Unique linearly convex support of an analytic functional

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0. Introduction

The objective of this paper is to prove a theorem that roughly says that if K is a compact linearly convex support for an analytic functional μ in $\mathbf{C}^n$, such that K is not too thin in a certain sense, and such that through each point $p\in\partial K$ there is at most one tangential complex hyperplane, then K is unique, i.e. μ admits no other linearly convex support. This theorem is analogous to a theorem of Kiselman and Martineau, [6] and [10], for convex supports and restricted to one complex variable, it reduces to a theorem of Kiselman [6] about polynomially convex supports.

We let $O'(\mathbf{C}^n)$ denote the space of analytic functionals in $\mathbf{C}^n$. A compact set K is a carrier for $\mu\in O'(\mathbf{C}^n)$ if for any open neighborhood ω of K there is a constant C_ω such that

$$|\mu\cdot f|\leq C_\omega(|f|_\omega), \quad f\in O(\mathbf{C}^n).$$

If K is a polynomially convex carrier, then μ has a continuous extension to an element μ_K in $O'(K)$, where $O'(K)$ is the dual of $O(K)$ and $O(K)$ is the inductive limit of the spaces $O(\omega)$ for open $\omega\supset K$.

A polynomially convex carrier K for μ is called a polynomially convex support for μ if no polynomially convex compact proper subset H of K carries μ. Since any $\mu\in O'(\mathbf{C}^n)$ has some polynomially convex carrier, it follows by Zorn's lemma that μ has at least one polynomially convex support. However, in general it is not unique as is shown by the following simple example:

Example 1. Let

$$\mu\cdot f=\int_0^1 f(t)\,dt, \quad f\in O(\mathbf{C}^1).$$

Then any simple curve from 0 to 1 is a polynomially convex support for μ.

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However, Kiselman proved in [6]:

Theorem 0.1. *Let K be a polynomially convex support for $\mu \in O'(\mathbf{C}^1)$. Suppose that for any open set ω intersecting ∂K, the interior of the union of K and an arbitrary component of $\omega \setminus K$ intersects ∂K. Then μ has unique polynomially convex support K.*

The requirement of K is a thickness assumption and for instance, it is fulfilled if for each $p \in \partial K$ there are arbitrary small neighborhoods V such that $V \setminus K$ is connected. However, the set $|y| \leq |x| \leq 1$ is thin at the origin but yet it satisfies the assumption in the theorem.

Remark. In [8] is also proved that the condition on K is necessary; i.e. if any $\mu \in O'(\mathbf{C}^1)$ having K as polynomially convex support has unique polynomially convex support, then K satisfies the condition in Theorem 0.1.

For $n > 1$ Kiselman proved in [6]:

Theorem 0.2. *If K is a polynomially convex support for $\mu \in O'(\mathbf{C}^n)$ and ∂K is C^2, then μ has unique polynomially convex support K.*

He also has analogous results for $O(\Omega)$-convex supports in a domain of holomorphy Ω in $\mathbf{C}^n$.

We also refer to [7] where uniqueness of support for μ with respect to a given class of compacts is proved to be equivalent to convexity of the indicator of a corresponding generalized Fourier–Laplace transform of μ.

A compact set K in $\mathbf{C}^n$ is linearly convex if $\mathbf{C}^n \setminus K$ is a union of complex hyperplanes. We shall work with the somewhat more restricted class LCC defined as follows: A compact set $K \subset \mathbf{C}^n$ is LCC if, through each point $p \in \mathbf{C}^n \setminus K$, there is a complex hyperplane which can be continuously moved to the hyperplane at infinity without intersecting K. By the Oka–Stoltzenberg theorem, see [11], it follows that any LCC set is polynomially convex. It is also clear that the class LCC is closed under intersections.

For motivation of this definition, note that a compact set K is convex if and only if through each point in its complement there is a real hyperplane that can be moved continuously to the hyperplane at infinity without intersecting K. Similarly, K is polynomially convex if and only if the same holds with hyperplane replaced by algebraic hypersurface, see [11].

A linearly convex compact set K is LCC if the set K^* of nonintersecting complex hyperplanes is connected, but the converse is not true, see Example 3 below. When $n=1$, LCC means precisely polynomial convexity.

We say that a LCC (or convex) compact carrier for $\mu \in O'(\mathbf{C}^n)$ is a LCC (or convex) support for μ if no LCC (or convex) compact proper subset H of K carries μ.

It is not hard to see that any convex support is unique if $n=1$. However, this is not true for $n>1$:

Example 2. Let $\mu\in O'(\mathbf{C}^2)$ be defined by

$$\hat{\mu}(z)=\mu(e^{\zeta\cdot z})=\cos\sqrt{z_1z_2}\,.$$

For each $t>0$,

$$2\sqrt{|z_1||z_2|}\le t|z_1|+|z_2|/t=\sup_{K_t}\operatorname{Re}(\zeta_1z_1+\zeta_2z_2),$$

where $K_t=\{|\zeta_1|<t,|\zeta_2|<1/t\}$ so it follows from the Pólya–Martineau theorem, see e.g. [5], that μ is carried by each K_t, and hence each K_t contains a LCC (or convex or polynomially convex) support for μ. However, if $K\subset K_t$ carries μ, then $(t,1/t)\in K$ so if $H\subset K_t$ and $H'\subset K_{t'}$ are supports and $t\neq t'$, then $H\neq H'$. Also note that there are several tangential hyperplanes through the point $(t,1/t)\in H$.

Theorem 0.3. *If K is a convex support for $\mu\in O'(\mathbf{C}^n)$ such that, through each extreme point $p\in\partial K$, there is at most one tangential hyperplane, then μ has unique convex support K.*

This theorem is due to Martineau [10]. It occurred first in [6] but with the somewhat stronger assumption that ∂K be C^1. This suggests that even in the polynomially convex case, cf. Theorem 0.2, it would be possible to weaken the assumption on ∂K. Theorem 0.4 below may be viewed as an attempt in that direction. In order to give the exact formulation we have to introduce some additional notation. We denote the elements in $\mathbf{P}^n=\mathbf{P}^n(\mathbf{C})$ by $[z]=[(z_0,\dots,z_n)]$. Consider $\mathbf{C}^n\hookrightarrow\mathbf{P}^n$ by $z'\to[(1,z')]$ and let $(\mathbf{P}^n)^*$ be the dual of $\mathbf{P}^n$ with respect to the pairing $[z],[w]\to[\langle z,w\rangle]=[\sum_0^n z_jw_j]$. There is a one-to-one correspondence between points $[a]\in(\mathbf{P}^n)^*$ and hyperplanes in $\mathbf{P}^n$,

$$[a]\sim\{[z]\in\mathbf{P}^n;\langle a,z\rangle=0\}.$$

Let

$$K^*=\{[a]\in(\mathbf{P}^n)^*;a\text{ does not intersect }K\}.$$

Here and in the sequel we identify $[a]\in(\mathbf{P}^n)^*$ with the corresponding hyperplane in $\mathbf{P}^n$. Also if $[a]=[(1,a')]$, we identify it with $a'\in(\mathbf{C}^n)^*$. In most situations we may assume that $0\in K\subset\mathbf{C}^n\hookrightarrow\mathbf{P}^n$ and then $K^*\subset(\mathbf{C}^n)^*\hookrightarrow(\mathbf{P}^n)^*$.

Note that $E\subset\mathbf{P}^n$ is linearly convex if and only if $E^{**}=E$, and that E^* is open (compact) if E is compact (open). For a thorough discussion of these concepts we refer to [4].

Theorem 0.4. *Let K be a* LCC *support for* $\mu \in O'(\mathbf{C}^n)$. *Suppose that*

(a) *through any point* $p \in \partial K$ *there is at most one tangential complex hyperplane,*

(b) *for any open set* V *intersecting* ∂K^* *in* $(\mathbf{C}^n)^*$ *the interior of the union of* $(\mathbf{C}^n)^* \setminus K^*$ *and an arbitrary component of* $V \cap K^*$ *intersects* ∂K^*.

Then μ *has unique* LCC *support* K.

The requirement (b) is fulfilled if

(b′) for each plane ξ tangential to K, there are arbitrary small neighborhoods $V \ni \xi$ in $(\mathbf{C}^n)^*$ such that $V \cap K^*$ is connected.

For instance, (b′) is fulfilled if ∂K has some regularity.

The condition (b) can be described as follows. If ξ_t is a curve in K^* from the hyperplane at infinity ξ_∞, ending up at a hyperplane $\xi_1 \in \partial K^*$, and V is a connected neighborhood of ξ_1 then there is another $\xi' \in \partial K^* \cap V$ having the property in (b') such that the curve ξ_t can be modified in $V \cap K^*$ to end up in ξ' instead of ξ_1.

It is clear that Theorem 0.4 reduces to Theorem 0.1 when $n=1$. However, the proof (see the final remark in Section 4) also contains Theorem 0.1 in the following sense. Let $K \subset \mathbf{C}^1$ be a support of $\mu \in O'(\mathbf{C}^1)$ and such that K satisfies the hypothesis of Theorem 0.1. Consider μ as an element $\tilde{\mu}$ in $O'(\mathbf{C}^n)$ in the obvious way. Then $K \subset \mathbf{C}^1 \hookrightarrow \mathbf{C}^n$ is a LCC support for $\tilde{\mu}$ in $\mathbf{C}^n$ and the proof of Theorem 0.4 applies to K and $\tilde{\mu}$.

In this context it is natural to state the following result of Martineau [9]:

Theorem 0.5. *Suppose K is a polynomially convex set in* $\mathbf{C}^n$. *Then there is a* $\mu \in O'(\mathbf{C}^n)$ *having K as unique polynomially convex support.*

In particular we have:

Corollary. *If* $K \subset \mathbf{C}^n$ *is* LCC (*or convex*) *then there is a* $\mu \in O'(\mathbf{C}^n)$ *having K as a* LCC (*or convex*) *support.*

Example 3. In [2] is an open connected set $D \subset \mathbf{C}^2$ constructed such that $D = D_0^{**} \subsetneq D^{**}$ where D_0^{**} is the component of D^{**} containing D. Now if $K = D^*$, then K is linearly convex and $K^* = D^{**}$ is disconnected. We claim that K is LCC with respect to a hyperplane at infinity $\xi_\infty \in D$.

Namely, if $p \in K^c$, then p belongs to a hyperplane $\xi \in K^* = D^{**}$, i.e. p is a hyperplane that intersects D^{**}. However, it must then also intersect $D = D_0^{**}$ in some point, say ξ'. This means that ξ' is a hyperplane through the point p that belongs to the ξ_∞-component of K^*. Thus K is LCC.

The paper is organized as follows. In Section 1 we briefly discuss the notion of $\mathbf{C}$-convexity and relates it to the class LCC. In Section 2 we sketch a proof of Theorem 1 (essentially the same as in [8]), based on Cauchy's formula in $\mathbf{C}^1$, in order to make the proof of Theorem 0.4 more comprehensive. In Section 3 we discuss a type of Cauchy–Fantappie–Leray formulas from [3] which are appropriate for linearly convex or LCC compacts K. Finally in Section 4 we conclude the proof of Theorem 0.4.

1. C-convexity and strongly linear convexity

A non-empty (compact or open) proper subset of $\mathbf{P}^1$ is said to be $\mathbf{C}$-convex if it is connected and its complement also is connected. A non-empty (compact or open) subset of $\mathbf{P}^n$ is called $\mathbf{C}$-convex if all its non-empty intersections with (projective) lines are $\mathbf{C}$-convex. For instance, if K is convex in some affinization of $\mathbf{P}^n$, then K is $\mathbf{C}$-convex.

If K is $\mathbf{C}$-convex, then it is linearly convex and K^* is also $\mathbf{C}$-convex, see [4], in particular K^* is non-empty and connected, so if K is a compact $\mathbf{C}$-convex set in $\mathbf{P}^n$ it follows that K is LCC in some affinization $\mathbf{C}^n=\mathbf{P}^n\setminus\xi_\infty$ of $\mathbf{P}^n$. The following characterization of $\mathbf{C}$-convexity is given by Zelinskij, [12]:

Theorem 1.1. *A linearly convex compact set $K\subset\mathbf{P}^n$ is $\mathbf{C}$-convex if and only if $\xi\cap\partial K$ is connected for each tangential hyperplane ξ.*

Suppose $0\in K\subset\mathbf{C}^n=\mathbf{P}^n\setminus\xi_\infty$, K is linearly convex. Then the Fantappie transform $\mathcal{F}\colon O'(K)\to O(K^*)$ is defined by

$$\mathcal{F}\mu(z)=\mu\left[\frac{1}{1+\zeta\cdot z}\right].$$

The following theorem connects the notion of $\mathbf{C}$-convexity to Martineau's notion of strongly linear convexity:

Theorem 1.2. *A linearly convex set $K\subset\mathbf{P}^n$ is $\mathbf{C}$-convex if and only if the Fantappie transform is an isomorphism.*

A proof of the if-part and an outline of a proof of the only if-part is given in [13] and [14]. A complete proof occurred in [3].

Note that for the $\mathbf{C}$-convex case, Theorem 0.5 immediately follows from Theorem 1.2, since there is some holomorphic function in K^* that cannot be continued anywhere over ∂K^*.

2. Idea of the proof and the one-variable case

In this section we sketch the proof of Theorem 0.4 in the case $n=1$, which is precisely Theorem 0.1 of Kiselman.

Let K be a polynomially convex compact set in $\mathbf{C}$, i.e. $\widehat{\mathbf{C}}\setminus K$ is open and connected. Note that if $\mu\in O'(\mathbf{C})$ is carried by K and μ_K denotes its extension to $O(K)$, then

$$\varphi(\zeta)=\frac{1}{2\pi i}\mu_K\left(\frac{1}{\zeta-\cdot}\right) \tag{1}$$

is holomorphic in $\widehat{\mathbf{C}}\setminus K$. Conversely, any $\varphi\in O(\widehat{\mathbf{C}}\setminus K)$ (or $\varphi\in O(\Omega), \Omega\supset K$) defines an element $\mu\in O'(\mathbf{C})$ by the formula

$$\mu\cdot f=\int_{\partial\omega} f(\zeta)\varphi(\zeta)\,d\zeta,\quad f\in O(\mathbf{C}) \tag{2}$$

if $\omega\supset K$, and μ is carried by K. By Cauchy's formula,

$$\mu\cdot f=\mu_K\cdot f=\mu_K\cdot\left[\frac{1}{2\pi i}\int_{\partial\omega}\frac{f(\zeta)d(\zeta)}{\zeta-\cdot}\right]=\int_{\partial\omega} f(\zeta)\varphi(\zeta)\,d\zeta$$

so (2) defines the inverse of (1) and we have an isomorphism $O'(K)\simeq O(\widehat{\mathbf{C}}\setminus K)$.

In order to prove Theorem 0.1 we must show that if K and H are polynomially convex carriers for $\mu\in O'(\mathbf{C})$, K having the additional property of Theorem 1, and $K\setminus H$ is non-empty, then there is a compact polynomially convex proper subset $\tilde{K}$ of K that also carries μ.

To this end we consider the function

$$\varphi(\zeta)=\mu_K\left(\frac{1}{\zeta-\cdot}\right).$$

Since $K\setminus H\neq\emptyset$ and $\widehat{\mathbf{C}}\setminus H$ is connected, there is a curve γ from ∞ to a boundary point $a\in\partial K\setminus H$, such that $\gamma\subset(\widehat{\mathbf{C}}\setminus H)\cap(\widehat{\mathbf{C}}\setminus K)$. Namely, if $\tilde{\gamma}$ is any curve in $\mathbf{C}\setminus H$ from ∞ to a point in $K\setminus H$, we can take a as the first point in K that $\tilde{\gamma}$ meets, and let γ be the part of $\tilde{\gamma}$ from ∞ to a. Let ω denote the component of $(\widehat{\mathbf{C}}\setminus K)\cap(\widehat{\mathbf{C}}\setminus H)$ that contains γ. By assumption, there is a point $p\in\partial K\cap\partial\omega$ and a neighborhood V of p such that $V\subset\omega\cup K$ and $\omega\cup V$ is connected. By uniqueness, $\mu_H(1/(\zeta-\cdot))=\mu_K(1/(\zeta-\cdot))$ for $\zeta\in\omega$, so $\mu_H(1/(\zeta-\cdot))$ provides a continuation of $\varphi(\zeta)$ to $(\widehat{\mathbf{C}}\setminus K)\cup V$. By the discussion above, this means that μ is carried by the polynomially convex compact set $\tilde{K}=K\setminus V$ which is a proper subset of K since $p\in K\setminus\tilde{K}$. Hence Theorem 1 is proved.

This proof is based on Cauchy's formula and in order to generalize the argument to $n>1$, we first discuss in Section 3 an appropriate analogue of Cauchy's formula for a linearly convex set in $\mathbf{C}^n$.

3. Cauchy–Fantappiè–Leray formulas for linearly convex sets

Let K be a polynomially convex compact subset of $\mathbf{C}^n$. If α is a closed $(n,n-1)$-form in $\mathbf{C}^n \setminus K$, then α defines an element $\mu \in O'(\mathbf{C}^n)$ by

$$\mu \cdot f = \int_{\partial\omega} f\alpha, \quad f \in O(\mathbf{C}^n),$$

where ω is any smooth set containing K, and it follows that K carries μ. Conversely, suppose K is a carrier for $\mu \in O'(\mathbf{C}^n)$ and let μ_K denote the extension of μ to $O(K)$. If $P(\zeta,z)$ is a reproducing kernel for $O(K)$, such that $\zeta \to P(\zeta,z)$ is a closed $(n,n-1)$-form in $\mathbf{C}^n \setminus K$ and $z \to P(\zeta,z)$ is in $O(K)$, and $f(z) = \int_{\partial\omega} P(\zeta,z) f(\zeta)$, $K \subset \omega$, then $\alpha(\zeta) = \mu_K \cdot P(\zeta,\cdot)$ is a closed $(n,n-1)$-form in $\mathbf{C}^n \setminus K$ that defines μ. More invariantly, and analogously to the one-variable case discussed in Section 2, one can consider $(n,n-1)$-forms in $\mathbf{P}^n \setminus K$ instead of $\mathbf{C}^n \setminus K$, and then one actually has an isomorphism $H^{n,n-1}(\mathbf{P}^n \setminus K) \simeq O'(K)$ (where $H^{n,n-1}$ denotes the Dolbeault cohomology group of bidegree $(n,n-1)$), but for our purposes it is enough to consider forms in $\mathbf{C}^n \setminus K$, though the forms that will be constructed below actually are forms in $\mathbf{P}^n \setminus K$.

Definition. A smooth mapping $s(\zeta): \omega \to (\mathbf{P}^n)^*$, $\omega \subset \mathbf{C}^n$ (or $\mathbf{P}^n$) is called a CL-section (Cauchy–Leray) if for each ζ, ζ belongs to the hyperplane $s(\zeta)$, i.e. $\langle \zeta, s(\zeta) \rangle = 0$.

In the sequel we think of $s(\zeta)$ as a mapping $\omega \to \mathbf{C}^n$ so that $s(\zeta)$ corresponds to the hyperplane $\{z; \langle s(\zeta), \zeta - z \rangle = 0\}$ through ζ. We also identify s with the $(1,0)$-form $\sum_1^n s_j d\zeta_j$. Now suppose K is linearly convex and that we have a CL-section $s(\zeta)$ defined in $\mathbf{C}^n \setminus K$ such that the planes $s(\zeta)$ avoid K. Then the Cauchy–Fantappiè–Leray kernel

$$H(s)(\zeta,z) = \left(\frac{1}{2\pi i}\right)^n \frac{s \wedge (\bar\partial s)^{n-1}}{\langle s(\zeta), \zeta - z \rangle^n}$$

is a reproducing kernel of the kind above. However if such a global CL-section $s(\zeta)$ exists for K, then it follows, see e.g. [3], that the Fantappiè transform is an isomorphism and this in turn implies that K is $\mathbf{C}$-convex. Though, as far as K is linearly convex, we can find such a CL-section at least locally in $\mathbf{C}^n \setminus K$, and we will recall from [3] how such local choices of sections can be used to construct a global kernel.

For CL-sections s_j, let

$$H^k(s_1, \ldots, s_k) = \left(\frac{1}{2\pi i}\right)^n s_1 \wedge \ldots \wedge s_k \sum_{|\alpha| = n-k} \frac{(\bar\partial s_1)^{\alpha_1} \wedge \ldots \wedge (\bar\partial s_k)^{\alpha_k}}{\langle s_1, \zeta - z \rangle^{\alpha_1 + 1} \ldots \langle s_k, \zeta - z \rangle^{\alpha_k + 1}}.$$

Then H^k is an $(n,n-k)$-form in ζ, alternating in s_j, vanishing if $k>n$, closed if $k=1$, and

$$dH^k(s_1,\dots,s_k)=\sum_{j=1}^{k}(-1)^{j+1}H^{k-1}(s_1,\dots,\hat{s}_j,\dots,s_k). \tag{1}$$

If $\{\omega_\alpha\}$ is a locally finite open covering of $\mathbf{C}^n\setminus K$, s_α CL-sections in ω_α avoiding K, and $\{\varphi_\alpha\}$ a partition of unity subordinated $\{\omega_\alpha\}$, then

$$P(\zeta,z)=\sum_{k=1}^{n}\sum_{\alpha\in\mathbf{N}^k}\varphi_{\alpha_k}d\varphi_{\alpha_{k-1}}\wedge\dots\wedge d\varphi_{\alpha_1}\wedge H^k(s_{\alpha_1},\dots,s_{\alpha_k})$$

is a reproducing kernel of the kind discussed above. Thus if $f\in O(\bar{\omega})$, $\omega\supset K$, then

$$f(z)=\int_{\partial\omega}P(\zeta,z)f(\zeta) \tag{2}$$

for z in some neighborhood of K.

Note that if $0\in K$ one can choose $s(\zeta)$ so that $\langle s(\zeta),\zeta\rangle=-1$ and hence $-\langle s(\zeta),\zeta-z\rangle=1+\langle s(\zeta),z\rangle$.

Thus it follows from (2) that if K is linearly convex, any $f\in O(K)$ can be written as a superposition of functions of the form

$$z\longrightarrow\prod_{j=1}^{n}\frac{1}{1+s_j\cdot z}$$

whereas in the $\mathbf{C}$-convex case one can use the simpler functions

$$z\longrightarrow\frac{1}{1+s\cdot z}.$$

Also, cf. [1] V, §24.

This formalism will be used in Section 4 in the proof of Theorem 0.4.

4. Proof of Theorem 0.4

Suppose H and K are LCC carriers for $\mu\in O'(\mathbf{C}^n)$, $K\setminus H$ is non-empty and K satisfies the requirements (a) and (b) of Theorem 0.4. Let ξ_1 be a plane (complex hyperplane) through $p\in K\setminus H$ that does not intersect H. Then there is a continuous curve ξ_t in H^* that joins ξ_1 to the plane at infinity ξ_∞. Note that $\xi_t\in K^*$ for large t since $\xi_\infty\in K^*$. Let t_0 be the least index such that $\xi_t\in K^*$ for each $t>t_0$. Then $\xi_{t_0}\in\partial K^*$ and, by assumption (b), the curve ξ_t can be changed in $H^*\cap K^*$ so that it ends up at a point $\xi_0\in\partial K^*\cap H^*$ with the property (b'), and so we have proved:

Claim I. *There is a $\xi_0 \in \partial K^* \cap H^*$ having a connected neighborhood $V \subset H^*$ such that $V \cap K^*$ is connected and such that ξ_0 can be joined to ξ_∞ by a curve in $K^* \cap H^*$.*

Clearly the plane ξ_0 is tangential to K, and we also have:

Claim II. *There is a neighborhood $\widetilde{\omega}$ of $K \cap \xi_0$ such that for any plane ξ, $\xi \in V$ if ξ intersects $\widetilde{\omega}$ but not K.*

Proof. Let $\{\omega_j\}$ be a basis of neighborhoods of $K \cap \xi_0$ and suppose that for each j, there is a plane $\xi_j \notin V$ that intersects ω_j but not K. By compactness, some subsequence converges to a plane $\xi' \notin V$ and thus $\xi' \neq \xi_0$. However, ξ' must be tangential to K and intersect $K \cap \xi_0$ and this contradicts assumption (a) about K. Thus some ω_j must have the proposed property.

We define a CL-section $s(\zeta)$ by letting $s(\zeta) = \xi_0$ for $\zeta \in K \cap \xi_0$ and then extend it in any smooth way to a neighborhood ω of $K \cap \xi_0$. We also assume that ω is contained in $\widetilde{\omega}$ of Claim II and that $s(\zeta)$ takes values in our fixed neighborhood V of ξ_0.

We now take a connected neighborhood $\widetilde{V}$ of ξ_0 in V such that $K \cap \xi \subset \omega$ if $\xi \in \widetilde{V}$. We define $\widetilde{K}$ as K minus the planes in $\widetilde{V}$.

Claim III. *$\widetilde{K}$ is a compact* LCC *proper subset of K.*

Proof. It is proper since $\emptyset \neq K \cap \xi_0 \subset K \setminus \widetilde{K}$. It is compact since $\widetilde{K} = (K^* \cup \widetilde{V})^*$ and $K^* \cup \widetilde{V}$ is open. Finally it is LCC since if $p \in \mathbf{C}^n \setminus \widetilde{K}$, say $p \in K \setminus \widetilde{K}$, then $p \in \xi$ for some $\xi \in \widetilde{V}$ and ξ can be joined to ξ_∞ in $K^* \cup \widetilde{V}$.

We now let $\omega_0 = \omega \setminus \widetilde{K}$. Then we have a CL-section $s_0(\zeta)$ in ω_0, taking values in our neighborhood V of ξ_0. Also note that $(\mathbf{C}^n \setminus \widetilde{K}) \subset \omega_0 \cup (\mathbf{C}^n \setminus K)$ so we can find a locally finite open cover $\{\omega_\alpha\}$ of $\mathbf{C}^n \setminus \widetilde{K}$ and CL-sections s_α in ω_α, such that if $\alpha \neq 0$, then $\omega_\alpha \subset \mathbf{C}^n \setminus K$ and s_α takes values in K^*.

We may now assume that $0 \in \widetilde{K}$.

Claim IV. *Let μ_K and μ_H be the extensions of μ to $O(K)$ and $O(H)$ respectively. If $a_j \in V \cap K^*$, $j = 1, \dots, n$, then*

$$\mu_H\Big(\prod_1^n \frac{1}{1 + a_j \cdot z}\Big) = \mu_K\Big(\prod_1^n \frac{1}{1 + a_j \cdot z}\Big).$$

Proof. By Claim I, all a_j belong to the ξ_∞-component A of $K^* \cap H^*$. Hence

$$(b_1, \dots, b_n) \to \mu_H\Big(\prod \frac{1}{1 + b_j \cdot z}\Big)$$

and

$$(b_1,...,b_n)\to\mu_K\Big(\prod\frac{1}{1+b_j\cdot z}\Big)$$

are both holomorphic in the connected set $A\times...\times A\subset((\mathbf{P}^n)^*)^n$, and coincide near $(\zeta_\infty,...,\xi_\infty)$. Hence they are equal at the point $(a_1,...,a_n)\in A\times...\times A$.

We can now define $(n,n-k)$-forms, cf. Section 3,

$$G^k(s_{\alpha_1},...,s_{\alpha_k})=\mu\cdot H^k(s_{\alpha_1},...,s_{\alpha_k}) \tag{1}$$

on $\bigcap_1^n\omega_{\alpha_j}$ by letting $\mu=\mu_K$ if all $\alpha_j\neq 0$ and $\mu=\mu_H$ if $\alpha_j=0$ for some j. Note that the dependence in z of $H^k(s_{\alpha_1},...,s_{\alpha_k})$ is of the form $\prod_1^n(1+s_{\alpha_j}\cdot z)^{-1}$.

Let $\{\varphi_\alpha\}$ be a partition of unity, subordinated $\{\omega_\alpha\}$ and put

$$\beta=\sum_{k=1}^n\sum_{\alpha\in\mathbf{N}^k}\varphi_{\alpha_k}d\varphi_{\alpha_{k-1}}\wedge...\wedge d\varphi_{\alpha_1}\wedge G^k(s_{\alpha_1},...,s_{\alpha_k}). \tag{2}$$

Claim V. *The form β is closed in $\mathbf{C}^n\setminus\widetilde{K}$ and represents the functional μ.*

Our last claim implies that μ is carried by the LCC set $\widetilde{K}$ and hence Theorem 0.4 is proved.

Proof of Claim V. It is clear that β is a smooth $(n,n-1)$-form in $\mathbf{C}^n\setminus\widetilde{K}$. The crucial point now is Claim II that ensures that all $s_{\alpha_j}\in V$ if some $\alpha_i=0$. Combined with Claim IV this implies that $G^k=\mu\cdot H^k$ is well defined in the sense that

$$dG^k(s_{\alpha_1},...,s_{\alpha_k})=\sum_{j=1}^k(-1)^{j+1}G^{k-1}(s_{\alpha_1},...,\hat{s}_{\alpha_j},...,s_{\alpha_k})$$

since this holds for H^k, cf. (1) in Section 3.

Note that

$$\sum_{\alpha\in\mathbf{N}^k}\varphi_{\alpha_k}\wedge d\varphi_{\alpha_{k-1}}\wedge...\wedge d\varphi_{\alpha_1}\wedge G^{k-1}(s_{\alpha_1},...,\hat{s}_{\alpha_j},...,s_{\alpha_k})$$

vanishes if $1\le j<k$ and equals

$$\sum_{\alpha\in\mathbf{N}^{k-1}}d\varphi_{\alpha_{k-1}}\wedge...\wedge d\varphi_{\alpha_1}\wedge G^{k-1}(s_{\alpha_1},...,s_{\alpha_{k-1}})$$

if $j=k$. We also note that in the definition (2) of β we may let k run from $-\infty$ to ∞, if we interprete G^k as zero for $k>n$ and $k\leq 0$. With this convention (3) holds for all k and we get

$$\begin{aligned} d\beta = &\sum_k \sum_{\alpha\in\mathbf{N}^k} d\varphi_{\alpha_k}\wedge\ldots\wedge d\varphi_{\alpha_1}\wedge G^k(s_{\alpha_1},\ldots,s_{\alpha_k}) \\ &+\sum_k (-1)^{k-1} \sum_{\alpha\in\mathbf{N}^k} \varphi_{\alpha_k}\wedge d\varphi_{\alpha_{k-1}}\wedge\ldots\wedge d\varphi_{\alpha_1}\wedge dG^k(s_{\alpha_1},\ldots,s_{\alpha_k}) \\ = &\sum_k \sum_{\alpha\in\mathbf{N}^k} d\varphi_{\alpha_k}\wedge\ldots\wedge d\varphi_{\alpha_1}\wedge G^k(s_{\alpha_1},\ldots,s_{\alpha_k}) \\ &-\sum_k \sum_{\alpha\in\mathbf{N}^{k-1}} d\varphi_{\alpha_{k-1}}\wedge\ldots\wedge d\varphi_{\alpha_1}\wedge G^{k-1}(s_{\alpha_1},\ldots,s_{\alpha_{k-1}}) = 0. \end{aligned}$$

Thus β is closed, and to see that it represents μ we can choose some large ball B containing $K\cup H$ and e.g. assume $s=s_\alpha=-\bar{\zeta}/|\zeta|^2$ in a neighborhood of ∂B. Hence by the usual Cauchy–Fantappiè–Leray formula, $f(z)=\int_{\partial B} H^1(s)f$ for $z\in B$, so that

$$\mu\cdot f = \int_{\partial B} \mu\cdot H^1(s)f = \int_{\partial B} G^1(s)f = \int_{\partial B} \beta f = \int_{\partial\omega} \beta f$$

if $\omega\supset\widetilde{K}$, since β is closed.

Final remark. An inspection of the proof reveals that the assumptions (a) and (b) of Theorem 0.4 can be replaced by

(ab′): For an arbitrary component $\widetilde{V}$ of $V\cap K^*$, where V is any open set intersecting ∂K^*, there is a point $\xi_0\in\partial K^*$ having a neighborhood $W\subset\widetilde{V}\cup(\mathbf{P}^n\setminus K^*)$, such that $\xi_0\cap K$ has a neighborhood ω in which there are CL-sections $s(\zeta)$ locally, taking values in W.

In particular this is fulfilled for a polynomially convex set $K\subset\mathbf{C}^1\hookrightarrow\mathbf{C}^n$ satisfying the condition in Theorem 0.1.

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