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A general discrepancy theorem

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Dedicated to Professor Meinardus on the occasion of his sixtyfifth birthday.

1. Introduction

Let E be a Jordan curve or a Jordan arc and let σ be a signed measure on E. We define the *discrepancy* of σ by

$$D[\sigma] := \sup |\sigma(J)| \tag{1.1}$$

where the supremum is taken over all subarcs $J \subseteq E$. If $\{\nu_n\}$ is a sequence of Borel measures on E converging to a Borel measure μ in the sense that $D[\nu_n - \mu] \to 0$ as $n \to \infty$, then $\{\nu_n\}$ converges to μ in the weak-star sense. Thus, the discrepancy *between* ν_n *and* μ defined by $D[\nu_n - \mu]$ serves as a measurement on the rate of the weak-star convergence. Therefore, many mathematicians ([1]–[4], [7], [8], [10], [12], [13], [15], [16], [18], [22], [30], [32], [34]) have studied the notion of discrepancy of a signed measure under various conditions. Often, the discrepancy is estimated in terms of the logarithmic potential $U(\sigma, z)$ defined for any signed measure σ by

$$U(\sigma, z) := \int \log \frac{1}{|z-t|}\, d\sigma(t). \tag{1.2}$$

A typical result is the following estimate due to Ganelius.

Theorem 1.1 ([13]). *Let ν be a positive unit Borel measure on the unit circle and $d\mu := dt/2\pi$. Then*

$$D[\nu - \mu] \le c \,|\inf_{|z|=1} U(\nu - \mu, z)|^{1/2}, \tag{1.3}$$

where c is an absolute constant independent of ν.

For example, consider a monic polynomial p_n of degree n having all of its zeros on the unit circle and let ν_n be the unit measure associating the mass $1/n$ with each of these zeros. If

$$A_n := \max_{|z|=1} |p_n(z)| \tag{1.4}$$

then

$$U(\nu_n - \mu, z) = -\frac{1}{n} \log |p_n(z)| + \log |z| \geq -\frac{\log A_n}{n}, \quad |z| = 1.$$

Hence, (1.3) yields an earlier estimate of Erdős and Turán, namely,

$$D[\nu_n - \mu] \leq c\sqrt{\frac{\log A_n}{n}}. \tag{1.5}$$

In particular, if $n \geq 2$ and $p_n(z) := z^n - 1$, then Theorem 1.1 implies that

$$D[\nu_n - \mu] \leq c(2/n)^{1/2},$$

whereas it is obvious that

$$D[\nu_n - \mu] \leq n^{-1}.$$

Nevertheless, Theorem 1.1 cannot be improved if the estimate is based on (1.4) only.

Recently, it was noticed that the estimate (1.5) can be considerably strengthened if, in addition to (1.4), a bound of the form

$$|p_n'(z_i)| \geq B_n^{-1} \tag{1.4a}$$

is known for all zeros z_i $(1 \leq i \leq n)$ of p_n, where $B_n > 0$.

Theorem 1.2 ([1]). *Let, for integer $n \geq 2$, p_n be a monic polynomial with n simple zeros on the unit circle such that* (1.4) *and* (1.4a) *hold. Let μ be as in Theorem* 1.1 *and ν_n be the measure that associates the mass $1/n$ with each of the zeros of p_n. Then there exists a positive constant c, independent of n, such that*

$$D[\nu_n - \mu] \leq c \log n \frac{\log C_n}{n}, \tag{1.6}$$

where

$$C_n := \max(A_n, B_n, n). \tag{1.7}$$

An analogous result is true for the case of an interval instead of the unit circle.

Theorem 1.3 ([1]). *Let p_n be a monic polynomial of degree n with simple zeros x_i on $[-1,1]$ such that*

$$\max_{-1\le x\le 1}|p_n(x)|\le A_n/2^n, \tag{1.8}$$

$$|p_n'(x_i)|\ge B_n^{-1}/2^n, \quad 1\le i\le n. \tag{1.8a}$$

Let ν_n be the measure which associates the mass $1/n$ with each of the points x_i and $d\mu(x):=dx/\pi\sqrt{1-x^2}$. Then, with

$$C_n:=\max(A_n,B_n,n),$$

we have

$$D[\nu_n-\mu]\le c\log n\frac{\log C_n}{n}, \tag{1.9}$$

where c is a positive constant independent of p_n.

In a recent paper [34], V. Totik has obtained sharp estimates in Theorem 1.3. He proved that

$$D[\nu_n-\mu]\le c\frac{\log C_n}{n}\log\frac{n}{\log C_n}, \tag{1.10}$$

if $\log C_n/n$ is less than a fixed constant less than 1, and that (1.10) is best possible.

The proof of (1.6) in [1] is of function theoretical nature and uses essentially the fact that the conditions (1.4), (1.4a) lead to

$$|U(\nu_n-\mu,z)|\le c\frac{\log C_n}{n}, \quad |z|\ge 1+n^{-\varkappa}, \tag{1.11}$$

where $\varkappa>0$ and the constant c may depend upon $\varkappa$ but is independent of n. Totik's proof is based much more on potential theory, but again the essential inequality of the form (1.11) is used.

The main object of this paper is to demonstrate how estimates such as (1.6), respectively (1.10), can be obtained in the case of Jordan curves and Jordan arcs, based only on the knowledge of a bound similar to (1.11). We shall apply our main theorem to get estimates for the distribution of Fekete points, extreme points of polynomials of best approximation and zeros of orthogonal polynomials on the unit circle and on compact intervals.

In Section 2, we develop some notation and formulate our main theorems. In Section 3, we discuss the applications. The proofs of the new results in Sections 2 and 3 are given in Section 4.

2. Main results

Let $K \subset \mathbf{C}$ be compact and $\mathcal{M}(K)$ denote the collection of all positive unit Borel measures supported on K. For $\sigma \in \mathcal{M}(K)$ the energy of σ is defined by the formula

$$I[\sigma] := \int U(\sigma, z)\, d\sigma(z)$$

where $U(\sigma, z)$ is the logarithmic potential of σ. If

$$W(K) := \inf_{\sigma \in \mathcal{M}(K)} I[\sigma]$$

then the (logarithmic) *capacity* of K is defined by

$$\operatorname{cap}(K) := \exp(-W(K)). \tag{2.1}$$

If $\operatorname{cap}(K) > 0$ then there exists (cf. [35, Chapter III]) a unique measure $\mu_K \in \mathcal{M}(K)$ such that

$$I[\mu_K] = W(K). \tag{2.2}$$

The measure μ_K is called the *equilibrium measure* of K. Let G be the Green's function of the unbounded component $\Omega(K)$ of $\mathbf{C} \cup \{\infty\} \setminus K$. If $\operatorname{cap}(K) > 0$, then G is connected with the logarithmic potential of μ_K by

$$U(\mu_K, z) = -G(z) - \log \operatorname{cap}(K), \quad z \in \Omega(K) \tag{2.3}$$

([35, Theorem III.37, p. 82]). Moreover, G tends to zero at all regular points of the boundary of $\Omega(K)$. In particular, if K is a Jordan curve or Jordan arc, then G can be continuously extended to K such that $G(z) = 0$ for $z \in K$ and (2.3) holds also for $z \in K$.

In the sequel, E will denote a Jordan curve or a Jordan arc of the class C^{1+}, i.e. the curve (arc) E is rectifiable and the first derivatives of the coordinates with respect to the arclength satisfy a Hölder condition with some positive exponent. Let Φ denote the conformal mapping of $\Omega := \Omega(E)$ onto

$$\Delta := \{ t \in \mathbf{C} \cup \{\infty\} : |t| > 1 \} \tag{2.4}$$

such that $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$. Then it is well known ([19, p. 172]) that

$$\log |\Phi(z)| = G(z) \quad \text{for} \quad z \in \overline{\Omega}. \tag{2.5}$$

For any $\alpha \geq 1$, let

$$\Gamma_\alpha := \{ z \in \mathbf{C} : G(z) = \log \alpha \} \tag{2.6}$$

denote the level curve of the Green's function G. Our main discrepancy theorem uses an upper bound

$$\varepsilon(\alpha) \geq \max_{z \in \Gamma_\alpha} |U(\sigma, z)| \tag{2.7}$$

of the modulus of the logarithmic potential of σ on such level lines Γ_α.

Theorem 2.1. *Let E be a Jordan curve or a Jordan arc of class C^{1+}, $\sigma=:\sigma^+-\sigma^-$ be a signed measure on E with positive part σ^+, negative part σ^- and $\sigma^+(E)=\sigma^-(E)=1$. Moreover, let $M>0$, $0<\gamma\le 1$ be constants such that for all subarcs J of E,*

$$\sigma^+(J)\le M\left(\int_J ds\right)^\gamma. \tag{2.8}$$

Then there exists a constant $c>0$ depending only on E, M, γ such that

$$D[\sigma]\le c\varepsilon(\alpha)\log(1/\varepsilon(\alpha)) \tag{2.9}$$

for all α with $\alpha\le 1+\varepsilon(\alpha)^{1+1/\gamma}$ and $\varepsilon(\alpha)<1/e$.

We remark that the function

$$f(\varepsilon):=\varepsilon\log(1/\varepsilon)$$

is monotonically increasing for $0<\varepsilon\le 1/e$. Hence,

$$\varepsilon(\alpha)\log(1/\varepsilon(\alpha))<e^{-1}<1$$

for $\varepsilon(\alpha)<1/e$.

In the applications in Section 3, the following consequence of Theorem 2.1 will be especially useful.

Theorem 2.2. *Let E be as in Theorem 2.1 and p_n a monic polynomial of degree n with zeros $z_i\in E$, $1\le i\le n$, such that*

$$\max_{z\in E}|p_n(z)|\le A_n\operatorname{cap}(E)^n, \tag{2.10}$$

$$|p_n'(z_i)|\ge B_n^{-1}\operatorname{cap}(E)^n, \tag{2.11}$$

$$C_n:=\max(A_n,B_n,n)\le e^{n/e}. \tag{2.12}$$

Let ν_n denote the measure which associates the mass $1/n$ with each of the zeros z_i. Then

$$D[\nu_n-\mu_E]\le c\frac{\log C_n}{n}\log\frac{n}{\log C_n} \tag{2.13}$$

where c is a positive constant depending only on E.

3. Applications

In the sequel, the symbols $c, c_1, \ldots$ will denote positive constants depending only on the Jordan curve (or Jordan arc) E.

Fekete points

Let E be a Jordan curve or arc as in Section 2 and $\mathcal{F}_n(E)$ be any n-point subset S of E for which the Vandermonde expression

$$V(S) := \left\{ \prod_{\substack{z,t \in S \\ z \neq t}} |z-t| \right\}^{1/2}$$

is as large as possible. The points in any such $\mathcal{F}_n(E)$ are called *Fekete points* of E and they are related to the capacity by

$$\lim_{n\to\infty} V(\mathcal{F}_n(E))^{2/n(n-1)} = \operatorname{cap}(E). \tag{3.1}$$

Let ν_n be the measure that associates the mass $1/n$ with each of the Fekete points in $\mathcal{F}_n(E)$. A previous result of Kleiner [18] implies that

$$D[\nu_n - \mu_E] \le c \frac{\log n}{\sqrt{n}}.$$

Pommerenke ([27], [28]) proved for analytic curves E that the distribution of the Fekete points is determined by a fixed analytic function and as a consequence that the optimal bound

$$D[\nu_n - \mu_E] \le c/n$$

can be obtained in this case. For smooth curves E, Theorem 2.2 yields a result which is not far away from the Pommerenke estimate for analytic curves.

Theorem 3.1. *Let E be as in Theorem 2.1 and for each integer $n \ge 2$, let ν_n denote the unit measure associated with an n-th Fekete point set of E. Then*

$$D[\nu_n - \mu_E] \le c \frac{(\log n)^2}{n}. \tag{3.2}$$

Extreme points

Next, we give an application of Theorem 2.2 to the distribution of extreme points in best complex polynomial approximation. If E is a Jordan curve, then let K be the closed region bounded by E. If E is a Jordan arc, then we define $K:=E$. We consider a continuous function f on K which is analytic in the two dimensional interior of K. Let p_n^* be the best Chebyshev approximation to f from the class Π_n of polynomials of degree at most n, i.e.,

$$\|f-p_n^*\|_E = \min_{p\in\Pi_n} \|f-p\|_E,$$

where $\|\cdot\|_E$ denotes the supremum norm of E. The distribution of the points in the extreme point set

$$A_n(f) := \{\, z\in E : |f(z)-p_n^*(z)| = \|f-p_n^*\|_E \,\} \tag{3.3}$$

was studied by Blatt, Saff and Totik in [4]. Let $\mathcal{F}_{n+2}(A_n(f))$ be any $n+2$ point Fekete set of $A_n(f)$ and let ν_{n+2} be the measure that associates the mass $1/(n+2)$ with each point of $\mathcal{F}_{n+2}(A_n(f))$. It was shown in [4] that a subsequence $\{\nu_{n_k+2}\}$ converges in the weak star sense to the equilibrium measure μ_E. Moreover, in the case when E is a Jordan curve of class C^{1+},

$$D[\nu_{n_k+2}-\mu_E] \le c\frac{\log n_k}{\sqrt{n_k}}. \tag{3.4}$$

In [4], it was essential that E be a *Jordan curve* because a technique of Kleiner [18] was used as an important tool in the proof. In the case when E is a *Jordan arc,* only the estimate

$$D[\nu_{n_k+2}-\mu_E] \le c\left(\frac{\log n_k}{n_k}\right)^{1/3} \tag{3.5}$$

is known [2] so far. As an application of Theorem 2.2, we give an estimate which is slightly weaker than (3.4), but sharper than (3.5), and is applicable in both situations.

Theorem 3.2. *Let E be as in Theorem 2.1, K be the closed Jordan region if E is a Jordan curve and $K=E$ when E is a Jordan arc. Let f be analytic in the interior of K and continuous on K and let $A_n(f)$ be defined by (3.3). If ν_{n+2} denotes the measure that associates the mass $1/(n+2)$ with each of the Fekete points in some $\mathcal{F}_{n+2}(A_n(f))$, then there exist infinitely many integers n satisfying*

$$D[\nu_{n+2}-\mu_E] \le c\frac{(\log n)^2}{\sqrt{n}}. \tag{3.6}$$

Orthogonal polynomials on the unit circle

Next, we study orthogonal polynomials on the unit circle. Let τ be a positive, unit Borel measure on $E:=\{z:|z|=1\}$ whose support is an infinite set. Then there exists [33] an infinite sequence of polynomials

$$\omega_n(z):=\omega_n(\tau,z):=\varkappa_n z^n+\ldots\in\Pi_n,\quad \varkappa_n>0 \tag{3.7}$$

such that

$$\int \omega_n\omega_m\,d\tau=\delta_{n,m},\quad n,m=0,1,2,\ldots\,. \tag{3.8}$$

For every integer $n\geq 1$, the zeros $\{z_{k,n}\}$ of ω_n are simple and lie in $|z|<1$. The asymptotic distribution of these zeros was recently studied in [23], [24]. In order to apply Theorem 2.1, we define the sequence ν_n of measures on the unit circle to be the balayage measures associated with these zeros as follows. Let

$$z_{k,n}=:r_{k,n}\exp(it_{k,n}),\quad k=1,\ldots,n,\ n=1,2,\ldots\,. \tag{3.9}$$

With the Poisson kernel

$$P(r,\theta):=\frac{1-r^2}{1-2r\cos\theta+r^2} \tag{3.10}$$

we define for any Borel measurable subset B of the unit circle

$$\nu_n(B):=\frac{1}{2\pi n}\sum_{k=1}^{n}\int_B P(r_{k,n},t-t_{k,n})\,dt. \tag{3.11}$$

We observe that if f is any function continuous on $|z|\leq 1$ and harmonic on $|z|<1$, then

$$\int f\,d\nu_n=\frac{1}{n}\sum_{k=1}^{n}f(z_{k,n}).$$

In particular,

$$U(\nu_n,z)=\frac{1}{n}\log\frac{\varkappa_n}{\omega_n(z)},\quad |z|>1. \tag{3.12}$$

Equation (3.12) persists on the unit circle as well. The results in [23] imply that, under some mild conditions on the *reflection coefficients* $\varkappa_k^{-1}|\omega_k(0)|$, the sequence ν_n converges to μ_E in the weak star sense, where we recall that $d\mu_E=dt/2\pi$.

Theorem 3.3. *With the measures ν_n and μ_E defined as above, we have*

$$D[\nu_n-\mu_E]\le c\log n\left\{\frac{\log(1+\varkappa_n)}{n}+\frac{1}{n}\sum_{k=0}^{n}\varkappa_k^{-1}|\omega_k(0)|+\frac{\log n}{n}\right\} \tag{3.13}$$

where c is an absolute constant.

Remarks. (1) If $\limsup_{n\to\infty}\varkappa_n^{-1}|\omega_n(0)|<1$, the relation

$$\varkappa_n^2=\varkappa_0^2\left\{\prod_{k=0}^{n-1}\left(1-\frac{|\omega_k(0)|^2}{\varkappa_k^2}\right)\right\}^{-1} \tag{3.14}$$

can be used to express (3.13) in the terms of the reflection coefficients as follows:

$$D[\nu_n-\mu_E]\le c\log n\left\{\frac{1}{n}\sum_{k=0}^{n}\varkappa_k^{-1}|\omega_k(0)|+\frac{\log n}{n}\right\}. \tag{3.15}$$

In the case when $\tau'>0$ a.e., Rahmanov [29] and Maté, Nevai and Totik [20] have proved that $\varkappa_n^{-1}(0)\omega_n(0)\to0$ as $n\to\infty$. Thus, the estimate (3.15) is applicable in this important case. However, using (3.14) it is easy to construct examples where (3.13) does not reduce to (3.15).

(2) If $\log\tau'$ is integrable, then $\varkappa_n$ is bounded and $\sum_{k=0}^{\infty}|\omega_k(0)|^2<\infty$ (cf. [14, Theorem 8.2]). Hence, (3.15) yields

$$D[\nu_n-\mu_E]\le c\frac{\log n}{\sqrt{n}}. \tag{3.16}$$

(3) In the "Jacobi case" where the measure τ is given by $d\tau:=|\sin(t/2)|^{2p}\,dt$, $p>0$, it is known [5], [24] that $\varkappa_k^{-1}\omega_k(0)=p/(k+p)$, so that (3.15) yields

$$D[\nu_n-\mu_E]\le c\frac{(\log n)^2}{n}. \tag{3.17}$$

The proof of Theorem 3.3 will also show that the estimate (3.17) is true generally when $\tau'\ge m>0$.

Orthogonal polynomials on a real interval

Let $E:=[-1,1]$ and τ be a positive, unit Borel measure on E with finite moments, i.e.,

$$\int_{-1}^{1}|x|^n\,d\tau(x)<\infty,\quad n=0,1,\dots\,.$$

Moreover, we assume that the support S of τ is infinite. Then there exists a unique system of orthonormalized polynomials

$$p_n(x) := p_n(\tau, x) := \gamma_n x^n + \ldots \in \Pi_n, \quad \gamma_n > 0, \tag{3.18}$$

such that

$$\int_{-1}^{1} p_n p_m d\tau = \delta_{n,m}, \quad n, m = 0, 1, \ldots . \tag{3.19}$$

For each integer $n \geq 1$, the polynomial p_n has n simple zeros in $[-1, 1]$.

Theorem 3.4. *Let τ be a positive, unit Borel measure on $[-1,1]$ with finite moments. Moreover, let the support S of τ be a finite union of compact, non-degenerate intervals. Let ν_n be the measure that associates the mass $1/n$ with each of the zeros of the orthogonal polynomial p_n. Then for all $n \geq 2$,*

$$D[\nu_n - \mu_S] \leq c \log n \left\{ \frac{\log n}{n} + \frac{\log(1 + \|p_n\|_S)}{n} \right\} \tag{3.20}$$

where c is a positive constant depending only on S.

The measure τ for which $\|p_n\|_S^{1/n} \to 1$ as $n \to \infty$ is sometimes known as a completely regular measure [30]. A typical example is where the Radon–Nykodym derivative τ' satisfies $\tau' > 0$ almost everywhere in $[-1,1]$ ([9]). When τ is completely regular, it is known [30] that $\nu_n \to \mu_S$ in the weak star sense. Under some mild additional hypothesis, (3.20) gives a rate of this convergence.

Corollary 3.5. *Under the conditions of Theorem 3.4, let $\tau' \geq \varkappa > 0$ on S. Then*

$$D[\nu_n - \mu_S] \leq c \frac{(\log n)^2}{n}. \tag{3.21}$$

Corollary 3.5 contains as a special case the estimate in [1] if $\tau' \geq \varkappa > 0$ on $[-1,1]$. In addition, Theorem 3.4 includes the case of generalized Jacobi weights. If $\log \tau'$ is integrable then in view of [14, p. 157], the right hand side of (3.20) reduces to $c(\log n)^2/\sqrt{n}$. This estimate holds also for the Pollaczek weights.

Conjecture

Let τ be any measure on a convex compact set $K \subset \mathbf{C}$, let $\{p_n(\tau, z)\}$ be the system of orthonormalized polynomials on K with respect to τ. For each integer $n \geq 1$, let ν_n be the measure that associates the mass $1/n$ with each of the zeros of $p_n(\tau, z)$, let $\tilde{\nu}_n$ be the balayage of ν_n to the outer boundary of K (i.e. the boundary of $\Omega(K)$) and let μ_K be the equilibrium measure of K. Then

$$D[\nu_n - \mu_K] \leq c \frac{\log(n(1 + \|p_n\|_K))}{n} \log \frac{n}{\log(n(1 + \|p_n\|_K))}.$$

4. Proofs

First, we recall some well known facts concerning the inverse conformal mapping Ψ (inverse to Φ) from Δ onto the exterior Ω of a closed Jordan curve E of class C^{1+}. According to a theorem due to Carathéodory (cf. [25, Theorem 9.10, p. 281]), the mapping Ψ can be extended to a homeomorphism of $\overline{\Delta}$ onto $\overline{\Omega}$. Moreover, a theorem due to Warschawski (cf. [25, Theorem 10.2, p. 298]) shows that the derivative Ψ' can be extended continuously to $\overline{\Delta}$ such that

$$\Psi'(t)\neq 0, \quad |t|=1. \tag{4.1}$$

Hence, for every $z\in E$,

$$\operatorname{dist}(z,\Gamma_\alpha)\leq c(\alpha-1), \quad \alpha\geq 1 \tag{4.2}$$

where c is a positive constant independent of z and α. Next, let us fix a point z_0 in the interior of E and define the conformal mappings ϕ_α of the interior of Γ_α onto the interior D of the unit circle,

$$D:=\{\, z\in \mathbf{C}:|z|<1\,\} \tag{4.3}$$

such that

$$\phi_\alpha(z_0)=0, \quad \phi_\alpha'(z_0)>0. \tag{4.4}$$

Let ψ_α denote the inverse mapping of ϕ_α. Then, again, ψ_α can be extended to a homeomorphism of $\overline{D}$ and the derivative ψ_α' can be extended as a continuous function on $\overline{D}$ which satisfies

$$\psi_\alpha'(t)\neq 0, \quad |t|=1.$$

It is possible to obtain a uniform bound in the above formula, independent of α, by using results of Warschawski about the behaviour of the conformal mapping and its derivative for variable regions.

Lemma 4.1 ([36, Theorem IV, p. 314]). *Let E be a Jordan curve of class C^{1+} and $\alpha\geq 1$ be fixed. Then there exists a constant $c>0$ such that for every $\beta\geq 1$, $|\alpha-\beta|<1$ and $|t|\leq 1$,*

$$|\psi_\alpha(t)-\psi_\beta(t)|\leq c|\alpha-\beta|, \quad |\psi_\alpha'(t)-\psi_\beta'(t)|\leq c|\alpha-\beta|\log\left(\frac{\pi}{|\alpha-\beta|}\right). \tag{4.5}$$

A direct application of this lemma leads to

Corollary 4.2. *Under the conditions of Lemma* 4.1, *there exist constants* $c, c_1 > 0$ *such that*

$$c \le |\psi'_\alpha(t)| \le c_1, \quad |t| \le 1, \quad 1 \le \alpha \le 2. \tag{4.6}$$

Next, we recall a well known estimate about the continuity properties for harmonic functions.

Lemma 4.3. *Let h be a 2π-periodic, continuously differentiable function. Let H denote the function harmonic in D and continuous in $\overline{D}$ which coincides with h on the unit circle. Then there exists an absolute constant $c > 0$ such that*

$$|H(t) - H(u)| \le c(1-r)\log\left(\frac{1}{1-r}\right)\|h'\| \tag{4.7}$$

where $\|h'\| := \max_{|\zeta|=1} |h'(\zeta)|$, $t = e^{i\zeta}$, $u = re^{i\zeta}$ *and* $r \ge \frac{1}{2}$.

While Lemma 4.3 is quite well known, we are unable to locate a reference where this is explicitly stated. For this reason and for the sake of completeness, we include a proof.

Proof of Lemma 4.3. We may assume that $t=1$, i.e. $\zeta = 0$. Then the Poisson integral formula implies that

$$|H(u) - H(1)| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |h(\theta) - h(0)| \frac{1-r^2}{1-2r\cos\theta + r^2}\, d\theta. \tag{4.8}$$

Let

$$I_1 := \frac{1}{2\pi}\int_{|\theta| \le 1-r} |h(\theta) - h(0)| \frac{1-r^2}{1-2r\cos\theta + r^2}\, d\theta.$$

Then it is easy to see that

$$I_1 \le (1-r)\|h'\|. \tag{4.9}$$

Let

$$I_2 := \frac{1}{2\pi}\int_{1-r \le |\theta| \le \pi} |h(\theta) - h(0)| \frac{1-r^2}{1-2r\cos\theta + r^2}\, d\theta.$$

Since

$$1 - 2r\cos\theta + r^2 \ge (1-r)^2 + 4r\sin^2(\theta/2) \ge c\theta^2,$$

we obtain

$$I_2 \le c(1-r)\|h'\| \int_{1-r}^{\pi} \frac{d\theta}{\theta} \le c(1-r)\log\left(\frac{1}{1-r}\right)\|h'\|. \tag{4.10}$$

We get (4.7) in view of (4.8), (4.9) and (4.10). □

Proof of Theorem 2.1. In this proof, $c, c_1, \ldots$ will denote positive constants which may depend only upon E, M and γ, but their values may be different at different occurrences, even within the same formula. First, let us note that the case of a Jordan arc E of class C^{1+} can be reduced to the case of a Jordan curve of class C^{1+} as follows. There exists a Jordan curve $\widetilde{E}$ of class C^{1+} such that the given arc E is a subarc of $\widetilde{E}$. If $\widetilde{G}(z)$ is the Green's function of the exterior of $\widetilde{E}$ then it is well known that $\widetilde{G}(z)<G(z)$ for all z exterior to $\widetilde{E}$. Hence, the level line

$$\widetilde{\Gamma}_\alpha := \{ z \in \mathbf{C} : \widetilde{G}(z) = \log \alpha \}$$

surrounds $\widetilde{E}$ and lies exterior to Γ_α. Then the maximum principle for harmonic functions yields

$$\tilde{\varepsilon}(\alpha) := \max_{z \in \widetilde{\Gamma}_\alpha} |U(\sigma, z)| \le \varepsilon(\alpha).$$

Since the discrepancy of σ on $\widetilde{E}$ is the same as on E, Theorem 2.1 follows for E if it is proved for $\widetilde{E}$. Therefore, without loss of generality, we may assume that E is a closed Jordan curve of class C^{1+}.

The basic idea of the proof of Theorem 2.1 is now the following. Given a subarc J of E, we construct a function h_o harmonic in the exterior of Γ_α and a function h_i harmonic in the interior of Γ_α such that h_o, $\operatorname{grad} h_o$, h_i and $\operatorname{grad} h_i$ can be extended continuously to Γ_α with

$$h_o(z) = h_i(z), \quad z \in \Gamma_\alpha$$

and the restriction of h_i to Γ_α is an approximation of the characteristic function of J. Then, applying the technique used by Sjögren [32, p. 67] we obtain

$$\frac{1}{2\pi} \int_{\Gamma_\alpha} \left[\frac{\partial h_o}{\partial n_o} + \frac{\partial h_i}{\partial n_i} \right] U(\tilde{\sigma}, z)\, ds = \int_{\Gamma_\alpha} h_i \, d\tilde{\sigma} \tag{4.11}$$

where n_o and n_i denote the outward and inward normals of Γ_α and $\tilde{\sigma}$ denotes the balayage of σ onto Γ_α. Since Γ_α is analytic, the balayage problem is solvable in the strict sense [19, p. 210, Theorem 3.4] and therefore,

$$U(\tilde{\sigma}, z) = U(\sigma, z) \quad \text{for all } z \in \Gamma_\alpha$$

and

$$\int_{\Gamma_\alpha} h_i \, d\tilde{\sigma} = \int_E h_i \, d\sigma.$$

Hence,

$$\int_E h_i \, d\sigma = \frac{1}{2\pi} \int_{\Gamma_\alpha} \left[\frac{\partial h_o}{\partial n_o} + \frac{\partial h_i}{\partial n_i}\right] U(\sigma, z) \, ds \tag{4.12}$$

and approximation arguments will lead to the estimate of $\sigma(J)$ on the left hand side of (4.12). Together with estimates of the right hand side of (4.12), we shall finally obtain the result of Theorem 2.1.

Before constructing the functions h_o and h_i, we remark that it is enough to prove that

$$\sigma(J) \geq -c\varepsilon(\alpha) \log(1/\varepsilon(\alpha)) \tag{4.13}$$

for any subarc J of E. Let

$$\Phi(J) = \{ t : t = e^{i\xi}, a \leq \xi \leq b \}. \tag{4.14}$$

Clearly, it is sufficient to prove (4.13) for all J with $b-a \leq \pi$.

Let us denote by $\chi(\xi)$ the characteristic, 2π-periodic function of the interval $[a-\delta/2, b+\delta/2]$ where

$$\delta := \varepsilon(\alpha)^{1/\gamma} \log(1/\varepsilon(\alpha)). \tag{4.15}$$

We set

$$u_\delta(\xi) := \frac{4}{\delta^2} \int_{-\delta/2}^{\delta/2} (\delta/2 - |s|) \chi(\xi - s) \, ds. \tag{4.16}$$

Then u_δ is continuously differentiable and u'_δ satisfies the Lipschitz condition

$$|u'_\delta(\xi) - u'_\delta(\tilde{\xi})| \leq \frac{4}{\delta^2} |\xi - \tilde{\xi}|. \tag{4.17}$$

Moreover,

$$0 \leq u_\delta(\xi) \leq 1, \tag{4.17a}$$

$$u_\delta(\xi) = 1, \quad \xi \in [a, b] \mod 2\pi, \tag{4.17b}$$

$$u_\delta(\xi) = 0, \quad \xi \notin [a-\delta, b+\delta] \mod 2\pi, \tag{4.17c}$$

$$0 \leq u'_\delta(\xi) \leq 2/\delta, \ \xi \in [a-\delta, a], \text{ and} \tag{4.17d'}$$

$$-2/\delta \leq u'_\delta(\xi) \leq 0, \ \xi \in [b, b+\delta], \tag{4.17d''}$$

$$\int_{-\pi}^{\pi} |u'_\delta(\xi)| \, d\xi \leq 2. \tag{4.17e}$$

Let m denote the integer part of $1/\delta$. Taking the convolution of u_δ with an appropriate Jackson kernel (cf. [21]), we get a trigonometric polynomial T of degree at most m^3 such that the following properties hold.

(4.18a) $$|u_\delta(\xi)-T(\xi)|\le cm^{-2}, \quad \xi\in\mathbf{R},$$

(4.18b) $$|u_\delta'(\xi)-T'(\xi)|\le cm^{-1}, \quad \xi\in\mathbf{R},$$

(4.18c) $$0\le T(\xi)\le 1, \quad \xi\in\mathbf{R},$$

(4.18d) $$T(\xi)\ge 1-c/m, \quad \xi\in[a,b] \mod 2\pi$$

(4.18e) $$T(\xi)\le c/m, \quad \xi\notin[a-\delta,b+\delta] \mod 2\pi,$$

(4.18f) $$\int_{-\pi}^{\pi}|T'(\xi)|\,d\xi\le c.$$

Moreover, we note for later use that because of (4.17c)–(4.17e) and the linearity of the convolution operator, the trigonometric polynomial T' can be expressed as a sum

(4.19) $$T'=T_1+T_2$$

where T_1, T_2 are trigonometric polynomials of degree at most m^3 such that

(4.20) $$1\le T_1(\xi)\le c/\delta, \quad -c/\delta\le T_2(\xi)\le -1, \quad \xi\in\mathbf{R}.$$

(4.20a) $$\int_{-\pi}^{\pi}|T_j(\xi)|\,d\xi\le c, \quad j=1,2.$$

Let p be an algebraic polynomial such that

(4.21) $$\operatorname{Re}p(e^{-i\xi})=T(\xi).$$

We define

(4.22) $$h_o(z):=\operatorname{Re}p\left(\frac{\alpha}{\Phi(z)}\right)$$

for $z\in\Gamma_\alpha$ and z exterior to Γ_α. Then h_o is harmonic in the exterior of Γ_α. Next, let h_i be the function, harmonic in the interior of Γ_α with boundary values

(4.23) $$h_i(z)=h_o(z), \quad z\in\Gamma_\alpha.$$

Due to a theorem of Kellogg [17], grad h_i is continuous on the closed interior of Γ_α. Of course, grad h_o is continuous on the closed exterior of Γ_α. Therefore, formula (4.12) is applicable and we have to estimate

$$\int_{\Gamma_\alpha} \frac{\partial h_o}{\partial n_o} U(\sigma, z)\, dz \quad \text{and} \quad \int_{\Gamma_\alpha} \frac{\partial h_i}{\partial n_i} U(\sigma, z)\, dz$$

and to compare $\int_E h_i\, d\sigma$ with $\sigma(J)$.

To begin with the last problem, let $z \in E$ be fixed and define

$$t = \Phi(z), \quad \tilde{t} = \alpha t = \alpha\Phi(z), \quad \tilde{z} = \Psi(\tilde{t}) \in \Gamma_\alpha.$$

Then, in view of (4.1)

$$|z - \tilde{z}| = |\Psi(t) - \Psi(\tilde{t})| \le c|t - \alpha t| = c(\alpha - 1) \tag{4.24}$$

and

$$\begin{aligned} h_i(\tilde{z}) &= h_i(\Psi(\alpha t)) = h_o(\Psi(\alpha t)) \\ &= \operatorname{Re} p\left(\frac{\alpha}{(\Phi\circ\Psi)(\alpha t)}\right) \\ &= \operatorname{Re} p(1/t) = T(\xi) \end{aligned}$$

where $t = e^{i\xi} = \Phi(z)$. Hence,

$$\begin{aligned} \int_E h_i(z)\, d\sigma(z) &= \int_E h_i(\tilde{z})\, d\sigma(z) + \int_E (h_i(z) - h_i(\tilde{z}))\, d\sigma(z) \\ &= \int_E T(\xi)\, d\sigma(z) + \int_E (h_i(z) - h_i(\tilde{z}))\, d\sigma(z) \end{aligned} \tag{4.25}$$

Because of (4.18c)–(4.18e), we get

$$\int_E T(\xi)\, d\sigma(z) \le \int_{J\cup J_1} d\sigma^+(z) - \left(1 - \frac{c}{m}\right) \int_J d\sigma^-(z) + \frac{c}{m} \int_{E\setminus(J\cup J_1)} d\sigma^+(z)$$

where

$$J_1 := \{\, z = \Psi(t) : t = e^{i\xi}, \xi \in [a-\delta, a] \cup [b, b+\delta]\,\}.$$

Then

$$\begin{aligned} \int_E T(\xi)\, d\sigma(z) &\le \sigma(J) + \frac{c}{m} + \int_{J_1} d\sigma^+(z) \\ &\le \sigma(J) + \frac{c}{m} + M\left(\int_{J_1} ds\right)^{\gamma}. \end{aligned}$$

Since

$$\int_{J_1} ds = \int_{\Phi(J_1)} |\Psi'(t)|\,|dt| \leq c\delta,$$

we obtain

$$\sigma(J) \geq \int_E T(\xi)\,d\sigma(z) - c(\delta+\delta^\gamma). \tag{4.26}$$

Next, we assert that

$$|h_i(z)-h_i(\tilde{z})| \leq \frac{c}{\delta}(\alpha-1)\log\frac{1}{\alpha-1}. \tag{4.27}$$

To prove this, we define

$$t_1 = \phi_\alpha(z), \quad t_2 = \phi_\alpha(\tilde{z}), \quad t_3 = \frac{\phi_\alpha(z)}{|\phi_\alpha(z)|} = t_1/|t_1|. \tag{4.28}$$

Then

$$t_1 - t_2 = \int_C \phi'_\alpha(y)\,dy$$

where C is the arc from z to $\tilde{z}$ on the trajectory orthogonal to the level lines of $G(z)$. Hence, (4.1) and Corollary 4.2 yield

$$|t_1-t_2| \leq |C| \max_{y\in C} |\phi'_\alpha(y)| \leq c(\alpha-1). \tag{4.29}$$

Let $z^*\in\Gamma_\alpha$ be such that $\operatorname{dist}(z,\Gamma_\alpha)=|z-z^*|$. Since $\operatorname{dist}(z,\Gamma_\alpha)\leq c(\alpha-1)$, we obtain by Corollary 4.2

$$c(\alpha-1) \geq |z-z^*| = |\psi_\alpha(t_1)-\psi_\alpha(t^*)| \geq c_1|t_1-t^*|$$

where $t^*=\phi_\alpha(z^*)$. Therefore,

$$|t_1-t_3| = 1-|t_1| \leq |t_1-t^*| \leq c(\alpha-1) \tag{4.30}$$

and by (4.29)

$$|t_2-t_3| \leq c(\alpha-1). \tag{4.31}$$

Now,

$$\begin{aligned} |h_i(z)-h_i(\tilde{z})| &= |(h_i\circ\psi_\alpha)(t_1)-(h_i\circ\psi_\alpha)(t_2)| \\ &\leq |H_\alpha(t_1)-H_\alpha(t_3)|+|H_\alpha(t_2)-H_\alpha(t_3)| \end{aligned} \tag{4.32}$$

where we have set

$$H_\alpha(t) := (h_i \circ \psi_\alpha)(t), \quad |t| \le 1. \tag{4.33}$$

Note that for $|t|=1$

$$H_\alpha(t) = \operatorname{Re} p\left(\frac{\alpha}{(\Phi\circ\psi_a)(t)}\right) = T(\theta)$$

where $e^{i\theta} = (1/\alpha)(\Phi\circ\psi_\alpha)(t)$. Consequently, if $t=e^{i\xi}$ then because of the definition of Φ and ψ_α, θ is an increasing function of ξ. Moreover,

$$\frac{d\theta}{d\xi} = (1/\alpha)e^{i(\xi-\theta)}\Phi'(\psi_\alpha(t))\psi'_\alpha(t).$$

Observing that the left hand side is real and positive, we get

$$\frac{d\theta}{d\xi} = (1/\alpha)|\Phi'(\psi_\alpha(t))\psi'_\alpha(t)|. \tag{4.34}$$

Because of (4.17d), (4.18b) and Corollary 4.2, it follows that

$$\left|\frac{\partial}{\partial\xi} H_\alpha(t)\right| \le c/\delta, \tag{4.35}$$

and Lemma 4.3, together with (4.30), yields

$$|H_\alpha(t_1) - H_\alpha(t_3)| \le \frac{c}{\delta}(\alpha-1)\log\frac{1}{\alpha-1}. \tag{4.36}$$

Moreover, we get by (4.31) and (4.35)

$$|H_\alpha(t_2) - H_\alpha(t_3)| \le \frac{c}{\delta}(\alpha-1). \tag{4.37}$$

The estimate (4.27) follows from (4.32), (4.36) and (4.37).

Thus, (4.25), (4.26) and (4.27) lead to

$$\sigma(J) \ge \int_E h_i(z)\,d\sigma(z) - c\left(\frac{1}{\delta}(\alpha-1)\log\frac{1}{\alpha-1} + \delta + \delta^\gamma\right). \tag{4.38}$$

Next, we want to estimate

$$\int_{\Gamma_\alpha} \frac{\partial h_i}{\partial n_i}(z) U(\sigma, z)\,ds.$$

The transformation $z=\psi_\alpha(t)$ leads to

$$\int_{\Gamma_\alpha} \frac{\partial h_i}{\partial n_i}(z)U(\sigma,z)\,ds=\int_{|t|=1}\frac{\partial h_i}{\partial n_i}(z)U(\sigma,z)|\psi'_\alpha(t)|\,|dt|. \tag{4.39}$$

Since

$$\frac{\partial h_i}{\partial n_i}(z)=\frac{1}{|\psi'_\alpha(t)|}\frac{\partial H_\alpha}{\partial n}(t)$$

where n is the normal to the unit circle directed into its interior, we obtain

$$\int_{\Gamma_\alpha} \frac{\partial h_i}{\partial n_i}(z)U(\sigma,z)\,ds=\int_{|t|=1}\frac{\partial H_\alpha}{\partial n}(t)U(\sigma,z)\,|dt|.$$

Using the bound $\varepsilon(\alpha)$ for the modulus of $U(\sigma,z)$ on Γ_α we get

$$\left|\int_{\Gamma_\alpha} \frac{\partial h_i}{\partial n_i}(z)U(\sigma,z)\,ds\right|\le\varepsilon(\alpha)\int_{|t|=1}\left|\frac{\partial H_\alpha}{\partial n}(t)\right|\,|dt|. \tag{4.40}$$

Let $(\partial/\partial s)H_\alpha(t)$ denote the tangential derivative of H_α at the point t, $|t|=r$ $(0<r\le1)$, along the circle of radius r passed in the positive direction. Then for $|t|=1$

$$\frac{\partial}{\partial s}H_\alpha(t)=\frac{\partial}{\partial\xi}H_\alpha(t)=T'(\theta)\frac{d\theta}{d\xi}=(T_1(\theta)+T_2(\theta))\frac{d\theta}{d\xi} \tag{4.41}$$

where T_1 and T_2 are the trigonometric polynomials defined in (4.19) and (4.20). Because of (4.34), we have

$$0<c_1\le\frac{d\theta}{d\xi}\le c_2. \tag{4.42}$$

For $|t|=1$, let

$$u_j(t):=T_j(\theta)\frac{d\theta}{ds},\quad j=1,2. \tag{4.43}$$

Then

$$0<c_1\le|u_j(t)|\le c_2/\delta,\quad j=1,2, \tag{4.44}$$

and, because of (4.20a),

$$\int_{|t|=1}|u_j(t)|\,|dt|\le c,\quad j=1,2. \tag{4.45}$$

Moreover, let u_j denote the harmonic function in D with boundary values (4.43), then (4.44) is true for all $t\in\overline{D}$. Fix r, $0<r<1$. Then

$$\int_0^{2\pi}\left|\frac{\partial}{\partial n}H_\alpha(re^{i\xi})\right|d\xi=\int_0^{2\pi}\left|\frac{\partial}{\partial s}\widetilde{H}_\alpha(re^{i\xi})\right|d\xi$$

where $\widetilde{H}_\alpha$ is a conjugate function of H_α. Now, $(\partial/\partial s)\widetilde{H}_\alpha=\tilde{u}_1+\tilde{u}_2$ where $\tilde{u}_j$ is the conjugate of u_j, with $\tilde{u}_j(0)=0$, $j=1,2$.

A theorem of Zygmund (cf. [6, Theorem 4.3, p. 58]) shows that

$$\int_0^{2\pi}|\tilde{u}_j(re^{i\xi})|\,d\xi\le\int_0^{2\pi}|u_j(re^{i\xi})|\log^+|u_j(re^{i\xi})|\,d\xi+6\pi e. \tag{4.46}$$

With (4.44) we finally get

$$\int_0^{2\pi}\left|\frac{\partial}{\partial n}H_\alpha(re^{i\xi})\right|d\xi\le c\log(1/\delta)\int_0^{2\pi}(|u_1(re^{i\xi})|+|u_2(re^{i\xi})|)\,d\xi. \tag{4.47}$$

Let $r\uparrow 1$. Then the uniform continuity of $(\partial/\partial n)H_\alpha$, u_1 and u_2 in $\overline{D}$ leads, together with (4.40) and (4.45), to

$$\left|\int_{\Gamma_\alpha}\frac{\partial h_i}{\partial n_i}(z)U(\sigma,z)\,ds\right|\le c\varepsilon(\alpha)\log(1/\delta). \tag{4.48}$$

By a similar argument, we can also deduce that

$$\left|\int_{\Gamma_\alpha}\frac{\partial h_o}{\partial n_o}(z)U(\sigma,z)\,ds\right|\le c\varepsilon(\alpha)\log(1/\delta). \tag{4.49}$$

Summarizing (4.12), (4.38), (4.48) and (4.49), we have obtained

$$\sigma(J)\ge -c\left(\delta+\delta^\gamma+\frac{1}{\delta}(\alpha-1)\log\frac{1}{\alpha-1}+\varepsilon(\alpha)\log(1/\delta)\right). \tag{4.50}$$

In view of the choice of δ in (4.15),

$$\sigma(J)\ge -c\varepsilon(\alpha)\log\frac{1}{\varepsilon(\alpha)}$$

and (4.13) is proved. □

Proof of Theorem 2.2. First, we observe that E is regular for the Dirichlet problem, and therefore

$$U(\mu_E,z)=\log(1/\operatorname{cap}(E)),\qquad z\in E. \tag{4.51}$$

Hence (2.10) yields

$$U(\mu_E, z) \le U(\nu_n, z) + \frac{1}{n} \log A_n, \quad z \in E. \tag{4.52}$$

By the maximum principle for potentials ([19, § I.5]), (4.52) holds for all $z \in \mathbf{C}$.

Next, the Lagrange interpolation formula and (2.11) gives for $z \in \Gamma_\alpha$

$$1 = \left| \sum_{k=1}^{n} \frac{p_n(z)}{p_n'(z_{k,n})(z - z_{k,n})} \right| \le \frac{nB_n |p_n(z)|}{\operatorname{cap}(E)^n \operatorname{dist}(E, \Gamma_\alpha)}.$$

Now, there exists a constant $c>0$ such that

$$\operatorname{dist}(E, \Gamma_\alpha) \ge c(\alpha - 1)^2$$

for all $1 \le \alpha \le 2$ (Siciak [31, Lemma 3.1]). Then we obtain for $\alpha = 1 + n^{-3}$,

$$0 \le U(\mu_E, z) - U(\nu_n, z) + \frac{\log B_n}{n} + c\frac{\log n}{n} \tag{4.53}$$

for $z \in \Gamma_\alpha$ or, together with (4.50),

$$\max_{z \in \Gamma_\alpha} |U(\mu_E, z) - U(\nu_n, z)| \le c \frac{\log C_n}{n} =: \varepsilon(\alpha) \tag{4.54}$$

where $c \ge 1$ is an absolute constant, $C_n := \max(A_n, B_n, n)$.

If E is a Jordan curve of class C^{1+} then $\partial G/\partial n$ is continuous on E with $\partial G / \partial n > 0$ for all $z \in E$ (cf. [2, Lemma 2]). Hence, in this case, for any subarc J of E

$$\mu_E(J) = \int_J \frac{\partial G}{\partial n}\, ds \le M \int_J ds$$

with some constant $M > 0$.

If E is a Jordan arc of class C^{1+} with endpoints a and b then the functions

$$h_\pm(z) := |(z-a)(z-b)|^{1/2} \frac{\partial G}{\partial n_\pm}(z)$$

are continuous at the interior points of E where n_+ and n_- denote the two normals at the point z directed into Ω. Moreover, $h_\pm$ can be continuously extended to E with $h_\pm(z) > 0$ for all $z \in E$ (cf. [2, Lemma 4]. We observe that for these results E needs to be only of class C^{1+} and not of the class C^{2+}.) Hence, for any subarc J of E

$$\mu_E(J) = \int_J \left(\frac{\partial G}{\partial n_+} + \frac{\partial G}{\partial n_-} \right) ds \le M \left(\int_J ds \right)^{1/2}$$

with some constant $M>0$. Therefore, with $\gamma=\frac{1}{2}$, the condition (2.8) of Theorem 2.1 is satisfied in any case. Since

$$\alpha-1=n^{-3}\le\varepsilon(\alpha)^3$$

for all $n\ge4$, Theorem 2.2 follows from Theorem 2.1. □

Proof of Theorem 3.1. Let $z_{1,n},\dots,z_{n,n}$ be an n-th Fekete point set, $p_n(z)=\prod_{k=1}^n(z-z_{k,n})$,

$$M_n:=\max_{z\in E}|p_n(z)|$$

and

$$m_n:=\min_{1\le k\le n}\prod_{j\ne k}|z_{j,n}-z_{k,n}|.$$

According to a result of Pommerenke ([25]),

$$\operatorname{cap}(E)^n\le M_n\le m_n\le(4e^{-1}\log n+4)n\operatorname{cap}(E)^{n-1}.$$

Therefore, the conditions (2.10) and (2.11) of Theorem 2.2 are satisfied with $B_n=1$ and $A_n=(4e^{-1}\log n+4)n/\operatorname{cap}(E)$. Hence, (3.2) follows immediately. □

Proof of Theorem 3.2. Let $\mathcal{F}_{n+2}(A_n(f))=\{z_0,\dots,z_{n+1}\}$ and

$$w_1(z):=\prod_{k=0}^{n+1}(z-z_k).$$

Then estimate (3.3) in [4] yields for infinitely many integers n

$$|w_1'(z_i)|\ge\frac{\operatorname{cap}(E)^{n+1}}{(n+2)^3},\quad 0\le i\le n+1 \tag{4.55}$$

since

$$\min_{p\in\Pi_n}\max_{z\in E}|z^{n+1}-p(z)|\ge\operatorname{cap}(E)^{n+1}.$$

Next, we define inductively

$$Z^1:=\mathcal{F}_{n+2}(A_n(f)),$$

and for $j\ge2$,

$$\begin{aligned} M_{j-1}&:=\max_{z\in E}|w_{j-1}(z)|=|w_{j-1}(z_{n+j})|,\\ Z^j&:=Z^{j-1}\cup\{z_{n+j}\},\\ w_j(z)&:=\prod_{w\in Z^j}(z-w).\end{aligned}$$

By the construction, we get for the Vandermonde expression

$$V(Z^{j+1}) = V(Z^j)M_j.$$

Let us assume now for simplicity that $\operatorname{cap}(E)=1$. We claim that for sufficiently large n there exists an index i, $1 \le i \le \sqrt{n}+1$, such that

$$\log M_i \le 3n^{1/2} \log n. \tag{4.56}$$

Because of (4.55),

$$\log V(Z^1) \ge -\frac{3(n+2)\log(n+2)}{2}.$$

If possible, let (4.56) be false. We fix $k := \lfloor \sqrt{n}+1 \rfloor$. Then

$$\begin{aligned} \log V(Z^{k+1}) &= \log V(Z^1) + \sum_{j=1}^{k} \log M_j \\ &\ge -\frac{3(n+2)\log(n+2)}{2} + 3n \log n \ge n \log n(\tfrac{3}{2} + o(1)). \end{aligned} \tag{4.57}$$

On the other hand, Pommerenke [25] has proved the upper bound

$$\log V_m \le (m/2)\log(4e^{-1}\log m + 4) + (m/2)\log m \tag{4.58}$$

where V_m is the Vandermonde expression for an m-point Fekete point set in E. For $m = n+2+k$, the upper bound in (4.58) is $n \log n(\frac{1}{2} + o(1))$. Comparing this upper bound with the lower bound (4.57) we obtain a contradiction for sufficiently large n. Hence (4.56) is true.

Now,

$$w_1(z) = \frac{w_i(z)}{(z - z_{n+2}) \dots (z - z_{n+i})}$$

and we consider the level line Γ_α with $\alpha = 1 + n^{-2}$. Then for $z \in \Gamma_\alpha$ we obtain, using again the estimate $\operatorname{dist}(E, \Gamma_\alpha) \ge c_1(\alpha - 1)^2$ of Siciak [31],

$$|(z - z_{n+2}) \dots (z - z_{n+i})| \ge (c_1 n^{-4})^{i-1}$$

or

$$\log |w_1(z)| \le c_2 n^{1/2} \log n \quad \text{for } z \in \Gamma_\alpha \tag{4.59}$$

where the constant $c_2 > 0$ is independent of n. Together with (4.55), we obtain by Theorem 2.2 that

$$D[\nu_{n+2} - \mu_E] \le c \frac{(\log n)^2}{\sqrt{n}}. \quad \square$$

Proof of Theorem 3.3. In this proof, Φ_n will denote the monic orthogonal polynomial $\varkappa_n^{-1}\omega_n$. Then (cf. [14])

$$|\Phi_n(z)| \le \exp\left(\sum_{k=0}^{n} |\Phi_k(0)|\right), \quad |z|=1$$

and Bernstein's inequality gives

$$|\Phi_n(z)| \le \exp\left(\sum_{k=0}^{n} |\Phi_k(0)|\right)|z|^n$$

or

$$\frac{1}{n}\log|\Phi_n(z)| - \log|z| \le \frac{1}{n}\sum_{k=0}^{n}|\Phi_k(0)|, \quad |z|\ge 1. \tag{4.60}$$

If $p\in\Pi_n$, we set

$$p^*(z) := z^n\overline{p(1/\bar{z})}.$$

According to [11, §V.2, (2.1)],

$$|\omega_n^*(z)| \ge cn^{-2}, \quad |z| = (1+n^{-2})^{-1}.$$

Since $\omega_n(z)=\varkappa_n\Phi_n^*(z)$, this yields

$$|\Phi_n^*(z)| \ge c\varkappa_n^{-1}n^{-2}, \quad |z| = (1+n^{-2})^{-1},$$

and hence

$$|z^n/\Phi_n(z)| \le c\varkappa_n n^2, \quad |z| = 1+n^{-2} \tag{4.61}$$

so that

$$\log|z| - \frac{1}{n}\log|\Phi_n(z)| \le c\left\{\frac{\log(1+\varkappa_n)}{n} + \frac{\log n}{n}\right\} \tag{4.62}$$

for $|z|=1+n^{-2}$. Theorem 3.3 now follows from (3.12), (4.60), (4.62) as an application of Theorem 2.1. □

In the case when $\tau' \ge m > 0$, it is known (cf. [14, p. 198]) that $|\Phi_n(z)| \le c|\omega_n(z)| \le c\sqrt{n}$ for all z with $|z|=1$ so that the right hand side of the estimate (4.60) can be replaced by $c\log n/n$. Moreover, we have

$$\frac{1}{\varkappa_n^2} = \inf_{P\in\Pi_{n-1}} \int |z^n+P(z)|^2\, d\tau(z) \ge m \inf_{P\in\Pi_{n-1}} \int |z^n+P(z)|^2\, |dz| = m$$

so that the right hand side of (4.61) can be estimated by cn^2. Therefore, in this case, the estimate (3.16) holds.

Proof of Theorem 3.4. The zeros x_i, $1\leq i\leq n$, of the orthogonal polynomial p_n are all simple and contained in $[-1,1]$. The Christoffel–Darboux formula (cf. [33, p. 43]) yields

$$\sum_{k=0}^{n-1} p_k^2(x_i) = \frac{\gamma_{n-1}}{\gamma_n} p_n'(x_i) p_{n-1}(x_i)$$

and therefore, using the fact that $\gamma_{n-1}\leq\gamma_n$ (cf. [11]), we have

$$\begin{aligned} 2\gamma_0 |p_{n-1}(x_i)| &= 2|p_0(x_i)p_{n-1}(x_i)| \\ &\leq p_0^2(x_i)+p_{n-1}^2(x_i) \\ &\leq \frac{\gamma_{n-1}}{\gamma_n} p_n'(x_i)p_{n-1}(x_i) \\ &\leq |p_n'(x_i)p_{n-1}(x_i)|. \end{aligned}$$

Hence,

$$|\gamma_n^{-1} p_n'(x_i)| \geq 2\gamma_0/\gamma_n.$$

On the other hand, let $T_{n,S}$ be the Chebyshev polynomial of degree n on the compact set S. Then the minimum property of $T_{n,S}$ yields

$$\gamma_n^{-1}\|p_n\|_S \geq \|T_{n,S}\| \geq \operatorname{cap}(S)^n$$

and therefore for $1\leq i\leq n$,

$$|\gamma_n^{-1} p_n'(x_i)| \geq \frac{2\gamma_0}{\|p_n\|_S} \operatorname{cap}(S)^n. \tag{4.63}$$

Moreover, the minimal property of the orthonormal polynomials p_n leads to

$$1 = \|p_n\|_2 \leq \gamma_n \|T_{n,S}\|_2. \tag{4.64}$$

According to a result of Widom [37, Theorem 11.5],

$$\|T_{n,S}\|_S \leq c \operatorname{cap}(S)^n$$

and

$$\gamma_n^{-1}\|p_n\|_S \leq \frac{c}{\gamma_0}\|p_n\|_S \operatorname{cap}(S)^n. \tag{4.65}$$

As in the proof of Theorem 2.2, we obtain by (4.65)

$$U(\mu_S,z)-U(\nu_n,z)\le c\frac{\log(1+\|p_n\|_S)}{n} \tag{4.66}$$

for all $z\in\mathbf{C}$. Moreover, analogous to (4.53), the inequalities (4.63) lead to

$$U(\nu_n,z)-U(\mu_S,z)\le c\left\{\frac{\log n}{n}+\frac{\log(1+\|p_n\|_S)}{n}\right\} \tag{4.67}$$

for $z\in\Gamma_\alpha$, $\alpha=1+n^{-3}$. Hence, (3.20) is a consequence of Theorem 2.1. □

Proof of Corollary 3.5. Let $S=:\bigcup_{i=1}^r[a_i,b_i]$, $a_i<b_i$ $(1\le i\le r)$. Fix $x_0\in S$ with

$$\|p_n\|_S=|p_n(x_0)|.$$

Then $x_0\in[a_j,b_j]$ for some j, $1\le j\le r$. According to Markov's inequality,

$$|p_n'(x)|\le\frac{2\|p_n\|_S}{b_j-a_j}n^2,\quad x\in[a_j,b_j].$$

Let $\delta:=mn^{-2}$ where $m=\min_{1\le i\le r}(b_i-a_i)/4$. Then at least one of the intervals $[x_0-\delta,x_0]$ or $[x_0,x_0+\delta]$, say the latter, lies in $[a_j,b_j]$ for all sufficiently large n. Therefore,

$$|p_n(x)|\ge\|p_n\|_S/2,\quad x\in[x_0,x_0+\delta]$$

and

$$1=\int_{-1}^1 p_n^2\,d\tau\ge\int_{x_0}^{x_0+\delta}p_n^2\,d\tau\ge(\varkappa\delta/4)\|p_n\|_S^2$$

or

$$\|p_n\|_S\le\frac{2n}{\sqrt{m\varkappa}}.$$

Hence (3.21) follows from (3.20). □

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