zv belongs to E_r. For example, let us choose $v = \exp(2z) \in E_2$. Then

$$\sup_{z\in\mathbf{C}} \left| z e^{2z} e^{-2|z|} \right| \geq \sup_{x\in R^+} \left| x \exp(2x) \exp(-2x) \right| = \infty$$

as x tends to infinity. But if we consider z as a mapping of E_r into $E_{r'}$ with $r' > r$, then

$$\|zv\|_{r'} = \sup_{z\in\mathbf{C}} \left| zve^{-r'|z|} \right| \leq \sup_{z\in\mathbf{C}} |ve^{-r|z|}| \sup_{z\in\mathbf{C}} |z| e^{-(r'-r)|z|} \leq \|v\|_r \frac{1}{e(r'-r)}.$$

Hence the multiplication operator $z\cdot$ is a bounded operator in the scale $\{B_s, \|\cdot\|_s\}$ with $\|zv\|_{s'} \leq \|v\|_s / (e(s_2 - s_1)(s - s'))$.

As in Lemma 5 one proves that the oprator $\mathcal{L}_0$ from (18) satisfies the conditions (5)–(7) from Theorem 1. The application of this Theorem to (18) and (19) with the scale $\{B_s, \|\cdot\|_s\}_{0<s\leq 1}$ yields the statement for $(\partial f(z,t)/\partial z)^{-1}$. This completes the proof.

Remark 6. Using the scale $\{B_s, \|\cdot\|_s\}_{0<s\leq 1}$ one can also get the univalence of $f(z,t)$ from that of $f_0(z)$. Let us suppose $|(\partial f_0/\partial z)^{-1}| \geq R > 0$ in $\overline{G}_{r_2}$ and fix the sphere $\|v - (\partial f_0/\partial z)^{-1}\|_s < R\exp(-r_2 s_2)$ around $(\partial f_0/\partial z)^{-1}$. Then the application of Theorem 1 to the problem (18) and (19) leads to

$$\|v(z,t) - (\partial f_0(z)/\partial z)^{-1}\|_s = \|(\partial f(z,t)/\partial z)^{-1} - (\partial f_0(z)/\partial z)^{-1}\|_s < Re^{-r_2 s_2}.$$

But this means that

$$\max_{G_{r_2}} |(\partial f(z,t)/\partial z)^{-1} - (\partial f_0(z)/\partial z)^{-1}| e^{-(s_1 + (s_2 - s_1)(1-s)|z|)} < Re^{-r_2 s_2},$$

and

$$\max_{\overline{G}_{r_2}} |(\partial f(z,t)/\partial z)^{-1} - (\partial f_0(z)/\partial z)^{-1}| < R.$$

Hence $\partial f(z,t)/\partial z \neq 0$ for all $z \in \overline{G}_{r_2}$ and all suitable $t \in [0, a_0(s_2))$. Then an upper bound for $\|(\partial f(z,t)/\partial z)^{-1}\|_{s_2}$ is $\|(\partial f_0(z)/\partial z)^{-1}\|_{s_1} + Re^{-r_2 s_2}$.

But we point out that the restriction to the above-introduced sphere around $(\partial f_0/\partial z)^{-1}$ can reduce the interval of existence of the solution with regard to t from Corollary 1.

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