Martin boundaries of sectorial domains

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Abstract. Let D be a domain in $\mathbf{R}^2$ whose complement is contained in a pair of rays leaving the origin. That is, D contains two sectors whose base angles sum to 2π. We use balayage to give an integral test that determines if the origin splits into exactly two minimal Martin boundary points, one approached through each sector. This test is related to other integral tests due to Benedicks and Chevallier, the former in the special case of a Denjoy domain. We then generalise our test, replacing the pair of rays by an arbitrary number.

1. Introduction

Consider a domain $D \subset \mathbf{R}^2 = \mathbf{C}$ which contains a pair of sectors whose base angles sum to 2π. That is, its complement is a pair of porous rays, radiating from the origin 0. A result of Ancona ((3.3) of [A1]) states that the Euclidean boundary point 0 gives rise to either one or two minimal points of the Martin boundary of D. The simplest version of our argument provides an integral test (Theorem (1.3)) that distinguishes between these two alternatives.

In reality what we will prove is a generalisation of this test. We will consider a domain $D \subset \mathbf{C}$ whose boundary is contained in the union of n rays leaving the origin. We will call such domains n-sectorial. Ancona's result now shows that the origin corresponds to at most n minimal points of the Martin boundary of D. In the case that the base angles of the sectors are all distinct, we will give a test (Theorem (3.6)) that determines whether or not there are exactly n such points. This provides a geometrically simple class of domains, whose Martin boundaries are non-trivial, yet for which explicit calculations can be made. We leave open the question of determining the number of minimal points when this number is less than n.

A modification of the test allows us to determine whether Brownian motion can

(1) Research supported in part by a grant from NSA

(2) Research supported in part by a grant from NSERC

be conditioned to travel between the various Martin boundary points corresponding to the origin (see Section 4). The remainder of this section will describe the application of Theorem (3.6) in the simpler context of the first paragraph. In combination with earlier integral tests, due to Benedicks and Chevallier, what emerges is a complete identification of which sectors a conditioned Brownian motion will visit.

In the basic integral test, the argument for sufficiency uses balayage. That for necessity relies on adaptions of arguments due to Chevallier [Ch] and Benedicks [Be].

Set

$$\begin{aligned} W(\phi,\psi) &= \{z\in \mathbf{C};\ \phi<\operatorname{Arg}(z)<\psi\},\\ V(\phi) &= \{re^{i\phi};\ r\ge 0\},\\ B(z,r) &= \{w\in\mathbf{C};\ |w-z|\le r\},\\ B(r) &= B(0,r). \end{aligned}$$

We start by recording the following simple fact.

(1.1) Lemma. *Let $0<\alpha<2\pi$ and set $B=\pi/\alpha$. Suppose that h is positive and harmonic in the sector $C=W(0,\alpha)$. Assume that h is bounded and continuous on the closure of C less any neighbourhood of the origin. Then*

$$h(re^{i\alpha/2}) = ar^{-B} + \alpha^{-1}\int_0^\infty \frac{r^B t^{B-1}}{r^{2B}+t^{2B}}\left[h(t)+h(te^{i\alpha})\right] dt$$

for some $a\ge 0$. Moreover, $r^B h(re^{i\alpha/2})\to a$ as $r\downarrow 0$.

Proof. This follows by a conformal transformation from the case $\alpha=\pi$ (Herglotz's theorem). For the asymptotic statement (as $r\downarrow 0$) note that $r^B h(re^{i\alpha/2})$ is the sum of a and an integral whose integrand decreases to 0. □

Let $D\subset\mathbf{C}$ be open and connected. Write $\mathcal{M}(D)$ for the set of all positive functions $D\to\mathbf{R}^+$ which are harmonic in D and bounded on D less some neighbourhood of the origin, and which converge to 0 at each regular point $z\in\partial D$, $z\ne 0$. Fix $z_0\in D$ and let $\mathcal{M}_1(D)=\{h\in\mathcal{M}(D); h(z_0)=1\}$. To each minimal element h of $\mathcal{M}_1(D)$ there corresponds a unique point z of the minimal Martin boundary of D, which we call the pole of h. h will be a multiple of the Martin function $K(z,\cdot)$.

Now let $0<\alpha\le\pi$, and take $C=W(0,\alpha)$. Suppose that the complement of D is a subset of ∂C. Let $\gamma_1=\alpha/2$, $\gamma_0=(\alpha+2\pi)/2$, $B_1=\pi/\alpha\ge 1$, and $B_0=\pi/(2\pi-\alpha)\le 1$. According to Proposition (3.1) and Theorem (3.6) there are at most two minimal elements of $\mathcal{M}_1(D)$, and there will be exactly two, if and only if

(1.2) $$\exists h_j\in\mathcal{M}(D)\quad\text{such that }\lim_{r\downarrow 0} r^{B_j}h_j(re^{i\gamma_j})>0,$$

for $j=0$ and $j=1$. Actually, Theorem (3.6) only applies when $\alpha<\pi$, but the case $\alpha=\pi$ is (after a conformal transformation) the one considered by Benedicks. By Proposition (2.9), the probabilistic interpretation of (1.2) is that

(i) any h_j-transform X_t of Brownian motion will a.s. approach 0 as t increases to the lifetime ζ of X, and that

(ii) for $j=0$ (resp. $j=1$) X_t will a.s. remain in C^c (resp. C) during some terminal interval $(\zeta-\delta,\zeta)$ of time.

In analytic terms, (ii) becomes that C (resp. C^c) is minimal-thin at the minimal Martin boundary point which is the pole of h_0 (resp. h_1). See [Do] for definitions of and information about h-transforms, Martin boundaries, and minimal-thinness. A shorter introduction to h-transforms may be found in [Du].

We wish to obtain an integral test, relating "local" information about ∂D to the existence of harmonic functions h_j as above. In the case of a Denjoy domain (that is, in the case $\alpha=\pi$), a conformal transformation reduces this test to one given by Benedicks (Theorem 4 of [Be]). Indeed, our argument for necessity will use that of Benedicks. Ancona [A2] obtained Green function estimates for a significantly more general class of domains. Based on these estimates, Chevallier [Ch] considered the case of general α and obtained an integral test, essentially for the existence of h_0. We will use balayage to give a test for the existence of h_1. In Section 3 we will give such a test in a more general setting, and will adapt arguments from [Ch] to relate such existence questions to the enumeration of the minimal elements of $\mathcal{M}_1(D)$. Stephen Gardiner has pointed out to us that the balayage portion of our argument is closely related to one he had earlier used in [Gd] to give a more concrete version of Benedicks's test. For related results on Martin boundaries or Brownian motion, see [A1], [A3], [AZ], [Bi1], [Bi2], [Bu], [MP], [Se1] and [Se2]. Other papers on Denjoy domains include [De], [Ca1], [Ca2], [GnJ], [RR], and [Z].

Let $0<2\varepsilon<\alpha\leq\pi$, $\varepsilon<1$, and define $G(re^{i\theta})$ to be the neighbourhood

$$G(re^{i\phi})=\{z;\ |z|\in\bigl(r(1-\varepsilon),r(1+\varepsilon)\bigr)\}\cap W(\phi-\varepsilon,\phi+\varepsilon)$$

of $re^{i\theta}$ (we are suppressing the dependence on ε). Let $g(w)=g(D,z;w)$ solve

$$\begin{cases} \Delta g=0 & \text{on } D\cap G(z),\\ g=0 & \text{on } \partial D\cap G(z),\\ g=1 & \text{on } D\cap\partial G(z),\end{cases}$$

and set $\eta(D,z)=g(D,z;z)$. Write $\varrho(r)=\eta(D,r)+\eta(D,re^{i\alpha})$. Combining the results of Section 3 with Chevallier's integral test yields the following classification theorem.

(1.3) Theorem. *Let $0<\alpha\leq\pi$, $C=W(0,\alpha)$, and let D be a domain with $D^c\subset\partial C$. Then*

(a) *If $\int_0^1 \varrho(t)t^{-1}\,dt=\infty$ then there is only one element h of $\mathcal{M}_1(D)$, and any h-transform a.s. visits the sector C together with its complement in every terminal interval $(\zeta-\delta,\zeta)$ of time. That is, neither C nor its complement is minimal-thin at the pole of h.*

(b) *If $\int_0^1 \varrho(t)t^{-1}\,dt<\infty$ but $\int_0^1 \varrho(t)t^{B_0-B_1-1}\,dt=\infty$ then there is only one element h of $\mathcal{M}_1(D)$, and any h-transform a.s. remains in the complement of the sector C during some entire terminal interval $(\zeta-\delta,\zeta)$ of time. That is, C is minimal-thin at the pole of h.*

(c) *If $\int_0^1 \varrho(t)t^{B_0-B_1-1}\,dt<\infty$ then there are exactly two minimal elements h_0 and h_1 of $\mathcal{M}_1(D)$. Any transform by h_0 (resp. h_1) a.s. remains in C^c (resp. C) during some entire terminal interval $(\zeta-\delta,\zeta)$ of time. That is, C (resp. C^c) is minimal-thin at the pole of h_0 (resp. h_1).*

Proof. The three cases are exhaustive, as $B_0-B_1\leq 0$. As remarked above, the case $\alpha=\pi$ is Benedicks's criterion, so we may assume that $\alpha<\pi$. Part (c) now follows by Theorem (3.6).

In the remaining cases, $\mathcal{M}_1(D)$ consists of a single function h. By (2-1) of [Ch], condition (1.2) fails for $j=1$ (the smaller angle). By Proposition (2.9), C^c is not minimal-thin at the pole of h. Moreover, C is minimal-thin at this pole if and only if (1.2) holds for $j=0$. By (2-5) of [Ch] this holds provided $\int_0^1 \varrho(t)t^{-1}dt<\infty$. The converse follows by Proposition (2.8). □

Note that when $\alpha<\pi$, symmetry considerations no longer rule out the possibility that (a) holds yet that $W(\delta,\alpha-\delta)$ is minimal-thin at the pole of h, for $\delta>0$. Chris Burdzy showed us an example in which this is the case, and we have since found an integral test that characterizes this behaviour.

2. Preliminary estimates

We take the convention that numbered constants $(c_1, c_2, \ldots)$ have specific values, usually depending on one or more parameters, but that c represents a constant whose value may change from line to line.

(2.1) Lemma. *Let $\varepsilon<\pi/2$ and suppose that $D\subset C=W(-2\varepsilon,2\varepsilon)$, with $\partial D\cap C\subset V(0)$. There is a constant c_1 (depending on ε but not otherwise on D) such that if h is harmonic in D and converges to 0 at each regular point $z\in V(0)\cap\partial D$, $z\neq 0$, then*

$$h(r)\leq c_1\eta(D,r)[h(re^{i\varepsilon})+h(re^{-i\varepsilon})],\quad \forall r\in D\cap V(0).$$

Proof. This follows immediately from the maximum principle and a weak Harnack inequality such as Proposition (1-1) of [Ch]. See also (4.8) of [Be] or (2.3) and (3.3) of [A2]. □

The simple form of the above result will suffice for most purposes. We will occasionally need the following stronger form, proved as in (1-2) of [Ch]. Write $A(r,r')=\{se^{i\theta};\ r<s<r'\}$.

(2.2) Lemma. *Let $\varepsilon<\pi/2$ and suppose that $U\subset C=A(r/4,4r)\cap W(-2\varepsilon,2\varepsilon)$, with $\partial U\cap C\subset V(0)$. There is a constant c (depending on ε but not otherwise on r or U) such that if h, u, and v are strictly positive and harmonic in U and converge to 0 at each nonzero regular $z\in V(0)\cap\partial U$, then*

$$h(z)\le c\left[\frac{h(re^{i\varepsilon})}{u(re^{i\varepsilon})}u(z)+\frac{h(re^{-i\varepsilon})}{v(re^{-i\varepsilon})}v(z)\right]$$

for $z\in A(r/2,2r)\cap W(-\varepsilon,\varepsilon)$.

For D open and u a positive measurable function defined on ∂D, define $H^D u$ to be the solution to the Dirichlet problem on D, with boundary data u. That is, $H^D u(z)$ is the integral of u with respect to harmonic measure μ_z.

Now let $\phi>0$, $\psi>0$, $\phi+\psi\le 2\pi$, and write $\Phi=\pi/\phi$, $\Psi=\pi/\psi$, $C=W(0,\phi)$. In the next result, we will be interested in domains D satisfying

$$\begin{gathered} D\subset W(-\psi,\phi),\\ W(-\psi,\phi)\backslash D\subset V(0),\\ V(0)\backslash B(1)\subset\partial D. \end{gathered}\tag{2.3}$$

For $q>0$ let $u_q(re^{i\phi})=r^{-q}$, and let $u_q=0$ on $\partial D\backslash V(\phi)$. For $x\in\mathbf{R}$ write

$$\Gamma(D,x)=\int_0^1\eta(D,t)t^{(x\wedge 0)-1}\,dt.$$

The following is our principal technical lemma. It establishes the estimates that allow us to carry out the balayage, at least when Γ is sufficiently small.

(2.4) Lemma. *Let ϕ and ψ be as above. Let $0<q<\Phi$. There are constants $c_2,c_3>0$ (depending only on q, ϕ, and ψ) such that if D satisfies (2.3), $0<s\le\Psi$, and $\Gamma(D,s-q)<c_2$ then*

$$H^D u_q(re^{-i\psi/2})\le c_3\Gamma(D,s-q)r^{-s},\quad\forall r>0,\tag{a}$$

$$(H^D-H^C)u_q(re^{i\phi/2})\le c_3\Gamma(D,s-q)r^{-q},\quad\forall r>0.\tag{b}$$

Remark. If $s>q$ then as $s\wedge q=q\wedge q$, the result applies both with s as given, and with s replaced by q. Therefore in this case, in fact

$$H^D u_q(re^{-i\psi/2})\le c_3\Gamma(D,0)[r^{-s}\wedge r^{-q}],\quad \forall r>0.$$

Proof. (a) We have not yet ruled out the possibility that $h=H^D u_q\equiv\infty$, so let $u_q^N=u_q\wedge N$. This makes each $h_N=H^D u_q^N$ bounded, with $h_N\uparrow h$. Let

$$m_N^+=\sup\{r^q h_N(re^{i\phi/2});\ 0<r<N\}<\infty$$
$$m_N^-=\sup\{r^s h_N(re^{-i\psi/2});\ 0<r<N\}<\infty.$$

Since $s\le\Psi$, we have by (2.3) and Lemma (1.1) that

$$\begin{aligned} r^s h_N(re^{-i\psi/2}) &= \psi^{-1}\int_0^1 \frac{r^{\Psi+s}t^{\Psi-1}}{r^{2\Psi}+t^{2\Psi}}h_N(t)\,dt \\ &= \psi^{-1}\int_0^1 \frac{(r/t)^{\Psi+s}}{1+(r/t)^{2\Psi}}t^{s-1}h_N(t)\,dt\le\psi^{-1}\int_0^1 t^{s-1}h_N(t)\,dt. \end{aligned}$$

Now choose $\varepsilon<(\phi+\psi)/2$. By Lemma (2.1) and Harnack's inequality, it follows that the above expression is

$$\begin{aligned} &\le\psi^{-1}c_1\int_0^1 t^{s-1}\eta(D,t)[h_N(te^{-i\varepsilon})+h_N(te^{i\varepsilon})]\,dt \\ &\le c\int_0^1 t^{s-1}\eta(D,t)[h_N(te^{-i\psi/2})+h_N(te^{i\phi/2})]\,dt \\ &\le c\int_0^1 t^{s-1}\eta(D,t)[t^{-s}m_N^-+t^{-q}m_N^+]\,dt \\ &= cm_N^-\int_0^1 t^{-1}\eta(D,t)\,dt+cm_N^+\int_0^1 t^{s-q-1}\eta(D,t)\,dt \\ &\le c\Gamma(D,s-q)[m_N^-+m_N^+]. \end{aligned}$$

Thus

$$m_N^-\le c\Gamma(D,s-q)[m_N^-+m_N^+]. \tag{2.5}$$

Now assume that $s\le q$. A change of variables, from t to $x=t/r$ shows that

$$\begin{aligned} \int_0^\infty \frac{r^{\Phi+q}t^{\Phi-1}}{r^{2\Phi}+t^{2\Phi}}u_q(t)\,dt &= \int_0^\infty \frac{x^{\Phi-q-1}}{1+x^{2\Phi}}\,dx \\ &\le\int_0^1 x^{\Phi-q-1}\,dx+\int_1^\infty x^{-\Phi-q-1}\,dx \\ &=(\Phi-q)^{-1}+(\Phi+q)^{-1}\le 2(\Phi-q)^{-1}. \end{aligned}$$

Thus, proceeding as before,

$$\begin{aligned} r^q h_N(re^{i\phi/2}) &= \phi^{-1}\int_0^1 \frac{r^{\Phi+q}t^{\Phi-1}}{r^{2\Phi}+t^{2\Phi}} h_N(t)\,dt + \phi^{-1}\int_0^\infty \frac{r^{\Phi+q}t^{\Phi-1}}{r^{2\Phi}+t^{2\Phi}} u_q(t)\,dt \\ &\le \phi^{-1}\int_0^1 t^{q-1}h_N(t)\,dt + 2\phi^{-1}(\Phi-q)^{-1} \\ &\le cm_N^+\int_0^1 t^{-1}\eta(D,t)\,dt + cm_N^-\int_0^1 t^{q-s-1}\eta(D,t)\,dt + c \\ &\le cm_N^+\int_0^1 t^{-1}\eta(D,t)\,dt + cm_N^-\int_0^1 t^{-1}\eta(D,t)\,dt + c \\ &\le c[1+\Gamma(D,s-q)(m_N^-+m_N^+)], \end{aligned}$$

here using that $q-s\ge 0$. Thus

$$m_N^+ \le c\big[1+\Gamma(D,s-q)(m_N^-+m_N^+)\big]. \tag{2.6}$$

Now choose c_2 such that for the common c of (2.5) and (2.6) we have $cc_2\le\frac{1}{3}$. Suppose that $\Gamma=\Gamma(D,s-q)\le c_2$. Then $m_N^-\le\frac{3}{2}c\Gamma m_N^+$ holds by (2.5), and substituting into (2.6) gives that $m_N^+\le c+\frac{1}{2}m_N^+$. Thus $m_N^+\le 2c$, so that $m_N^-\le c_3$, here choosing $c_3=3c^2$. Letting $N\to\infty$ now gives part (a) of the lemma, at least when $s\le q$.

Now suppose that $s>q$. Then (2.5) holds, so again $m_N^-\le\frac{3}{2}c\Gamma m_N^+$ holds if $cc_2\le\frac{1}{3}$. As before, the result will follow once we show that $m_N^+\le 2c$. But this follows from what we have already proved, as the hypotheses of the lemma still hold if we decrease s and replace it by q. Thus part (a) is proved in general.

(b) The second part of the lemma follows immediately. In fact, we have that

$$(H^D-H^C)u_q(re^{i\phi/2}) = \phi^{-1}r^{-q}\int_0^1 \frac{r^{\Phi+q}t^{\Phi-1}}{r^{2\Phi}+t^{2\Phi}} h_N(t)\,dt.$$

Our work above provides the upper bound $cr^{-q}\Gamma[m^+ + m^-]$ (where m^+ (resp. m^-) is $\lim_{N\to\infty} m_N^+$ (resp. m_N^-)), which in turn gives part (b), after perhaps increasing the value of c_3. □

Remark. In our applying this result, it would in fact be enough to assume that $\phi=\psi$, and to replace $x\wedge 0$ by $-|x|$ in the definition of $\Gamma(D,x)$. Our weaker hypotheses cause no significant complications however, and help clarify the asymmetric roles of s and q.

(2.7) Lemma. *Let ϕ,ψ,Φ,Ψ, and Γ be as above, and let D satisfy* (2.3). *Take $C=W(0,\phi)$. Let $u=0$ on $V(\phi)$, and suppose that $u(r)\le\eta(D,r)[r^{-\Phi}+r^{-\Psi}]$ on $V(0)$. Then*

$$H^C u(re^{i\phi/2}) \le 2\phi^{-1}r^{-\Phi}\Gamma(D,\Phi-\Psi).$$

Proof.

$$\begin{aligned}
r^{\Phi} H^C u(re^{i\phi/2}) &= \phi^{-1} \int_0^1 \frac{r^{2\Phi} t^{\Phi-1}}{r^{2\Phi}+t^{2\Phi}} u(t)\, dt \\
&\le \phi^{-1} \int_0^1 t^{\Phi-1} \eta(D,t)[t^{-\Phi}+t^{-\Psi}]\, dt \\
&\le 2\phi^{-1}\Gamma(D, \Phi-\Psi). \quad \square
\end{aligned}$$

The following asserts that finiteness of certain integrals is a necessary condition for there to exist harmonic functions with given growth rates.

(2.8) Proposition. *Let ϕ, ψ, Φ, Ψ, and Γ be as above. Suppose that D is a domain containing $W(-\psi, 0) \cup W(0, \phi)$, and that $h \in \mathcal{M}(D)$ satisfies*

$$\lim_{r \downarrow 0} r^{\Phi} h(r \exp(i\phi/2)) > 0.$$

Then $\Gamma(D, \Psi-\Phi) < \infty$.

Proof. Suppose first that $\phi \le \psi$ so that $\Psi - \Phi \le 0$. Then by Lemma (1.1),

$$\begin{aligned}
h(e^{-i\psi/2}) &\ge \psi^{-1} \int_0^\infty \frac{t^{\Psi-1}}{1+t^{2\Psi}} h(t)\, dt \\
&\ge \psi^{-1} \int_0^1 t^{\Psi-1} h(t)\, dt.
\end{aligned}$$

Because h is harmonic on $G(t)$, it follows from the maximum principle and Harnack's inequality that this is

$$\begin{aligned}
&\ge c \int_0^1 t^{\Psi-1} \eta(D,t) h(te^{i\phi/2})\, dt \\
&\ge c \int_0^1 \eta(D,t) t^{\Psi-\Phi-1}\, dt,
\end{aligned}$$

the latter by our hypothesis on h. Thus Γ must be finite.

If $\phi \ge \psi$, then by a similar argument,

$$\begin{aligned}
h(e^{i\phi/2}) &\ge \phi^{-1} \int_0^1 t^{\Phi-1} h(t)\, dt \\
&\ge c \int_0^1 t^{\Phi-1} \eta(D,t) h(te^{i\phi/2})\, dt \\
&\ge c \int_0^1 \eta(D,t) t^{-1}\, dt.
\end{aligned}$$

Thus Γ is finite in this case as well. $\square$

Finally, we record the relationship between minimal-thinness and the existence of harmonic functions having prescribed rates of growth.

(2.9) Proposition. *Let D be a domain containing $C=W(0,\alpha)$. Set $\gamma=\alpha/2$ and $B=\pi/\alpha$. The following two conditions are equivalent*

(a) *There is a point ξ of the minimal Martin boundary of D that is a minimal fine limit point of $W(0,\alpha)\cap B(\delta)$ for every $\delta>0$, and at which C^c is minimal-thin.*

(b) $\exists h\in\mathcal{M}(D)$ *s.t.* $\lim_{r\downarrow 0} r^B h(r\exp(i\gamma))>0$.

Moreover, in this case, ξ is unique.

Proof. Assume (a) and let h be the harmonic function with pole at ξ. Then an h-transform converges to the vertex of C and stays entirely within C with positive probability. The same is therefore true for a Brownian motion on C, transformed by the restriction of h to C. Lemma (1.1) now implies that $h(re^{i\gamma})\sim cr^{-B}$.

Conversely, suppose (b) holds. Then by Lemma (1.1), an h-transform X_t of Brownian motion has positive probability of reaching the vertex of C before leaving C. If h is not minimal, we may still find a minimal function for which this property holds. Let ξ be its pole. Then (a) holds for this ξ.

Moreover, the law of X, restricted to the tail σ-field, must be the same as that of a transform in C by some element of $\mathcal{M}_1(C)$. Since this set is a singleton, we conclude that ξ is unique. □

3. Sectorial domains

We now turn to the class of domains for which our principal theorem will be stated. Let $0=\alpha_0<\alpha_1<\ldots<\alpha_n=2\pi$ where $n\geq 2$, and let D be an open subset of $\mathbf{C}$ for which $D^c\subset V(\alpha_1)\cup\ldots\cup V(\alpha_n)$. We say that D is n-sectorial. This hypothesis and notation will be in effect for the remainder of the paper. It will be convenient to write $\alpha_{n+1}=\alpha_1+2\pi$ and $\alpha_{-1}=\alpha_{n-1}-2\pi$. Set $\beta_j=\alpha_j-\alpha_{j-1}$, $B_j=\pi/\beta_j$, and $\gamma_j=\alpha_j-(\beta_j/2)$.

(3.1) Proposition. *Let $D\subset\mathbf{C}$ be an n-sectorial domain.*

(a) *$\mathcal{M}_1(D)$ has at most n minimal elements. The pole of each such element is a minimal fine limit point of the set $B(r)\cap\bigcup_{j=1}^n V(\gamma_j)$, for every $r>0$.*

(b) *If $\mathcal{M}_1(D)$ has exactly n minimal elements, then their poles are exactly the points $\xi_j=\lim_{r\downarrow 0} r\exp(i\gamma_j)$. In this case, ξ_j is a minimal fine limit point of $V(\gamma_j)\cap B(\delta)$, for every $\delta>0$.*

Proof. The first assertion in (a) is (3.3) of [A2]. To show the remainder of the proposition, we adapt an argument of Chevallier. Lemma (2.2) allows us to estimate $h(re^{i\theta})$ for $|\theta-\gamma_j|<\varepsilon$, where $j=1,\ldots,n$. Combining these estimates with Harnack's inequality yields a constant c such that if $h,v_1,\ldots,v_n$ are strictly positive and harmonic in $D\cap A(r/4,4r)$ then

$$h(z)\le c\sum_{j=1}^{n}\frac{h(re^{i\gamma_j})}{v_j(re^{i\gamma_j})}v_j(z),\quad \forall z\in D\cap A(r/2,2r).$$

Now the argument of (1-3) of [Ch] applies, and shows that if $r_k\downarrow 0$ and if δ is sufficiently small, then

$$\bigcup_{k=1}^{\infty}\bigcup_{j=1}^{n}B\big(r_ke^{i\gamma_j},\delta r_k\big) \tag{3.2}$$

is not minimal-thin at the pole of any minimal element of $\mathcal{M}_1$. In particular, the second statement of (a) is now immediate.

For the first statement of (b), we adapt the argument of (1-4) of [Ch]. Notice that with n now possibly bigger than 2, it would be false to conclude that $r\exp(i\gamma_j)$ converges to a minimal point of the Martin boundary as $r\downarrow 0$. Let the n minimal elements of $\mathcal{M}_1(D)$ be $h_1,\dots,h_n$ with $h_j=K(\xi_j,\cdot)$. Choose $r_k\downarrow 0$, and write $z_k=r_k\exp(i\gamma_n)$. Suppose that the Martin function $K(z_k,\cdot)$ converges to some $h\neq h_1,\dots,h_{n-1}$. The set in (3.2) is not minimal-thin at any ξ_j for $j\neq n$, so we may choose a subsequence of r_k (which we assume to be the original sequence), together with indices $J(1),\dots,J(n-1)$, so that $r_k\exp(i\gamma_{J(j)})\to\xi_j$ as $k\to\infty$, $j=1,\dots,n-1$. Because the set in (3.2) is not minimal-thin at ξ_n, it follows that h is a multiple of h_n. Thus no non-minimal point can be a limit point of $r\exp(i\gamma_n)$ as $r\downarrow 0$. Being the intersection of a decreasing sequence of compact connected sets, the set of such limit points is also connected, so consists of the singleton ξ_n. This shows the first assertion of (b).

The second follows by Harnack's inequality, as it is now clear that

$$\bigcup_{k=1}^{\infty}B\big(r_ke^{i\gamma_j},\delta r_k\big)$$

is not minimal-thin at ξ_j. $\square$

Now choose $\varepsilon<(1\wedge\beta_1\wedge\dots\wedge\beta_n)/4$ and set $\eta_j(r)=\eta(D,re^{i\alpha_j})$. The integral criterion we will consider is that

$$\Gamma_j=\int_0^1\eta_j(t)t^{-|B_{j+1}-B_j|-1}dt<\infty, \tag{3.3}$$

for $j=1,\dots,n$. Note that by Harnack's inequality, the validity of this condition does not depend on the choice of the small parameter ε.

(3.4) Corollary. *Let $D \subset \mathbf{C}$ be an n-sectorial domain, and suppose that*

$$\exists h_j \in \mathcal{M} \quad \textit{such that} \ \lim_{r \downarrow 0} r^{B_j} h_j(re^{i\gamma_j}) > 0, \tag{3.5}$$

for $j=1,\dots,n$. Then $\mathcal{M}_1(D)$ has n minimal elements with poles $\xi_1,\dots,\xi_n$. Each $W(\alpha_{j-1},\alpha_j)^c$ is minimal-thin at ξ_j, and (3.3) *holds for each j.*

Proof. This follows immediately from Proposition (2.8), (2.9), and (3.1). □

We are ready for the main result of this section.

(3.6) Theorem. *Let $D \subset \mathbf{C}$ be an n-sectorial domain. Assume that the β_j are all distinct. For $\mathcal{M}_1(D)$ to have exactly n minimal elements, it is necessary and sufficient that* (3.3) *holds for $j=1,\dots,n$. Moreover* (3.5) *holds in this case, so that $W(\alpha_{j-1},\alpha_j)^c$ is minimal-thin at ξ_j.*

Remarks. (1) Even if the β_j are not all distinct, the argument will show that (3.3) and (3.5) are equivalent, in the sense that if one holds for every j, then so does the other. Thus (3.3) is in general sufficient for the existence of n minimal elements of $\mathcal{M}_1(D)$.

(2) Using similar arguments, it is possible to weaken the hypothesis that the β_j be distinct. This condition will only be used in the proof of Lemma (3.3), and there it could be dispensed with provided either $n=2$ or $V(0) \subset D^c$. In the case where $n=3$ or $n=4$, this makes it possible to prove the theorem provided there are not two adjacent sectors whose base angles β are equal and are at least as big as the remaining β's.

(3) We'd like to thank Chris Burdzy for pointing out a shortcoming in the original proof.

(3.7) Lemma. *Let $D \subset \mathbf{C}$ be an n-sectorial domain, for which $\mathcal{M}_1(D)$ has exactly n minimal elements. Assume that the β_j are all distinct, and suppose that β_1 is the biggest of the β_j. Then $W(\alpha_0,\alpha_1)$ is minimal-thin at each ξ_j, $j \neq 1$.*

Proof. If $\alpha_1 \geq \pi$, this is just the statement following (2-3) of [Ch]. Since this is always the case if $n=2$, we may assume that $n \geq 3$ and $\alpha_1 < \pi$. It will suffice to show minimal-thinness at ξ_2, as the same argument will apply to ξ_n, and the general case will then follow.

Choose $\delta > 0$ so that $\beta_2 + \delta < \alpha_1$. This is possible since $\beta_1 > \beta_2$. Let us define $D' = D \cap W(\alpha_0, \alpha_2 + \delta)$. Because $re^{i\gamma_1} \to \xi_1 \neq \xi_2$ as $r \downarrow 0$, it follows that $V(\gamma_1)$ is minimal-thin at ξ_2. The same is true for $V(\alpha_2+\delta)$. Thus the complement of D' is minimal-thin at ξ_2. In particular, the limit ξ_2' of $r\exp(i\gamma_2)$ exists in the Martin topology of D', and by localization, it will suffice to show that $W(\alpha_0,\alpha_1)$ is minimal-thin at ξ_2', relative to D'.

Let D'' be the image of D' under the conformal transformation $g(z) = z^{\pi/\alpha_1}$. The map g is one-to-one on D', by our choice of δ. The sector $g(W(\alpha_0, \alpha_1))$ now has base angle equal to π. By Chevallier's remark, it is minimal-thin at $g(\xi_2')$ relative to D''. We obtain the desired conclusion by conformal invariance. □

Proof of theorem (necessity). Fix n and let ξ_j be as in Proposition (3.1). For ease of exposition, we will abandon the assumption that D is connected. In other words, D will be open but it need not be a domain. This necessitates several trivial changes, for example to our definition of $\mathcal{M}_1(D)$. Write h_j for the minimal element of $\mathcal{M}_1(D)$ with pole at ξ_j. Our proof will proceed by induction on $n-m$, where m is the number of connected components of D. The inductive hypothesis is that if D is n-sectorial with more than m or more components, and if $\mathcal{M}_1(D)$ has n minimal elements, then

$$\lim_{r \downarrow 0} r^{B_j} h_j(re^{i\gamma_j}) > 0 \tag{3.8}$$

holds for each j. This is trivial if $m = n$. We assume, without loss of generality, that β_1 is the largest of the β_j, and that $W(\alpha_0, \alpha_1)$ is not itself a component of D.

Let $D' = D \setminus [V(\alpha_0) \cup V(\alpha_1)]$. By Lemma (3.7) $\mathcal{M}_1(D')$ has the same number of minimal elements as $\mathcal{M}_1(D)$, yet it has at least one more connected component. By induction, (3.8) holds for $j = 2, ..., n$. Moreover, by Proposition (2.8), we have that (3.3) holds for each j.

Finally, we must show that (3.8) holds for $j = 1$. By (3.3) we have that

$$\int_0^1 t^{-1}[\eta_1(t) + \eta_0(t)]dt < \infty.$$

If $\alpha_1 \geq \pi$, then (3.8) follows immediately as in the proof of part 2 of Proposition (2-5) of [Ch]. If not, then restrict h_1 to $D' = D \cap W(-\delta, \alpha_1 + \delta)$ for some small δ, and apply a conformal mapping as in Lemma (3.7) to reduce to this case. Alternatively, once the sufficiency half of the theorem has been established, it could be used to reach the same conclusion. □

In order to show the sufficiency of (3.3), we will need to construct the harmonic functions appearing in (3.5). Without loss of generality, the n functions of (3.5) may be taken to be equal (if not, consider their sum), and it is this function that we will construct. We will break the argument up into three parts: showing monotonicity of the balayage; estimating its successive terms, at least when D has a special form; and approximating a general domain D by domains of this form.

Let $Y(j) = D \cap W(\alpha_j - 2\varepsilon, \alpha_j + 2\varepsilon)$, $Z(j) = W(\alpha_{j-1}, \alpha_j)$, and

$$Y = Y(1) \cup ... \cup Y(n),$$
$$Z = Z(1) \cup ... \cup Z(n).$$

Let $f_{-1}=f_{-2}=0$ and

$$f_0(re^{i\theta})=r^{-B_j}\sin(B_j(\theta-\alpha_{j-1})),\quad \text{if } \alpha_{j-1}\le\theta\le\alpha_j\,.$$

Given f_{2k}, define

$$f_{2k+1}=\begin{cases} f_{2k}, & \text{on } D\setminus Y,\\ H^Y f_{2k}, & \text{on } Y,\end{cases}$$

$$f_{2k+2}=\begin{cases} f_{2k+1}, & \text{on } D\setminus Z,\\ f_0+H^Z f_{2k+1}, & \text{on } Z.\end{cases}$$

Where needed, we take the f's to be zero on $\partial Y\setminus D$ and $\partial Z\setminus D$.

(3.9) Lemma. *Let D be an n-sectorial domain. Then the functions f_k increase with k. Let h be their limit. Then either $h\equiv\infty$ or $h\in\mathcal{M}(D)$.*

Proof. In order to eliminate problems with infinities of the f's, we should first truncate f_0 at some large value N, prove the result in this case, and then recover the general case by letting $N\to\infty$. We leave this to the reader, and instead simply proceed as if each f were bounded, and hence (by induction) continuous.

Let $C=W(0,2\varepsilon)$, and suppose that $f_0\le f_1\le\ldots\le f_{2k}$. Since $f_{2k+1}=H^Y f_{2k}$ on Y, also $f_{2k+1}=H^Y f_{2k+1}$ on Y. Hence $f_{2k+1}=H^C f_{2k+1}$ on C. Being finite,

$$f_{2k}(r\exp(i\gamma_1))=O(r^{-B_1})=o(r^{-\pi/2\varepsilon}).$$

It is harmonic on C, so by Lemma (1.1), also $f_{2k}=H^C f_{2k}$ on C. By induction, $f_{2k+1}=H^Y f_{2k}\ge H^Y f_{2k-2}=f_{2k-1}$ on Y. In particular, $f_{2k+1}\ge f_{2k-1}=f_{2k}$ on $V(0)$. By definition $f_{2k+1}=f_{2k}$ on $V(2\varepsilon)$. Thus $f_{2k+1}\ge f_{2k}$ on ∂C, and so on C as well. The same is true on $W(-2\varepsilon,0)$ and hence on all of $Y(0)$. Arguing similarly, this is also true on each $Y(j)$. But $f_{2k+1}=f_{2k}$ on $D\setminus Y$, so in fact $f_{2k+1}\ge f_{2k}$ on all of D.

Almost the same argument works when we consider f_{2k+2}. As above, on C both $f_{2k+2}=H^C f_{2k+2}$ and $f_{2k+1}=H^C f_{2k+1}$. Moreover,

$$f_{2k+2}=f_0+H^Z f_{2k+1}\ge f_0+H^Z f_{2k-1}=f_{2k}$$

on Z, so $f_{2k+2}\ge f_{2k}=f_{2k+1}$ on $V(2\varepsilon)$. Since $f_{2k+2}=f_{2k+1}$ on $V(0)$, in turn $f_{2k+2}\ge f_{2k+1}$ on C. Arguing similarly, the same is true on Y. But $f_{2k+2}\ge f_{2k}=f_{2k+1}$ on $D\setminus Y$, so in fact $f_{2k+2}\ge f_{2k+1}$ on all of D.

This shows that the f's are increasing. Let h be their limit, and assume it is not identically infinite. Because it is a monotone limit of functions harmonic on Y, it also is harmonic on Y. Similarly it is harmonic on Z, and hence on D as well.

To see that it belongs to $\mathcal{M}(D)$, apply Lemma (2.1) to f_{2k+1}. This gives an inequality

$$f_{2k+1}(t)\le c\eta(Y(0),t)\left[f_{2k+1}(te^{i\varepsilon})+f_{2k+1}(te^{-i\varepsilon})\right].$$

Letting $k\to\infty$ produces the same inequality, but for h. It now follows that h vanishes at regular points $z\neq 0$ of ∂D. $\square$

We will show that $h\not\equiv\infty$ under an additional condition on D, imposed to make Lemma (2.4) apply

(3.10) Lemma. *There is a constant $c_4>0$ such that if $\lambda<1$ and if D is an n-sectorial domain and satisfies*

$$V(\alpha_j)\setminus B(1)\subset\partial D \quad \textit{and} \quad \Gamma_j\le c_4\lambda; \quad j=1,\dots,n \tag{3.11}$$

then

$$(f_{2k+2}-f_{2k})(re^{i\gamma_j})\le\lambda^k r^{-B_j}; \quad k=1,2,\dots,\ j=1,\dots,n.$$

Proof. Let the constants c_2 and c_3 of Lemma (2.4) be chosen to work for $\phi=\psi=2\varepsilon$, and for q any of the B_j. By Harnack, there is a constant c_5 such that if f is harmonic on $Z(j)$ and r is such that $f(re^{i\gamma_j})\le 1$, then $f(re^{i(\alpha_j-2\varepsilon)})$ and $f(re^{i(\alpha_{j-1}+2\varepsilon)})$ are both $\le c_5$. Also suppose that (3.11) holds for some c_4 to be chosen later.

Assume that the $f_{k'}$ are all finite, for $k'\le 2k$, and that

$$(f_{2k}-f_{2k-2})(re^{i\gamma_j})\le\lambda^k r^{-B_j} \quad \forall j.$$

Then

$$f_{2k+1}=f_{2k-1}+H^{Y(0)}(f_{2k}-f_{2k-2}),$$

on $Y(0)$, so in particular f_{2k+1} is finite. Moreover, we can break this up as

$$f_{2k+1}-f_{2k-1}=H^{Y(0)}\big((f_{2k}-f_{2k-2})1_{V(-2\varepsilon)}\big)+H^{Y(0)}\big((f_{2k}-f_{2k-2})1_{V(2\varepsilon)}\big).$$

Let $C=W(0,2\varepsilon)$. Applying Lemma (2.4) with $\phi=\psi=2\varepsilon$ gives that

$$\begin{aligned}(f_{2k+1}-f_{2k})(re^{i\varepsilon})\le{}& c_3c_5\lambda^k\Gamma(D,B_1-B_0)r^{-B_1}\\ &+c_3c_5\lambda^k\Gamma(D,B_0-B_1)r^{-B_1}+H^C\big((f_{2k}-f_{2k-2})1_{V(2\varepsilon)}\big)(re^{i\varepsilon}).\end{aligned}$$

But $f_{2k}-f_{2k-1}$ equals 0 on $V(0)$ and $f_{2k}-f_{2k-2}$ on $V(2\varepsilon)$, so in fact the latter term is just $(f_{2k}-f_{2k-1})(re^{i\varepsilon})$. In short,

$$(f_{2k+1}-f_{2k})(re^{i\varepsilon})\le 2c_3c_4c_5\lambda^{k+1}r^{-B_1}.$$

A similar inequality holds at $re^{-i\varepsilon}$, so by Lemma (2.1),

$$(f_{2k+1}-f_{2k})(r)\le 2c_1c_3c_4c_5\lambda^{k+1}\eta(D,r)[r^{-B_1}+r^{-B_0}].$$

Applying Lemma (2.7) now gives that f_{2k+2} is finite, and that

$$\begin{aligned}(f_{2k+2}-f_{2k})(re^{i\gamma_1}) &= H^{Z(1)}(f_{2k+1}-f_{2k})(re^{i\gamma_1})\\ &\le \frac{4}{\alpha_1}c_1c_3c_4c_5\lambda^{k+1}\Gamma(D,B_1-B_0)r^{-B_1}.\end{aligned}$$

Taking c_4 sufficiently small makes this $\le\lambda^{k+1}r^{-B_1}$, and doing the same for $j\ne 1$ completes the proof. □

Proof of theorem (sufficiency). Assume first that (3.11) holds. Summing over k in Lemma (3.10) gives that $h\not\equiv\infty$, so that $h\in\mathcal{M}(D)$ by Lemma (3.9). Because $h\ge f_0$, it follows that (3.5) holds for each j, with $h_j=h$.

To remove the restriction (3.11), note that if $\Gamma_j<\infty$, $j=1,\dots,n$, then (3.11) holds for the open set

$$D'=D\setminus\big[\big(V(\alpha_1)\cup\ldots\cup V(\alpha_n)\big)\setminus B(r)\big],$$

for some $r<1$. Adding back a small portion of the removed set, if necessary, gives a domain with the same property. Let $h\in\mathcal{M}(D')$ be the function constructed above. Let $g=H^{D'}\tilde{g}$, where $\tilde{g}=1$ on $D\cap\partial D'$ and $\tilde{g}=0$ elsewhere on $\partial D'$. By the boundary Harnack principle for D', it follows that there is a c such that $cg>h$ on $D'\setminus B(r)$. Since g is superharmonic on D it follows that so is $h+cg$ (setting $h=0$ on $D\setminus D'$). If $h^0+G^D\nu$ is its Riesz decomposition then ν charges only $B(r)^c$, so that $G^D\nu$ is bounded. Thus $h^0(re^{i\gamma_j})\sim h(re^{i\gamma_j})$ as $r\downarrow 0$, for $j=1,\dots,n$, showing (3.5). □

4. Conditional Brownian motion

Let P^x be the measure on Brownian paths started at x and killed on exiting $D\subset\mathbf{C}$. Given a positive harmonic function h on D a new measure P^x_h is constructed as follows: let $\mathcal{F}_t=\sigma(X_s;s\le t)$, where X is the Brownian path and define $\tau_D=\inf\{t>0;X_t\notin D\}$. Define the measure P^x_h by specifying that for $\Lambda\in\mathcal{F}_t$,

$$P^x_h(\Lambda\cap\{\tau_D>t\})=E^x\left(\frac{h(X_t)}{h(x)};\ \Lambda\cap\{\tau_D>t\}\right).$$

Then X under P^x_h is called conditional (or h-transformed) Brownian motion. More information may be found in [Do]. When h is taken to be a minimal harmonic function with pole at $\xi, h(\cdot)=K(\xi,\ \cdot)$, then under P_h^x, the process X_t converges a.s. in the Martin topology to ξ as $t\uparrow\tau_D$. In this case we shall write P_ξ^x in place of P_h^x. This process (measure) always exists for $x\in D$ and ξ a point on the minimal Martin boundary. The transition density $p^\xi(t,x,y)$ of X under P_ξ^x is given by

$$p^\xi(t,x,y)=\frac{p(t,x,y)K(\xi,y)}{K(\xi,x)}\,.$$

Let η be a minimal point of the Martin boundary of D. We say that P_η^ξ exists if there is a probability measure P_η^ξ under which X_t has the transition density p^η and $X_t\to\xi$ in the Martin topology, as $t\downarrow 0$. As one might expect, P_η^ξ exists, and is a limit of P_η^x as $x\to\xi$, provided ξ and η are distinct minimal points and the boundary Harnack principle for parabolic functions holds at ξ. Further discussion may be found in [Sa]. In that paper, an example is given of a $D\subset\mathbf{C}$ which has distinct minimal boundary points ξ_1 and ξ_2 for which $P_{\xi_2}^{\xi_1}$ does not exist. What goes wrong is that the process, if it existed, would go from ξ_1 to ξ_2 instantaneously. Another example of nonexistence, this time for a Denjoy domain, was given in [Bu]. Burdzy's example also features two minimal points which are so close together that any conditional Brownian motion started at one and conditioned to converge to the other must make the trip instantaneously.

Consider the Green function for X under P_ξ^x, namely

$$G^\xi(x,y)=\frac{G(x,y)K(\xi,y)}{K(\xi,x)}$$

where $G(x,y)$ is the Green function for D. Also recall that ξ is said to be attainable if $P_\xi^x(\tau_D<\infty)=1$ for some (and hence all) $x\in D$. Then we have the following result of Walsh from [Sa].

(4.1) Theorem. *Let $z\in D$ be fixed and let ξ and η be distinct attainable minimal Martin boundary points. Set*

$$f(w)=\frac{K(\xi,w)}{G(z,w)}\,.$$

Then the following are equivalent:

(a) *the process P_η^ξ exists*

(b) $\liminf_{t\uparrow\tau_D} f(X_t)<\infty$, *$P_\eta^w$ a.s., for some $w\in D$,*

(c) $\lim_{t\uparrow\tau_D} f(X_t)$ *exists and is finite P_η^w a.s. for all $w\in D$.*

Walsh's Theorem will be applied to an n-sectorial domain D satisfying (3.3) for each j. Let $\xi_1,\dots,\xi_n$ be the poles of the n minimal elements of $\mathcal{M}_1(D)$. In particular, we are concerned with existence of the $P_{\xi_j}^{\xi_k}$ process when $\xi_k \neq \xi_j$. Note that all the ξ_j are attainable, since attainability in D is equivalent to that in $D \cap B(0,1)$, and in the latter all minimal points are attainable (see [CrM]).

The verification of the condition

$$\liminf_{t \uparrow \tau_D} f_k(X_t) < \infty, \quad P_{\xi_j}^w \text{ a.s.}$$

where $f_k(\cdot) = K(\xi_k, \cdot)/G(z, \cdot)$ may be accomplished by checking that

$$\liminf_{r \downarrow 0} f_k(re^{i\gamma_j}) < \infty.$$

To see this, we first prove the following:

(4.3) Lemma. *Suppose that $\mathcal{M}_1(D)$ has n minimal elements, whose poles are $\xi_1, \dots, \xi_n$. Suppose that $r_m \to 0$ and that $\delta > 0$ is such that $B(r_m e^{i\gamma_j}, \delta r_m) \subset D$ for $j = 1, 2, \dots, n$. Then $\bigcup_{m=1}^{\infty} B(r_m e^{i\gamma_j}, \delta r_m)$ is not minimal thin at ξ_j.*

Proof. By the proof of Proposition (3.1) $\bigcup_{m=1}^{\infty} \bigcup_{j=1}^{n} B(r_m e^{i\gamma_j}, \delta r_m)$ is not minimal thin at ξ_k. By part (b) of that proposition $\bigcup_{j=1, j \neq k}^{n} B(r_m e^{i\gamma_j}, \delta r_m)$ is minimal thin at ξ_k. The result now follows. $\square$

Now we claim:

$$P_{\xi_j}^{\xi_k} \text{ exists} \tag{4.4}$$

if and only if

$$\liminf_{r \downarrow 0} f_k(re^{i\gamma_j}) < \infty. \tag{4.5}$$

Assume (4.4), and let $r_m \downarrow 0$. Then, by Lemma (4.3), under $P_{\xi_j}^w$ the path X will hit infinitely many balls $B_m^j = B(r_m e^{i\gamma_j}, \delta r_m)$. By Theorem (4.1) (c) and Harnack's inequality on each B_m^j which is hit by X, it follows that (4.5) holds. Conversely, if (4.5) holds, let $r_m \downarrow 0$ satisfy $\liminf_{m \to \infty} f_k(r_m e^{i\gamma_j}) < \infty$. Again by the fact that $\bigcup_{m=1}^{\infty} B_m^j$ is not minimal thin at ξ_j, the process X hits infinitely many of B_m^j, $P_{\xi_j}^w$ a.s. Thus by Harnack's inequality applied in the balls hit by X, condition (b) of Theorem (4.1) is satisfied and so (4.4) holds.

This result will be used to prove our criterion for existence. The criterion is an integral test which determines when the boundary near the origin is thick enough to prevent the conditional process from instantaneously flashing from ξ_k to ξ_j.

(4.6) Theorem. *Let $D \subset \mathbf{C}$ be n-sectorial. Assume that* (3.3) *holds for each* j. *Then all the conditional Brownian motions* $P_{\xi_j}^{\xi_k}$ *exist for* $k \neq j$, $1 \leq k, j \leq n$, *if and only if*

$$\int_0^1 \eta_j(t) t^{-(B_j+B_{j+1}+1)} dt < \infty \quad \text{for } j = 1, \ldots, n. \tag{4.7}$$

(*Recall* $B_1 = B_{n+1}$.)

Proof. It is now clear that we need only show the equivalence of (4.5) and (4.7). Since (3.3) holds for each j, Theorem (3.6) and Theorem 1.XII.14 of [Do] imply that

$$G(re^{i\gamma_j}, w) \sim c(w) G_j(re^{i\gamma_j}, w) \quad \text{as } r \downarrow 0,$$

where G_j is the Green function for $W(\alpha_{j-1}, \alpha_j)$. But $G_j(re^{i\gamma_j}, w) \sim c(w) r^{B_j}$, hence $G(re^{i\gamma_j}, w) \sim c(w) r^{B_j}$ too.

By Lemma (1.1) we have the bound

$$K(\xi_k, re^{i\gamma_j}) \leq cr^{-B_j}, \quad \forall r < 1, \forall j \ \forall k.$$

Moreover, by Proposition (2.9)

$$K(\xi_j, re^{i\gamma_j}) \geq cr^{-B_j}, \quad \forall r < 1, \forall j.$$

Now, in one direction, suppose that for some j,

$$\int_0^1 \eta_j(t) t^{-(B_j+B_{j+1}+1)} dt = \infty.$$

Then, by Lemma (1.1),

$$\begin{aligned} K(\xi_j, re^{i\gamma_{j+1}}) &\geq c \int_0^\infty \frac{r^{B_{j+1}} t^{B_{j+1}-1}}{r^{2B_{j+1}} + t^{2B_{j+1}}} K(\xi_j, te^{i\alpha_j})\, dt \\ &\geq c \int_0^1 \frac{r^{B_{j+1}} t^{B_{j+1}-1}}{r^{2B_{j+1}} + t^{2B_{j+1}}} \eta_j(t) K(\xi_j, te^{i\gamma_j})\, dt, \quad \text{(using Harnack)} \\ &\geq cr^{B_{j+1}} \int_0^1 \frac{t^{B_{j+1}-1}}{r^{2B_{j+1}} + t^{2B_{j+1}}} \eta_j(t) t^{-B_j}\, dt, \end{aligned}$$

again by Lemma (1.1). Thus

$$\begin{aligned} \liminf_{r \downarrow 0} \frac{K(\xi_j, re^{i\gamma_{j+1}})}{G(re^{i\gamma_{j+1}}, w)} &\geq c \liminf_{r \downarrow 0} \frac{K(\xi_j, re^{i\gamma_{j+1}})}{r^{B_{j+1}}} \\ &= c \int_0^1 \frac{\eta_j(t)}{t^{B_{j+1}+B_j+1}} dt \\ &= \infty. \end{aligned}$$

Thus $P_{\xi_{j+1}}^{\xi_j}$ does not exist.

Conversely, assume that

$$\int_0^1 \frac{\eta_j(t)}{t^{B_j+B_{j+1}+1}}\,dt<\infty \quad \text{for } j=1,\dots,n\,.$$

Given $N>0$, let u_N be the harmonic function on

$$D'=W(\alpha_{j-2}+\tfrac{1}{4}\beta_{j-1},\alpha_{j+2}-\tfrac{1}{4}\beta_{j+1})\cap D$$

which agrees with $K(\xi_k,\cdot)\wedge N$ on $\partial D'$. Then by Lemma (2.1),

$$\begin{aligned}
u_N(re^{i\gamma_j}) &= \frac{1}{\beta_j}\int_0^1 \frac{r^{B_j}t^{B_j-1}}{r^{2B_j}+t^{2B_j}}\left[u_N(te^{i\alpha_j})+u_N(te^{i\alpha_{j-1}})\right]dt\\
&\le \frac{c_1}{\beta_j}\int_0^1 \frac{r^{B_j}t^{B_j-1}}{r^{2B_j}+t^{2B_j}}\left[\eta_j(t)\big(u_N(te^{i\gamma_j})+u_N(te^{i\gamma_{j+1}})\big)\right.\\
&\quad \left.+\eta_{j-1}(t)\big(u_N(te^{i\gamma_{j-1}})+u_N(te^{i\gamma_j})\big)\right]dt\\
&\le r^{B_j}\left[c\int_0^1 \frac{\eta_j(t)}{t^{B_{j+1}+B_j+1}}\,dt+c\int_0^\delta \frac{\eta_{j-1}(t)}{t^{B_j+B_{j-1}+1}}\,dt\right.\\
&\quad \left.+c_6\int_0^\delta t^{-B_j}u_N(te^{i\gamma_j})\frac{\eta_j(t)+\eta_{j-1}(t)}{t}\,dt\right]
\end{aligned}$$

where we have used the bounds

$$u_N(te^{i\gamma_l})\le ct^{-B_l}.$$

Here c may depend on D, but $c_6=c_1/\min_j\beta_j$ depends only on the α's. Thus

$$r^{-B_j}u_N(re^{i\gamma_j})\le c+c_6\int_0^1 t^{-B_j}u_N(te^{i\gamma_j})\frac{\eta_j(t)+\eta_{j-1}(t)}{t}\,dt.$$

Write $m_N=\sup_{r>0} r^{-B_j}u_N(re^{i\gamma_j})$. Since $u_N\le N$ and

$$\int_0^1 \frac{\eta_j(t)+\eta_{j-1}(t)}{t^{1+B_j}}\,dt<\infty$$

it follows that $m_N<\infty$, and hence

$$m_N\le c\left(1+m_N\int_0^1 \frac{\eta_j(t)+\eta_{j-1}(t)}{t}\,dt\right).$$

If, in addition,

$$c_6 \int_0^1 \frac{\eta_j(t)+\eta_{j-1}(t)}{t}\, dt \leq \frac{1}{2} \tag{4.8}$$

this results in

$$m_N \leq 2c\,.$$

Letting $N\uparrow\infty$ and noting $u_N(\cdot)\uparrow K(\xi_k,\cdot)$, we have

$$2c \geq m_N \uparrow \sup_{r>0} r^{-B_j} K(\xi_k, re^{i\gamma_j})\,.$$

Since $G(re^{i\gamma_j},w)\sim c(w)r^{B_j}$ it follows that $P_{\xi_j}^{\xi_k}$ exists. $\square$

Finally, we must remove the additional condition (4.8). To do this we use the following lemma.

Lemma 4.9. *Let D be n-sectorial. Let $D'\subset D$ be a domain satisfying*

$$D\backslash D' \subset \left[\bigcup_{j=1}^{n} V(\alpha_j)\right]\backslash B(\delta).$$

Then there is a constant c such that

$$G'(x,y) \leq G(x,y) \leq cG'(x,y), \quad \forall x,y \in D\cap B(\delta/8)$$

(*where G, G' are the Green functions for D and D'*).

In fact, as at the end of Section 3, it is straightforward to construct such a domain D', and have (4.8) hold also. Condition (3.3) is immediate, so the argument just given shows (4.5) for D'. The estimate of Lemma (4.9) now guarantees that (4.5) holds for D as well, so that $P_{\xi_j}^{\xi_k}$ exists, showing the theorem.

Proof of (4.9). The first inequality is clear. Let G_0 be the Green function of $D\cap B(\delta/2)$. Let $\mu_2^x(dz)$ be harmonic measure for this domain, and let $\mu_4^x(dz)$ be harmonic measure for $D\cap B(\delta/4)$. Then

$$\begin{aligned} G(x,y) &= G_0(x,y) + \int_{\partial B(\delta/2)} G(z,y)\mu_2^x(dz) \\ &= G_0(x,y) + \int_{\partial B(\delta/2)} \int_{\partial B(\delta/4)} G(z,w)\mu_2^x(dz)\mu_4^y(dw)\,. \end{aligned}$$

Similarly,

$$G'(x,y)=G_0(x,y)+\int_{\partial B(\delta/2)}\int_{\partial B(\delta/4)} G'(z,w)\mu_2^x(dz)\mu_4^y(dw).$$

Therefore it will suffice to show that

$$G(z,w)\leq cG'(z,w)$$

for $z\in D\cap\partial B(\delta/2)$, $w\in D\cap\partial B(\delta/4)$. By Lemma (2.2) it will be enough to show this for $z=(\delta/2)e^{i\gamma_j}$, $w=(\delta/4)e^{i\gamma_k}$, $1\leq j,k\leq n$. Since this involves only finitely many points, the inequality becomes a trivial consequence of the connectedness of D and D'. □

Remarks and Examples

(1) Theorem (4.6) in the case of Denjoy domains ($\alpha_1=\pi$) asserts that the P_{0-}^{0+} process exists if and only if

$$\int_{-1}^{1}\frac{\eta(t)}{|t|^3}\,dt<\infty.$$

This is in contrast to the requirement

$$\int_{-1}^{1}\frac{\eta(t)}{|t|}\,dt<\infty$$

for the existence of two minimal points 0+ and 0− corresponding to the origin.

The two tests can be generalized to Denjoy domains in $\mathbf{R}^{n+1}$. Benedicks showed in [Be] that the condition for the origin to split into a pair of minimal points is that

$$\int_{[-1,1)^n}\frac{\eta(t)}{|t|^n}\,dt<\infty.$$

The corresponding condition for the existence of P_{0+}^{0-} turns out to be that

$$\int_{[-1,1)^n}\frac{\eta(t)}{|t|^{2n+1}}\,dt<\infty.$$

(2) We now (briefly) modify an example of [Ch] showing what the integral test means if the holes in ∂D are highly regular.

Take $D=\mathbf{C}\setminus(e^{i\alpha}\mathbf{R}^+\cup(\mathbf{R}\setminus\bigcup_m((1-\alpha_m)2^{-m},2^{-m}))$. Then we claim that

$$\int_0^1 \frac{\eta_0(t)}{t^{\pi/\alpha+\pi/(2\pi-\alpha)+1}}\,dt<\infty \quad\Leftrightarrow\quad \sum_m \alpha_m^2 2^{m(\pi/\alpha+\pi/(2\pi-\alpha))}<\infty .$$

This follows from these observations

(a) $\eta_0(t)\le c\eta_0((1-\frac{1}{2}\alpha_m)2^{-m})$, $t\in((1-\alpha_m)2^{-m},2^{-m})$,

(b) $\eta_0(t)\ge(1/c)\eta_0((1-\frac{1}{2}\alpha_m)2^{-m})$, $t\in((1-\frac{3}{4}\alpha_m)2^{-m},(1-\frac{1}{4}\alpha_m)2^{-m})$,

(c) $\eta_0((1-\frac{1}{2}\alpha_m)2^{-m})\cong\alpha_m$, where $\cong$ means that there is a two-sided inequality between the two quantities.

These follow from boundary Harnack, Harnack and conformal mapping, respectively.

Thus,

$$\begin{aligned}\int_0^1 \frac{\eta_0(t)}{t^{\pi/\alpha+\pi/(2\pi-\alpha)+1}}\,dt &\cong \sum_m \frac{\alpha_m\cdot\alpha_m 2^{-m}}{((1-\alpha_m)2^{-m})^{\pi/\alpha+\pi/(2\pi-\alpha)+1}}\\ &\cong \sum_m \alpha_m^2 2^{m(\pi/\alpha+\pi/(2\pi-\alpha))}.\end{aligned}$$

This proves the claim.

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Received August 28, 1991

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