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Removability theorems for solutions of degenerate elliptic partial differential equations

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1. Introduction

Removable singularities for Hölder continuous harmonic functions are completely known, see [C_1], [C_2, p. 91] and [KW].

Theorem A. *Let Ω be an open set in $\mathbf{R}^n$ and let E be a relatively closed subset of Ω. Then E is removable for harmonic functions of $\Omega\setminus E$ which are locally Hölder continuous in Ω with exponent $0<\alpha<1$ if and only if the $(n-2+\alpha)$-dimensional Hausdorff measure of E is zero.*

We recall that a function $u:\Omega\to\mathbf{R}$ is said to be locally Hölder continuous in Ω with exponent $0<\alpha\le 1$ if for each compact subset K of Ω there is $M<\infty$ such that

$$|u(x)-u(y)|\le M|x-y|^\alpha \tag{1.1}$$

for all x and y in K.

In this paper we consider an analogous question for solutions of second order degenerate elliptic partial differential equations. For linear equations we refer the reader to [HP]. We call a function u $\mathcal{A}$-harmonic if u is a continuous weak solution of the equation

$$\operatorname{div}\mathcal{A}(x,\nabla u(x))=0 \tag{1.2}$$

with $|\mathcal{A}(x,\xi)|\approx|\xi|^{p-1}$, $p>1$. For the exact requirements on the mapping $\mathcal{A}$ we refer the reader to Section 3. Here we point out that the prototype of equation (1.2) is the p-harmonic equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u)=0. \tag{1.3}$$

Since quasiconformal mappings do not preserve any Hausdorff dimension s in the range $0<s<n$ and since for $p=n$ equations (1.2) have a quasiconformal invariance property, see [R, p. 146], [HKM, Ch. 14], removability theorems for $\mathcal{A}$-harmonic functions seem to be problematic in terms of the Hausdorff measure. However, using a concept somewhat more restrictive than the Hausdorff dimension and closely related to the Minkowski dimension we establish a result similar to Theorem A for $\mathcal{A}$-harmonic functions. For the use of the Minkowski content in the study of removable singularities for solutions of linear equations with restricted growth see [B] and [L]. It is remarkable that the removability depends only on α and p and not on the structure of a particular mapping $\mathcal{A}$.

By an exhaustion (K_i) of $E\subset\mathbf{R}^n$ we mean an increasing sequence of compact sets K_i such that $\bigcup K_i=E$.

Theorem B. *Let $\Omega\subset\mathbf{R}^n$ be open and let E be a relatively closed subset of Ω. If for some exhaustion (K_i) of E*

$$(1.4)\qquad \begin{cases} (1.4.1) & \displaystyle\int_{\{0<d(x,K_i)<1\}} d(x,K_i)^{p(\alpha-1)}\,dm(x)<\infty, \\ (1.4.2) & \displaystyle\liminf_{r\to 0}\frac{m(\{0<d(x,K_i)<r\})}{r^b}=0, \end{cases}$$

$b=p-\alpha(p-1)$, then E is removable for $\mathcal{A}$-harmonic functions of $\Omega\setminus E$ which are locally Hölder continuous in Ω with exponent $0<\alpha\le 1$.

For certain regular sets K_i, for example for self-similar sets, condition (1.4) for $p=2$ is equivalent to $H^{n-2+\alpha}(K_i)=0$, and hence for these sets the sufficiency part of Theorem A follows from Theorem B. In particular, (1.4) holds if

$$(1.5)\qquad \int_{\{0<d(x,K_i)<1\}} d(x,K_i)^{-b}\,dm(x)<\infty,\quad b=p-\alpha(p-1),$$

which is true, for example, if the Minkowski dimension $\dim_M(K_i)$ of K_i is strictly less than $n-p+\alpha(p-1)$, see Section 2. Theorem B is a consequence of a stronger result, where the Hölder continuity is studied in the set $\Omega\setminus E$ only. We say that a function $u\colon\Omega\to\mathbf{R}$ belongs to $\operatorname{locLip}_\alpha(\Omega)$, $0<\alpha\le 1$, if there exists $M<\infty$ such that for each $x\in\Omega$ and each y with $|x-y|\le d(x,\partial\Omega)/2$ we have

$$(1.6)\qquad |u(x)-u(y)|\le M|x-y|^\alpha.$$

For the properties of the class $\operatorname{locLip}_\alpha(\Omega)$ see [GM]. We remark that a function u, locally Hölder continuous with exponent α in Ω, is always in $\operatorname{locLip}_\alpha(G)$ for any open set $G\subset\subset\Omega$. Since the $\mathcal{A}$-harmonicity is a local property, Theorem B is a consequence of the following result, which for $\alpha=1$ can also be deduced from [HK, Corollary 4.5].

Theorem C. *Suppose that E satisfies* (1.4) *for $b=p-\alpha(p-1)$. Then E is removable for $\mathcal{A}$-harmonic functions of $\Omega\setminus E$ in the class* $\operatorname{locLip}_\alpha(\Omega\setminus E)$.

In fact, Theorem C holds for $\mathcal{A}$-superharmonic functions. This is Theorem E in Section 3. Note that there is no condition for the smoothness of an $\mathcal{A}$-superharmonic function $u:\Omega\setminus E\to\mathbf{R}$ on the set E in Theorem E. For removability results of ordinary superharmonic functions we refer the reader to [KW]. Theorem C leads to interesting non-smoothness results for $\mathcal{A}$-superharmonic functions for certain values of p, see Theorems F and G in Section 4.

We show by an example in Section 4 that Theorem C is essentially sharp. The relations between the Minkowski and the Hausdorff dimension of the set E and condition (1.4) are explained in Section 2.

The limiting case $\alpha=0$ in Theorem B, or in Theorem C, deserves a special attention. This case corresponds to locally bounded $\mathcal{A}$-harmonic functions and it is well known that for locally bounded functions $u:\Omega\to\mathbf{R}$, which are $\mathcal{A}$-harmonic in $\Omega\setminus E$, E is removable if and only if E is of p-capacity zero ([HKM, Theorem 7.36]). The next theorem extends this result.

Theorem D. *Suppose that u is $\mathcal{A}$-harmonic in $\Omega\setminus E$ and that u belongs to* $\mathrm{BMO}(\Omega\setminus E)$. *If E satisfies* (1.4) *with $\alpha=0$, then u extends to a function $\mathcal{A}$-harmonic in Ω.*

Take notice that for $\alpha=0$, (1.4.2) follows from (1.4.1). The proof of Theorem D is given in Section 5.

2. Condition (1.4)

In this short section we study condition (1.4).

Let K be a closed set in $\mathbf{R}^n$. In the well known Whitney decomposition, see [St], $\mathbf{R}^n\setminus K$ is represented as a union of non-overlapping closed cubes Q with edge length $l(Q)$ equal to 2^{-k}, $k\in\mathbf{Z}$, and $d(Q,K)/\operatorname{dia}(Q)\in[1,4]$. We let N_k be the number of those cubes Q with $l(Q)=2^{-k}$; we write Q_i^k, $i=1,\dots,N_k$, for the collection of these cubes.

If $A\subset\mathbf{R}^n$ and $r>0$, then we let $A(r)$ denote the open set $A+B(r)$, i.e.

$$A(r)=A+B(r)=\bigcup_{x\in A}B(x,r)$$

is the r-inflation of A.

The next lemma relates (1.4.1) to the Whitney decomposition of $\mathbf{R}^n\setminus K$.

2.1. Lemma. *Let* $\gamma \le 0$ *and* $j \in \mathbf{Z}$. *Then*

$$\int_{K(c2^{-j})\setminus K} d(x,K)^\gamma \, dm(x) \ge c^\gamma \sum_{k=j}^{\infty} N_k 2^{-k(\gamma+n)} \tag{2.2}$$

where $c = 5\sqrt{n}$ *and*

$$\int_{K(2^{-j})\setminus K} d(x,K)^\gamma \, dm(x) \le \sum_{k=j}^{\infty} N_k 2^{-k(\gamma+n)}. \tag{2.3}$$

Proof. First note that

$$d(Q_i^k, K) \le 4 \operatorname{dia}(Q_i^k) = 4\sqrt{n} 2^{-k},$$

and hence the interior of each Q_i^k, $k \ge j$, $i = 1, \dots, N_k$, lies in $K(c2^{-j})\setminus K$. For each $x \in Q_i^k$ we have

$$d(x,K) \le \operatorname{dia}(Q_i^k) + d(Q_i^k, K) \le c2^{-k},$$

and thus we obtain

$$\begin{aligned} \int_{K(c2^{-j})\setminus K} d(x,K)^\gamma dm(x) &\ge \sum_{k=j}^{\infty} \sum_{i=1}^{N_k} \int_{Q_i^k} d(x,K)^\gamma dm(x) \\ &\ge c^\gamma \sum_{k=j}^{\infty} N_k 2^{-k(\gamma+n)}. \end{aligned}$$

This is inequality (2.2). The proof of (2.3) is completely analogous and left to the reader.

For $A \subset \mathbf{R}^n$ we let $H^s(A)$ denote the s-dimensional Hausdorff measure of A; $\dim_H(A)$ denotes the Hausdorff dimension of A. For $r > 0$ we set

$$M_s(A,r) = \frac{m(A(r))}{r^{n-s}}$$

and call this quantity the s-dimensional Minkowski precontent of A. Next, the Minkowski dimension of A is

$$\dim_M(A) = \inf\{\, s : \limsup_{r\to 0} M_s(A,r) < \infty \,\},$$

and we set

$$\underline{M}_s(A) = \liminf_{r\to 0} \frac{m(A(r)\setminus \bar{A})}{r^{n-s}}.$$

Note that $\underline{M}_s(K_i) = 0$ is the same as (1.4.2) for $b = n - s$.

Clearly, $\dim_H(A) \le \dim_M(A)$; the converse holds for certain regular sets, cf [MV, Section 4].

2.4. Lemma. *Suppose that K is a compact subset of $\mathbf{R}^n$ and that*

$$\underline{M}_s(K)=0. \tag{2.5}$$

If $s\le n-1$, then

$$\liminf_{r\to 0} M_s(K,r)=H^s(K)=0. \tag{2.6}$$

In particular, for $s\le n-1$, (2.5) implies $H^{n-1}(K)=0$.

Proof. We first show that for $s\le n-1$, (2.5) implies $m(K)=0$. For this we may assume that $s=n-1$. By the Brunn–Minkowski inequality [F, Corollary 3.2.42, p. 278]

$$m(K)\le c(n)[m(K(r)\setminus K)/r]^{n/(n-1)}$$

and hence (2.5) implies that $m(K)=0$.

Thus we obtain for $s\le n-1$ and $0<r$

$$\frac{m(K(r))}{r^{n-s}}=\frac{m(K(r)\setminus K)}{r^{n-s}}.$$

Then [MV, Lemma 3.1] implies $\liminf_{r\to 0} H_s(K,r)=0$, where

$$H_s(K,r)=\inf\left\{kr^s : K\subset\bigcup_{i=1}^{k} B(x_i,r)\right\}.$$

This clearly yields $H^s(K)=0$. The lemma follows.

2.7. *Remarks.*

(a) For compact sets K, $H^s(K)=0$ does not, in general, imply that $\underline{M}_s(K)=0$. However, if K is sufficiently regular, then $H^s(K)=0$ implies the stronger condition $\limsup_{r\to 0} M_s(K,r)=0$, see [MV, Section 4].

(b) Let $\gamma<0$ and suppose that

$$\int_{K(1)\setminus K} d(x,K)^\gamma\, dm(x)<\infty.$$

Then $\underline{M}_s(K)=0$, where $s=n+\gamma$. Hence (1.5) yields (1.4). Moreover, (1.4.1) implies that $\underline{M}_s(K_i)=0$, where $s=n-p(1-\alpha)$. This is weaker than (1.4.2). Conversely, one can construct Cantor sets for which (1.4.2) is satisfied but (1.4.1) fails.

(c) If $\dim_M(K)=\lambda\le n-1$, then

$$\int_{K(1)\setminus K} d(x,K)^\gamma\, dm(x)<\infty$$

for $\gamma>\lambda-n$; this follows from [MV, Theorem 3.12] and Lemma 2.1. In particular, if K is a self-similar fractal set with $\dim_H(K)<n-p+p(1-\alpha)$, then condition (1.4) holds. For this result see [MV, Section 4].

3. $\mathcal{A}$-supersolutions and proofs for Theorems B and C

We consider mappings $\mathcal{A}:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^n$ which satisfy the following assumptions for some $p>1$ and $0<\beta_1\le\beta_2$:

(a) the mapping $x\mapsto\mathcal{A}(x,\xi)$ is measurable for all $\xi\in\mathbf{R}^n$ and the mapping $\xi\mapsto\mathcal{A}(x,\xi)$ is continuous for a.e. $x\in\mathbf{R}^n$;

for all $\xi\in\mathbf{R}^n$ and a.e. $x\in\mathbf{R}^n$

(b) $\mathcal{A}(x,\xi)\cdot\xi\ge\beta_1|\xi|^p$;

(c) $|\mathcal{A}(x,\xi)|\le\beta_2|\xi|^{p-1}$;

(d) $(\mathcal{A}(x,\xi_1)-\mathcal{A}(x,\xi_2))\cdot(\xi_1-\xi_2)>0$ whenever $\xi_1\ne\xi_2$; and

(e) $\mathcal{A}(x,\lambda\xi)=|\lambda|^{p-2}\lambda\mathcal{A}(x,\xi)$ for $\lambda\in\mathbf{R}$, $\lambda\ne0$.

The constant p is always associated with the mapping $\mathcal{A}$ as in (b) and (c), and we write $e_{\mathcal{A}}=\beta_2/\beta_1$.

Let Ω be an open set in $\mathbf{R}^n$. A function $u\in C(\Omega)\cap W^{1,p}_{\text{loc}}(\Omega)$ is called *$\mathcal{A}$-harmonic* if u is a weak solution of (1.2), i.e. if for all $\psi\in C_0^\infty(\Omega)$

$$\int_\Omega A(x,\nabla u(x))\cdot\nabla\psi(x)\,dm(x)=0. \tag{3.1}$$

It is important to notice that continuity is superfluous in the definition of an $\mathcal{A}$-harmonic function. More precisely, if a function $u\in W^{1,p}_{\text{loc}}(\Omega)$ satisfies (3.1), then after a change in a set of measure zero u is $\mathcal{A}$-harmonic in Ω; see [S] or [HKM].

A lower semicontinuous function $v:\Omega\to\mathbf{R}\cup\{\infty\}$ is *$\mathcal{A}$-superharmonic* in Ω if for all domains $D\subset\subset\Omega$ and for all functions $u\in C(\overline{D})$, $\mathcal{A}$-harmonic in D, the condition $v\ge u$ in ∂D yields $v\ge u$ in D and if $v\ne\infty$ in every component of Ω. If $\mathcal{A}(x,\xi)=\xi$, i.e. if we consider the ordinary Laplace equation $\Delta u=0$, then $\mathcal{A}$-harmonicity and $\mathcal{A}$-superharmonicity reduces to ordinary harmonicity and superharmonicity, respectively.

For our removability results a solution class between $\mathcal{A}$-harmonic and $\mathcal{A}$-superharmonic functions is of importance. A function $v\in W^{1,p}_{\text{loc}}(\Omega)$ is an *$\mathcal{A}$-supersolution* of (1.2) if

$$\int_\Omega A(x,\nabla u(x))\cdot\nabla\psi(x)\,dm(x)\ge0 \tag{3.2}$$

for all non-negative $\psi\in C_0^\infty(\Omega)$. Then every $\mathcal{A}$-supersolution is $\mathcal{A}$-superharmonic, after a change in a set of measure zero if necessary. Conversely, every locally bounded $\mathcal{A}$-superharmonic function is an $\mathcal{A}$-supersolution. For these results see [HKM]. In the classical case smooth $\mathcal{A}$-supersolutions are functions $v\in C^2(\Omega)$ with $\Delta v\le0$ in Ω.

The following is a key lemma.

3.3. Lemma. *Let Ω be an open set in $\mathbf{R}^n$ and let E be a relatively closed subset of Ω. Suppose that u is an $\mathcal{A}$-supersolution in $\Omega\setminus E$, and that for some $a\le n$*

$$\int_Q |\nabla u|^p\,dm \le c_1\,\mathrm{dia}(Q)^a \tag{3.4}$$

for each cube Q in a Whitney decomposition of $\Omega\setminus E$. If for some exhaustion (K_i) of E

$$\begin{cases} \int_{K_i(1)\setminus K_i} d(x,K_i)^{a-n}\,dm(x)<\infty, \\ \underline{M}_s(K_i)=0,\ \ s=(a(p-1)+n)/p-1, \end{cases} \tag{3.5}$$

then u extends to an $\mathcal{A}$-supersolution in Ω.

Proof. Since $a\le n$, $s\le n-1$, and it follows from Lemma 2.4 and (3.5) that $H^{n-1}(E)=0$. To prove that u extends to an $\mathcal{A}$-supersolution in Ω it suffices to show that $u\in W^{1,p}_{\mathrm{loc}}(\Omega)$ and that u satisfies (3.2). Since $H^{n-1}(E)=0$, u is ACL in Ω and in order to show that $u\in W^{1,p}_{\mathrm{loc}}(\Omega)$ it thus suffices to show that for each point $x_0\in E$ there is $r>0$ such that

$$\int_{B(x_0,r)} |\nabla u|^p\,dm<\infty. \tag{3.6}$$

To this end, fix $x_0\in E$, and let $r=(1/5\sqrt{n})\min\{1,d(x_0,\partial\Omega)\}$. Now we use the fact that $K=E\cap\overline{B}(x_0,4r)$ is a compact subset of E and choose K_i such that $K\subset K_i$. Let W_0 be the collection of those cubes in the Whitney decomposition of $\Omega\setminus E$ which meet $B=B(x_0,r)$. Then each $Q\in W_0$ lies in $K_i(1)\setminus K_i$. Since $m(E)=0$, we obtain from (3.4)

$$\begin{aligned}\int_{B(x_0,r)} |\nabla u|^p\,dm &\le c_1\sum_{Q\in W_0}\mathrm{dia}(Q)^a\le c_2\sum_{Q\in W_0} d(Q,E)^a\\ &\le c_3\sum_{Q\in W_0}\int_Q d(x,E)^{a-n}\,dm(x)\\ &\le c_3\int_{K_i(1)\setminus K_i} d(x,K_i)^{a-n}\,dm(x),\end{aligned}$$

which is finite by assumption (3.5); here $c_3=c_3(c_1,a,n)<\infty$. This shows that $u\in W^{1,p}_{\mathrm{loc}}(\Omega)$.

Next we consider inequality (3.2); rather sharp estimates are needed for this. The problem is again local and thus it suffices to show that (3.2) holds whenever

$\psi \in C_0^\infty(B(x_0,r))$ is non-negative, $x_0 \in E$ and $r>0$ is sufficiently small. Let r, K_i, and W_0 be as in the previous consideration and write $B=B(x_0,r)$. We let W_j, $j=1,2,\dots$, be the set of those cubes $Q\in W_0$ with $2^{-j-1}(5\sqrt{n})^{-1}\le l(Q)\le 2^{-j}$, and we denote their union by $\bigcup W_j$.

Fix a non-negative function $\psi\in C_0^\infty(B)$. For $j\ge 1$ consider the Lipschitz functions

$$\phi_j=\min\{1,\max\{(2^{-j}-d(x,K_i))/2^{-j-1},0\}\}.$$

Since $\phi_j(x)=1$ for $x\in K_i$, the non-negative function $\psi(1-\phi_j)$ is in the Sobolev space $W_0^{1,p}(B\setminus E)$, cf. [HKM, Ch. 1], and thus it can be used in (3.2) as a non-negative $C_0^\infty(B\setminus E)$-function. Now

$$\begin{aligned}\int_B A(x,\nabla u(x))\cdot\nabla\psi(x)\,dm(x) &= \int_{B\setminus E} A(x,\nabla u(x))\cdot\nabla(\psi(1-\phi_j))\,dm\\ &\quad+\int_B A(x,\nabla u(x))\cdot\nabla(\psi\phi_j)\,dm\\ &=I'+I'',\end{aligned}$$

and since u is an $\mathcal{A}$-supersolution in $B\setminus E$, the integral I' is non-negative. It remains to show that $I''\to 0$ as $l\to\infty$ for some sequence (j_l) of positive integers.

To this end, write

$$\begin{aligned}I''&=\int_B \psi A(x,\nabla u(x))\cdot\nabla\phi_j\,dm+\int_B \phi_j A(x,\nabla u(x))\cdot\nabla\psi\,dm\\ &=I_1+I_2.\end{aligned}$$

We estimate the integrals I_1 and I_2 separately. First, $|\psi|\le c_2$ for some constant c_2, and hence by the Hölder inequality and (3.4)

$$\begin{aligned}|I_1|&\le c_2\sum_{Q\in W_j}\int_Q |A(x,\nabla u(x))||\nabla\phi_j|\,dm\\ &\le c_2\beta_2\sum_{Q\in W_j}\left(\int_Q|\nabla u|^p\,dm\right)^{(p-1)/p}\left(\int_Q|\nabla\phi_j|^p\,dm\right)^{1/p}\\ &\le c_3\sum_{Q\in W_j}\operatorname{dia}(Q)^{a(p-1)/p}2^j\operatorname{dia}(Q)^{n/p}\\ &\le c_3 2^j\sum_{Q\in W_j}\operatorname{dia}(Q)^{[a(p-1)+n]/p};\end{aligned}$$

here $c_3=c_3(c_1,c_2,\beta,p)$ and we have also used the fact that $|\nabla\phi_j|\le 2^j$. Since for $x\in Q\in W_j$, $d(x,E)$ is bounded from above and from below by a multiple of $\operatorname{dia}(Q)$

and since $2^{-j-1}/5 \le \mathrm{dia}(Q) \le \sqrt{n}2^{-j}$ for each $Q \in W_j$, we obtain

$$
\begin{aligned}
|I_1| &\le c_3\sqrt{n} \sum_{Q\in W_j} \mathrm{dia}(Q)^{-1+[a(p-1)+n]/p} \\
&\le c_4 m\Big(\bigcup W_j\Big) 2^{-j([a(p-1)+n]/p-1-n)} \le c_5 M_s(K_i, 5\sqrt{n}2^{-j}),
\end{aligned}
$$

with $s=(a(p-1)+n)/p-1$ and $c_4=c_4(c_1, c_2, \beta_2, p, a, n)$. By Lemma 2.4 and (3.5) there is a sequence (r_l) with

$$
\lim_{l\to\infty} r_l = \liminf_{l\to\infty} M_s(K_i, r_l) = 0.
$$

Select for each l a positive integer j_l with $5\sqrt{n}2^{-j_l} < r_l \le 5\sqrt{n}2^{-j_l+1}$. Then we have $M_s(K_i, 5\sqrt{n}2^{-j_l}) \le c_6 M_s(K_i, r_l)$, and it follows that $I_1 \to 0$ as $l \to \infty$.

For the second integral I_2 we again use the Hölder inequality to obtain

$$
\begin{aligned}
|I_2| &\le c\Big(\int_{\bigcup W_j} |\nabla u|^p \, dm\Big)^{(p-1)/p} m\Big(\bigcup W_j\Big)^{1/p} \\
&\le c\Big(\int_B |\nabla u|^p \, dm\Big)^{(p-1)/p} m\Big(\bigcup W_j\Big)^{1/p},
\end{aligned}
$$

where $c=c(p, \beta, \sup|\psi|)$. Since $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and since $m(\bigcup W_j) \to 0$ as $j \to \infty$, $I_2 \to 0$ as $j \to \infty$. Thus $I'' \to 0$, and the proof is complete.

Theorem E. *Let E be a relatively closed subset of an open set $\Omega \subset \mathbf{R}^n$. Suppose that $u \in \mathrm{locLip}_\alpha(\Omega \setminus E)$, $0 < \alpha \le 1$, is $\mathcal{A}$-superharmonic in $\Omega \setminus E$. If for some exhaustion (K_i) of E*

$$
\begin{cases} \int_{K_i(1)\setminus K_i} d(x, K_i)^{p(\alpha-1)} \, dm(x) < \infty, \\ \underline{M}_s(K_i) = 0, \quad s = n-p+\alpha(p-1), \end{cases} \tag{3.7}
$$

then u extends to an $\mathcal{A}$-superharmonic function of Ω.

3.8. *Remarks.*

(1) It follows from the proof that E is removable for $\mathcal{A}$-supersolutions $u \in \mathrm{locLip}_\alpha(\Omega \setminus E)$ under condition (3.7). In fact, the extended function will be an $\mathcal{A}$-supersolution in Ω.

(2) The proof of Lemma 3.3 and the proof below show that the condition (3.7) can be replaced by a weaker set of conditions: $H^{n-1}(E)=0$ and E has an exhaustion K_i such that (1.4.1) holds and

$$
\liminf_{r\to 0} \frac{m(\{r/2 < d(x, K_i) < r\})}{r^b} = 0
$$

for $b=p-\alpha(p-1)$. Theorems B and C also remain valid under these assumptions.

Proof of Theorem E. Let $u \in \operatorname{locLip}_\alpha(\Omega\setminus E)$ be $\mathcal{A}$-superharmonic in $\Omega\setminus E$. Since u is continuous in $\Omega\setminus E$ and hence locally bounded in $\Omega\setminus E$, u is an $\mathcal{A}$-supersolution in $\Omega\setminus E$ [HKM, Corollary 7.19]. Let Q be a cube in a Whitney decomposition of $\Omega\setminus E$. Then $\frac{3}{2}Q \subset\subset \Omega\setminus E$ and we pick a point $y \in \frac{3}{2}Q$ such that

$$u(y) = \min_{\frac{3}{2}Q} u.$$

The Caccioppoli type estimate [HKM, Lemma 3.53] for positive $\mathcal{A}$-supersolutions v in the interior of $\frac{3}{2}Q$ reads

$$\int_{\frac{3}{2}Q} |\nabla v|^p v^{-1-\varepsilon} |\eta|^p \, dm \le c_2 \int_{\frac{3}{2}Q} v^{p-1-\varepsilon} |\nabla \eta|^p \, dm, \tag{3.9}$$

where $\varepsilon > 0$, $c_2 = (pe_{\mathcal{A}}/\varepsilon)^p$ and $\eta \in C_0^\infty(\frac{3}{2}Q)$. Choosing $\varepsilon = (p-1)/2$, $\eta = 1$ on Q and $|\nabla\eta| \le 4/l(Q)$ and letting $v = u - u(y) + \delta$, $\delta > 0$, we obtain from (3.9)

$$\int_Q |\nabla u|^p \, dm \le c \max_{\frac{3}{2}Q} (u - u(y) + \delta)^p \operatorname{dia}(Q)^{n-p}, \tag{3.10}$$

where $c = c(p, n, e_{\mathcal{A}})$; note that

$$v^{-1-\varepsilon} \ge \max_{\frac{3}{2}Q} (u - u(y) + \delta)^{-1-\varepsilon}.$$

Since $u \in \operatorname{locLip}_\alpha(\Omega\setminus E)$, it follows from [GM, Theorem 2.13] that

$$\max_{\frac{3}{2}Q} (u - u(y)) \le M_1 \operatorname{dia}(Q)^\alpha, \tag{3.11}$$

where $M_1 = M_1(M, \alpha)$ and M is the constant in (1.6). Now $\delta \to 0$ in (3.10) together with (3.11) yields

$$\int_Q |\nabla u|^p \, dm \le c \operatorname{dia}(Q)^{\alpha p + n - p},$$

where $c = c(p, n, e_{\mathcal{A}}, M, \alpha)$. Since no non-empty compact set satisfies (3.7) for $\alpha < (p-n)/(p-1)$, we may assume that $\alpha \ge (p-n)/(p-1)$, and in particular that $0 < \alpha p + n - p \le n$. Hence, letting $a = \alpha p + n - p$, we obtain from Lemma 3.3 that u extends to an $\mathcal{A}$-supersolution of Ω; note that

$$s = \frac{a(p-1)+n}{p} - 1 = n - p + \alpha(p-1)$$

as required. Finally, every $\mathcal{A}$-supersolution can be made $\mathcal{A}$-superharmonic after a redefinition on a set of measure zero [HKM, Corollary 7.17]. The proof is complete.

Proof of Theorem C. Since both u and $-u$ are $\mathcal{A}$-superharmonic in $\Omega\setminus E$, it follows from Theorem E that they extend to $\mathcal{A}$-superharmonic functions u^* and $(-u)^*$, respectively, of Ω. Since $m(E)=0$, [HKM, Theorem 3.66] yields for each $x\in E$

$$u^*(x)=\lim_{r\to 0}\frac{1}{m(B(x,r))}\int_{B(x,r)}u^*\,dm=\lim_{r\to 0}\frac{1}{m(B(x,r))}\int_{B(x,r)}u\,dm.$$

The same applies to $(-u)^*$ and hence $u^*=-(-u)^*$ which means that u^* is both $\mathcal{A}$-superharmonic and $\mathcal{A}$-subharmonic in Ω. Consequently u^* is $\mathcal{A}$-harmonic and the theorem follows.

Proof of Theorem B. This is a direct consequence of Theorem C; note that if u is locally Hölder continuous in Ω with exponent $0<\alpha\leq 1$, then for each open $D\subset\subset\Omega$, u belongs to $\operatorname{locLip}_\alpha(D\setminus E)$.

4. Examples and non-smoothness results

Our non-smoothness results imply, for example, that if $u:\Omega\to\mathbf{R}$ is $\mathcal{A}$-superharmonic in Ω and $\mathcal{A}$-harmonic in $\Omega\setminus E$ and if E is thin, then u cannot be smooth unless u is $\mathcal{A}$-harmonic in Ω. The first theorem is a consequence of Theorem C.

Theorem F. *Suppose that u is locally Hölder continuous with exponent α, $0<\alpha\leq 1$, in Ω and let $E\subset\Omega$ satisfy* (1.5) *for some $\gamma\leq -1$. If u is $\mathcal{A}$-harmonic in $\Omega\setminus E$, then either u is $\mathcal{A}$-harmonic in Ω or $\alpha<(p+\gamma)/(p-1)$.*

Proof. Since u belongs to $\operatorname{locLip}_\alpha(D\setminus E)$ for each open $D\subset\subset\Omega$, Theorem C yields that u is $\mathcal{A}$-harmonic in Ω provided that $\alpha\geq(p+\gamma)/(p-1)$.

If $p>n$, then even the set $E=\{x_0\}$ is of interest.

Theorem G. *Suppose that u is locally Hölder continuous with exponent α, $0<\alpha\leq 1$ in Ω and let $\mathcal{A}$ satisfy* (a)–(e) *for $p>n$. If u is $\mathcal{A}$-harmonic in $\Omega\setminus\{x_0\}$, then either u is $\mathcal{A}$-harmonic in Ω or $\alpha\leq(p-n)/(p-1)$.*

Proof. Suppose that u is not $\mathcal{A}$-harmonic in Ω. Fix γ, $-1\geq\gamma>-n$. Now $E=\{x_0\}$ satisfies (1.5) and hence it follows from Theorem F that $\alpha<(p+\gamma)/(p-1)$. Letting $\gamma\to -n$ we obtain $\alpha\leq(p-n)/(p-1)$ as desired.

The function $u(x)=-|x|^{(p-n)/(p-1)}$, $p>n$, is $\mathcal{A}$-harmonic (p-harmonic),

$$\mathcal{A}(x,\xi)=|\xi|^{p-2}\xi,$$

in $\mathbf{R}^n\setminus\{0\}$, but u is not $\mathcal{A}$-harmonic in $\mathbf{R}^n$ (u is $\mathcal{A}$-superharmonic). Since u is Hölder continuous with exponent $\alpha=(p-n)/(p-1)$ in $\mathbf{R}^n$, Theorem G is sharp. Note that the upper bound $(p-n)/(p-1)$ in Theorem G is independent of a particular mapping $\mathcal{A}$, i.e. it does not depend on $e_{\mathcal{A}}$. A careful study of isolated singularities of a p-harmonic function in the plane is made in [M].

Next we present an example which shows that Theorem C is essentially sharp.

4.1. *Example.* For each $p>1$ and $0<\alpha<1$ there is a compact set K of the unit ball B of $\mathbf{R}^n$ with $\dim_H(K)=0$ and with

$$\int_{K(1)\setminus K} d(x,K)^{\gamma}\,dm(x)<\infty \tag{4.2}$$

for some $\gamma<0$ and an $\mathcal{A}$-harmonic function (p-harmonic), $\mathcal{A}(x,\xi)=|\xi|^{p-2}\xi$, which does not extend to an $\mathcal{A}$-harmonic function of B.

In fact, our construction shows that for $1<p<n$ one may take any number γ, $\gamma>-(1-\alpha)(p-1)/(n-1)$. Fix $1<p<n$ and $0<\alpha<1$. Then the function $v(x)=|x|^{(p-n)/(p-1)}$ is $\mathcal{A}$-harmonic in $B\setminus\{0\}$. Set

$$R_j=B(2^{-j})\setminus\overline{B}(2^{-j-1}),\quad j=1,2,\dots\,. \tag{4.3}$$

Select for each j a set K_j consisting of N_j points in R_j with

$$d(x,K_j)\le|x|^{a},\quad a=\frac{(n-1)}{(p-1)(1-\alpha)}, \tag{4.4}$$

for each $x\in R_j$ and

$$N_j\le c(n)2^{bj},\quad b=\frac{n(n-1)}{(p-1)(1-\alpha)}-n; \tag{4.5}$$

this follows, for example, from a packing argument via the Besicovitch covering theorem. Define

$$K=\{0\}\cup\bigcup_{j=1}^{\infty}K_j \tag{4.6}$$

and let u be the restriction of v to $B\setminus K$. Then K is a compact, countable subset of B and, in particular, $\dim_H(K)=0$. Moreover, u is $\mathcal{A}$-harmonic in $B\setminus K$ with

$$|\nabla u(x)|\le c_1|x|^{(1-n)/(p-1)}, \tag{4.7}$$

where $c_1=(p-n)/(p-1)$, and since

$$d(x,K)\le |x|^a, \quad a=\frac{(n-1)}{(p-1)(1-\alpha)},$$

see (4.3), (4.5) and (4.6), the mean value theorem shows that $u\in\operatorname{locLip}_\alpha(B\setminus K)$.

Since u does not extend to an $\mathcal{A}$-harmonic function of B, it suffices to verify that (4.2) holds for some $\gamma<0$. Clearly, it is enough to show that there is $\lambda>0$ such that

$$m(K(r))\le c_2 r^\lambda;$$

then (4.2) holds for any $\gamma>-\lambda$. Fix $0<r<1$. We may assume that $r=2^{-m}$ for some positive integer m. Now

$$K(r)\subset B(2^{-ma+1})\cup\bigcup_{j\le ma} K_j(2^{-m}),$$

where $a=(1-\alpha)(p-1)/(n-1)$. Thus (4.5) gives

$$m(K(r))\le c_2 2^{-mna}=c_2 r^\lambda,$$

here $c_2=c_2(n)$ and $\lambda=n(1-\alpha)(p-1)/(n-1)$. The claim follows.

The construction for $p\ge n$ is similar and left to the reader (for $p=n$ begin with $v(x)=\log(1/|x|)$); see also the comments following Theorem G.

5. Removability in the BMO class

A function $u\in L^1_{\mathrm{loc}}(\Omega)$ is of bounded mean oscillation in Ω if

$$\|u\|_*=\sup_{Q\subset\Omega}\frac{1}{m(Q)}\int_Q |u-u_Q|\,dm<\infty;$$

here Q is any cube and u_Q is the average of u over Q, i.e.

$$u_Q=\frac{1}{m(Q)}\int_Q u\,dm.$$

If $\|u\|_*<\infty$, then we say that $u\in\mathrm{BMO}(\Omega)$. It is a well known consequence of the John–Nirenberg lemma that

$$\sup_{Q\subset\Omega}\left(\frac{1}{m(Q)}\int_Q |u-u_Q|\,dm\right)^{1/p}\le c_1\|u\|_*$$

holds for any $u\in\mathrm{BMO}(\Omega)$ for all $p>1$.

Proof of Theorem D. Let $u\in\mathrm{BMO}(\Omega\setminus E)$ be $\mathcal{A}$-harmonic in $\Omega\setminus E$. If Q is a cube in a Whitney decomposition of $\Omega\setminus E$, then the standard Caccioppoli estimate yields

$$\int_{2Q}|\nabla u|^p|\psi|^p\,dm\le c\int_{2Q}|u-u_Q|^p|\nabla\psi|^p\,dm \tag{5.1}$$

for any $\psi\in C_0^\infty(2Q)$, where c depends only on p, e_A, and n (see [S, pp. 255–261], [BI, p. 290], [GLM], [HKM]). Choosing ψ such that $\psi=1$ on Q and $|\nabla\psi|\le 2/l(Q)$ we obtain from (5.1)

$$\int_Q|\nabla u|^p\,dm\le c_1 l(Q)^{-p}\int_{2Q}|u-u_{2Q}|^p\,dm\le C_1 m(Q)^{(n-p)/n}\|u\|_*.$$

By Lemma 3.3, u extends to an $\mathcal{A}$-supersolution of Ω and the same reasoning applied to $-u$ yields that $-u$ extends to an $\mathcal{A}$-supersolution in Ω. This means that u extends to an $\mathcal{A}$-harmonic function of Ω, as required.

Theorem E is sharp at least for the borderline case $p=n$. Then only $E=\emptyset$ satisfies (1.4) for $\alpha=0$. On the other hand, the function $u(x)=\log(1/|x|)$ is $\mathcal{A}$-harmonic (n-harmonic), $\mathcal{A}(x,\xi)=|\xi|^{n-2}\xi$, in $\mathbf{R}^n\setminus\{0\}$ and $u\in\mathrm{BMO}(\mathbf{R}^n)$ ([RR, p. 5]), but u is not $\mathcal{A}$-harmonic in $\mathbf{R}^n$.

Added in proof. T. Kilpeläinen (Hölder continuity of solutions to quasilinear elliptic equations involving measures) has constructed examples showing that the exponent b in (1.5) is sharp.

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