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Special results in adjunction theory in dimension four and five

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Introduction

Let $L^\wedge$ be a very ample line bundle on an n-dimensional projective manifold $(X^\wedge, L^\wedge)$. Assume $n=\dim X^\wedge \geq 3$. In [1], Beltrametti, Fania, and Sommese give a very explicit structure theory for pairs $(X^\wedge, L^\wedge)$ as above, and such that the Kodaira dimension $\varkappa(K_{X^\wedge}+(n-3)L^\wedge)<n$ and $n\geq 6$. Moreover, if $n\geq 6$ and $\varkappa(K_{X^\wedge}+(n-3)L^\wedge)=n$ it is shown there is a very simple birational morphism $f: X^\wedge \to X$ with X having at most 2-factorial isolated terminal singularities, and $K_X+(n-3)L$ nef and big where $L=(f_*L^\wedge)^{**}$ is at worst 2-Cartier.

Partial results are given for dimensions $n=3,5$ by Beltrametti, Fania, and Sommese [1] and for $n=4$ by Fania and Sommese [4].

In this paper we extend the structure theory of Beltrametti, Fania, and Sommese [1] to $n=5$ in the same form as the structure theorem when $n\geq 6$. See (1.1) and (1.2) for a statement. We also extend the structure theorem if $n=4$, to cover pairs $(X^\wedge, L^\wedge)$ where $\varkappa(3K_{X^\wedge}+4L^\wedge)<4$. If $\varkappa(3K_{X^\wedge}+4L^\wedge)=4$ there is a very simple morphism $\psi: X^\wedge \to Z$ with Z having at most Gorenstein, 2-factorial, isolated terminal singularities, and $3K_Z+4L$ nef and big where $L=(\psi_*L^\wedge)^{**}$ is at worst 2-Cartier. See Theorems (2) and (2.5) for a complete statement.

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After this paper was finished we received a preprint of T. Fujita, *On the Kodaira energy and adjoint reduction of polarized manifolds*, (which will appear in Manuscripta Math.) that overlaps with our paper. In particular T. Fujita has shown that in case (1.1.2) of Theorem (1.1), $(X, \mathcal{K})$ is the projective cone over the Veronese 4-fold, $(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(2))$.

0. Background material

(0.1) Notation. We work over the complex field **C**. By *variety* we mean an irreducible and reduced projective scheme, V. We denote its structure sheaf by $\mathcal{O}_V$.

Basically we use the standard notation from algebraic geometry. We almost always follow the notation of Beltrametti, Fania, and Sommese [1]. We refer to it, to Beltrametti–Sommese [2] and to Kawamata, Matsuda, and Matsuki [6] for definitions of the following: **Q**-divisor, **Q**-Cartier divisor, **Q**-factorial, numerically effective (nef, for short), big, numerical equivalence (denote by $\sim$), linear equivalence (denote by $\approx$) of **Q**-divisors, intersection of cycles, canonical divisor, terminal and canonical singularities.

We say that a normal variety V is *r-Gorenstein* if rK_V is a line bundle, where K_V is the canonical divisor. The smallest positive integer, r, such that rK_V is a line bundle is called the *index* of V.

For a reflexive sheaf $\mathcal{L}$ of rank 1, we denote by $|\mathcal{L}|$ the complete linear system associated to $\mathcal{L}$ and by $\Gamma(\mathcal{L})$ the space of the global sections of $\mathcal{L}$. We say that $\mathcal{L}$ is *spanned* if it is spanned by $\Gamma(\mathcal{L})$.

Linear equivalence classes of Weil divisors on a normal variety and isomorphism classes of reflexive sheaves of rank 1 are used with little (or no) distinction. Hence we shall freely switch between the multiplicative and the additive notation for divisors.

(0.2) Nef values. We state the following result in the case of terminal singularities which occur in the adjunction theory (see e.g. Beltrametti, Fania, and Sommese [1]), even though it holds true in the more general case of log-terminal singularities (see Kawamata, Matsuda, and Matsuki [6], 4.1).

(0.2.1) Kawamata rationality theorem. *Let V be a normal variety of dimension n with terminal singularities and let r be the index of V. Let $\psi: V \to S$ be a projective morphism onto a variety S. Let L be a ψ-ample line bundle on V. If K_V is not ψ-nef, then*

$$\tau := \min\{t \in \mathbf{R}, K_V + tL \textit{ is } \psi\textit{-nef}\}$$

is a positive rational number. Furthermore expressing $rt - u/v$ with u, v positive coprime integers, we have $u \leq r(b+1)$ where $b = \max_{s\in S}\{\dim \psi^{-1}(s)\}$.

With the notation as in (0.2.1) we say that the rational number τ is the *ψ-nef value* of (V, L). If S is a point, τ is called the *nef value* of (V, L). Note also that, if S is a point, then $K_V + \tau L$ is nef and hence by the Kawamata–Shokurov base point free theorem (Kawamata, Matsuda, and Matsuki [6], §3) we know that for $m \gg 0$ with $m\tau$ integral and $r|m$, $|m(K_V + \tau L)|$ is base point free, and defines a morphism, ϕ, with connected fibers onto a normal variety, which we call the *nef value morphism* of (V, L).

Note that the definition of nef value given here differs from that one, say τ', given by Beltrametti, Fania, and Sommese [1], (1.1). In fact, $\tau = 1/\tau'$.

(0.2.2) *Remark.* Let (V, L) be as in (0.2.1). Let τ be the nef value of (V, L) and let ϕ be the nef value morphism of (V, L). Then

$$\tau = \min\{t \in \mathbf{R}, K_V + tL \text{ is nef}\} = \min\{t \in \mathbf{R}, K_V + tL \text{ is } \phi\text{-nef}\}.$$

That is, τ coincides with the ϕ-nef value of (V, L).

(0.2.3) **Lemma** (Beltrametti–Sommese [2], (0.8.3)). *Let V be a normal variety with terminal singularities. Let L be an ample line bundle on V. A rational number τ is the nef value of (V, L) if and only if $K_V + \tau L$ is nef but not ample.*

(0.3) Reductions (Sommese [9], (0.5), Fania and Sommese [4], §0 and Beltrametti, Fania, and Sommese [1], §0). Let $X^\wedge$ be a smooth connected variety of dimension $n \geq 3$ and let $L^\wedge$ be a very ample line bundle on $X^\wedge$. We say that a pair (X', L') with X' smooth is a (first) *reduction* of $(X^\wedge, L^\wedge)$ if L' is ample and

(i) there exists a morphism $\pi: X^\wedge \to X$ expressing $X^\wedge$ as X' with a finite set B blown up, $L' := (\pi_* L)^{**}$;

(ii) $L^\wedge = \pi^* L' - [\pi^{-1}(B)]$ (equivalently $K_{X^\wedge} + (n-1)L^\wedge \approx \pi^* K_{X'} + (n-1)L'$).

Except for an explicit list of well understood pairs $(X^\wedge, L^\wedge)$ we can assume that (see Sommese [9], Sommese and Van de Ven [10], and Fania and Sommese [4]):

(a) $K_{X^\wedge} + (n-1)L^\wedge$ is spanned and big and $K_{X'} + (n-1)L'$ is ample. Note that in this case such a reduction (X', L') is unique up to isomorphism. We will refer to this reduction, (X', L'), as *the reduction* of $(X^\wedge, L^\wedge)$.

(b) $K_{X'} + (n-2)L'$ is nef and big.

Then from the Kawamata–Shokurov base point free theorem (Kawamata, Matsuda, and Matsuki [6], §3) we know that $|m(K_{X'} + (n-2)L'|$ gives rise to a morphism for some $m > 0$, say $\varphi: X' \to X$, and for m large enough we can assume that φ has connected fibers and normal image. Thus there is an ample line bundle $\mathcal{K}$ on X such that $\varphi^* \mathcal{K} \approx K_{X'} + (n-2)L'$. The pair $(X, \mathcal{K})$ is known as the *second reduction* of $(X^\wedge, L^\wedge)$. The morphism φ is very well behaved (see Sommese [8], Fania [3], Fania and Sommese [4], and Beltrametti, Fania, and Sommese [1]). Let us recall the following results we need.

(0.3.1) **Proposition** ([1], (0.2.4)). *Let $(X^\wedge, L^\wedge)$, (X', L') and $(X, \mathcal{K})$ be as in (0.3) with $n \geq 3$. Then X has isolated rational terminal singularities. If $n=3$, X is 2-Gorenstein while, for $n \geq 4$, X is 2-factorial and it is Gorenstein in even dimensions.*

(0.3.2) **Proposition** ([1],(0.2.6) and [2], (4.4)). *Let* $(X^\wedge, L^\wedge)$, (X', L') *be as in* (0.3) *with* $n \geq 3$ *and* $(X, \mathcal{K})$, $\varphi: X' \to X$ *be the 2nd reduction of* $(X^\wedge, L^\wedge)$. *Let* $L := (\varphi_* L')^{**}$. *Let* τ *be the nef value of* $(X, \mathcal{K})$ *and let* $\phi: X \to W$ *be the nef value morphism of* $(X, \mathcal{K})$. *Then we have:*

(0.3.2.1) L *is a 2-Cartier divisor and* $\mathcal{K} \approx K_X + (n-2)L$;

(0.3.2.2) L *is* ϕ*-ample;*

(0.3.2.3) *the* ϕ*-nef value of* $2L$ *is* $\tau(2L) = \tau(n-2)/(2(\tau+1))$.

(0.3.3) **Lemma.** *Let* $(X, \mathcal{K})$, L *be as in* (0.3.2). *Then* $aK_X + bL$ *is a line bundle if* $b - an$ *is even.*

Proof. Note that $aK_X + bL \approx a\mathcal{K} + (b - a(n-2))L$. We are then reduced to showing that $(b - a(n-2))L$ is a line bundle. If $b - an$ is even, then so is $b - (n-2)a$. Then by (0.3.2.1) we are done. □

For further details and properties of the 2nd reduction, $(X, \mathcal{K})$, we refer to Beltrametti, Fania, and Sommese [1], (0.2). See also Beltrametti and Sommese [2] for a partial extension to the case when $L^\wedge$ is merely ample.

(0.4) Special varieties. Let V be a normal r-Gorenstein variety of dimension n, L an ample line bundle on V. We say that V is an *r-Gorenstein Fano* (or simply an *r-Fano*) *variety* if $-rK_V$ is ample. We say that (V, L) is a *Del Pezzo variety* (respectively a *Mukai variety*) if $rK_V \approx -(n-1)rL$ (respectively $rK_V \approx -(n-2)rL$).

We also say that (V, L) is a *scroll* (respectively a *quadric fibration*; respectively a *Del Pezzo fibration*; respectively a *Mukai fibration*) over a normal variety Y of dimension m if there exists a surjective morphism with connected fibers $p: V \to Y$, such that $r(K_V + (n-m+1)L) \approx p^*\mathcal{L}$ (respectively $r(K_V + (n-m)L) \approx p^*\mathcal{L}$; respectively $r(K_V + (n-m-1)L) \approx p^*\mathcal{L}$; respectively $r(K_V + (n-m-2)L) \approx p^*\mathcal{L}$) for some *ample* line bundle $\mathcal{L}$ on Y.

(0.5) A part of Mori's theory of extremal rays will be used throughout the paper. We will use freely the notion of extremal ray, as well as the basic theorems as Cone theorem and Contraction theorem. We refer the reader to Mori [7] and Kawamata, Matsuda, and Matsuki [6].

In particular if V is a normal variety with canonical singularities we will denote by $\varrho = \mathrm{cont}_R: V \to Y$ the morphism given by the contraction of an extremal ray R. We will denote by $E(R)$, or simply by E, the locus of R, that is the locus of curves whose numerical classes are in R. Equivalently, $E := \{v \in V,\ \varrho \text{ is not an isomorphism at } v\}$.

For any further background material we refer to Beltrametti, Fania, and Sommese [1] and Beltrametti and Sommese [2].

1. The case $n=5$

Let $X^\wedge$ be a smooth connected projective variety of dimension $n=5$ and let $L^\wedge$ be a very ample line bundle on $X^\wedge$. Let $(X,\mathcal{K})$, $\mathcal{K}\approx K_X+3L$, be the second reduction of $(X^\wedge,L^\wedge)$ as in (0.3). In this section we study the nefness and bigness of $K_X+(n-3)\mathcal{K}=K_X+2\mathcal{K}$. Note that, since $(n-2)(K_X+(n-3)L)\approx K_X+(n-3)\mathcal{K}$, the nefness and bigness of $K_X+2\mathcal{K}$ is equivalent to that of K_X+2L.

(1.1) **Theorem.** *Let $(X^\wedge,L^\wedge)$, $(X,\mathcal{K})$, L be as above. Let τ be the nef value of $(X,\mathcal{K})$. Then $K_X+2\mathcal{K}$ is nef unless either:*

(1.1.1) *$\tau=5$ and either $(X,\mathcal{K})\cong(Q,\mathcal{O}_Q(1))$, Q a hyperquadric in $\mathbf{P}^6$ or $(X,\mathcal{K})$ is a scroll over a smooth curve C. In the latter case X is a $\mathbf{P}^4$ bundle over C and the restriction $\mathcal{K}_{\mathbf{P}^4}$ of $\mathcal{K}$ to a fiber is isomorphic to $\mathcal{O}_{\mathbf{P}^4}(1)$;*

(1.1.2) *$\tau=\frac{7}{2}$ and X is a singular 2-Gorenstein Fano 5-fold with ${L^\wedge}^5=121$, $2K_X\approx-7\mathcal{K}$ and $2L\approx3\mathcal{K}$.*

Proof. From Beltrametti, Fania, and Sommese, [1], (4.2), we know that $K_X+3\mathcal{K}$ is nef unless $(X,\mathcal{K})$ is either as in (1.1.1) or (1.1.2). Thus we can assume $\tau\leq3$. Let $\phi: X\to W$ be the nef value morphism of $(X.\mathcal{K})$. We assume that the index of X is 2 since the argument is parallel but simpler if the index is one.

Step I: $\tau<3$. By the above we can assume $\tau=3$. Therefore

$$K_X+3\mathcal{K}\approx4K_X+9L\approx\phi^*H \tag{1.1.3}$$

for some ample line bundle H on W. Since $(4,9)=1$, there exist positive integers $(a,b)=(7,3)$ such that $4a-9b=1$. From Lemma (0.3.3) we know that $\mathcal{L}:=3K_X+7L$ is a line bundle. Note also that $\mathcal{L}$ is ample since $4K_X+9L$ is nef and $\frac{7}{3}>\frac{9}{4}$. By (1.1.3) we have

$$K_X+9\mathcal{L}=28K_X+63L=7(4K_X+9L)\approx\phi^*(7H),$$

hence $K_X+9\mathcal{L}$ is nef but not ample. It thus follows that $\theta=9$ is the nef value of $(X,\mathcal{L})$ (and ϕ is the nef value morphism of $(X,\mathcal{L})$). Then by the Kawamata rationality theorem (0.2.1) we know that $2\theta=p/q$ for some coprime positive integers p,q such that $p\leq2(n+1)$. This leads to the contradiction $p=18<12$.

Step II: $\tau\leq2$. By the above we can assume $2<\tau<3$. From the rationality theorem we know that $2\tau=u/v$ for some coprime positive integers u,v such that $u\leq2(n+1)=12$. We also have $4<u/v<6$. This shows that $v\leq2$, otherwise $u>12$. If $v=1$, then $u=5$. If $v=2$, then $8<u<12$. Since $(u,v)=1$, then either $u=9$ or $u=11$. Thus the only possible cases are $(u,v)=(5,1),(9,2),(11,2)$. Recall that by (0.3.2.3) the ϕ-nef value of $2L$ is $\tau(2L)=3\tau/(2(\tau+1))$ and by (0.2.1), $2\tau(2L)=a/b$ where a,b are coprime positive integers with $a\leq2(\max_{w\in W}\{\dim\phi^{-1}(w)\}+1)=12$.

If $(u,v)=(5,1)$, we have $\tau=\frac{5}{2}$ and $2\tau(2L)=\frac{15}{7}$. This contradicts the bound $a\leq 12$.

If $(u,v)=(9,2)$, we have $\tau=\frac{9}{4}$ and $2\tau(2L)=\frac{23}{13}$. This contradicts again the bound $a\leq 12$.

Let $(u,v)=(11,2)$. Since τ coincides with the ϕ-nef value morphism of $(X,\mathcal{K})$ (see (0.2.2)), we have in this case $2\tau=\frac{11}{2}$ with $11\leq 2(\max_{w\in W}\{\dim\phi^{-1}(w)\}+1)$. Therefore, since $n=5$, the morphism ϕ contracts X to a point so that $K_X+\tau\mathcal{K}\approx\mathcal{O}_X$, or

$$4K_X+11\mathcal{K}\approx 3(5K_X+11L)\approx\mathcal{O}_X.$$

Hence in particular $3(5(2K_X)+11(2L))\approx\mathcal{O}_X$. By Lemma (0.11) of Beltrametti and Sommese [2] it thus follows that $5(2K_X)+11(2L)\approx\mathcal{O}_X$ and $2K_X\approx -11M$, $2L\approx 5M$ for some ample line bundle M on X. Let

$$p(t):=\chi(tM).$$

Since $tM-K_X\sim(t+\frac{11}{2})M$ is ample for $t>-\frac{11}{2}$, we see by the Kodaira vanishing theorem that $p(t)=h^0(tM)$ for $t>-\frac{11}{2}$ so that $p(t)=0$ for $t=-1,\dots,-5$ and $p(0)=1$. Therefore we have

$$p(t)=(t+1)\dots(t+5)/5!.$$

Hence in particular the coefficient $c_1(M)^5/5!$ of the leading term is $1/5!$ or $c_1(M)^5=1$. Furthermore, we have $h^0(M)=p(1)=6=\dim X+1$. Thus we conclude by Fujita [5], I, (1.1) (see also Beltrametti and Sommese [2], (1.2)) that $(X,M)\cong(P^5,\mathcal{O}_{\mathbf{P}^5}(1))$. Then the relation $2K_X\approx -11M$ leads to a numerical contradiction. This completes the proof of Step II and hence of the theorem. □

(1.2) **Proposition.** *Let $(X^\wedge,L^\wedge)$, $(X,\mathcal{K})$, be as in* (1.1). *Assume $K_X+2\mathcal{K}$ to be nef. Let ϕ be the nef value morphism of $(X,\mathcal{K})$. Then $K_X+2\mathcal{K}$ is also big unless either:*

(1.2.1) *$(X,\mathcal{K})$ is a scroll under ϕ over a normal 4-fold and in particular the restriction $\mathcal{K}_F$ of $\mathcal{K}$ to a general fiber F is $\mathcal{O}_{\mathbf{P}^1}(1)$;*

(1.2.2) *$(X,\mathcal{K})$ is a quadric fibration under ϕ over a normal 3-fold;*

(1.2.3) *$(X,\mathcal{K})$ is a Del Pezzo fibration under ϕ over a smooth surface;*

(1.2.4) *$(X,\mathcal{K})$ is a Mukai fibration under ϕ over a smooth curve; or*

(1.2.5) *X is a 2-Gorenstein Fano 5-fold with $2K_X\approx -4\mathcal{K}$.*

Proof. Let τ be the nef value of $(X,\mathcal{K})$. If $K_X+2\mathcal{K}$ is not big, then not ample, we have $\tau=2$. Hence in particular ϕ is the morphism given by $|N(K_X+2\mathcal{K})|$ for $N\gg 0$. If $\dim\phi(X)>0$ we have the cases (1.2.1), (1.2.2), (1.2.3), (1.2.4). If $\dim\phi(X)=0$ we fall in case (1.2.5). □

2. The case $n=4$

Let $X^\wedge$ be a smooth connected projective variety of dimension $n=4$ and let $L^\wedge$ be a very ample line bundle on $X^\wedge$. Let $(X,\mathcal{K})$, $\mathcal{K}\approx K_X+(n-2)L$, be the second reduction of $(X^\wedge,L^\wedge)$ as in (0.3). In this section we show (2.2) and (2.3). The result (2.2) states that if $K_X+3\mathcal{K}$ is nef and big there is a first reduction, $(Z,\mathcal{K}(Z))$, of $(X,\mathcal{K})$, that is closely related to $(X,\mathcal{K})$ and such that $K_Z+3\mathcal{K}(Z)$ is ample. The result (2.3) states that under the same conditions $K_Z+2\mathcal{K}(Z)$ is nef. Recall also that X is Gorenstein since $n=4$.

We need the following result due to Fania and Sommese.

(2.1) **Theorem** ([4], (1.1)). *Let $(X^\wedge,L^\wedge)$, $(X,\mathcal{K})$, L be as above. Let τ be the nef value of $(X,\mathcal{K})$. Then $K_X+3\mathcal{K}$ is nef and big unless either:*

(2.1.1) *$\tau=5$, $(X,\mathcal{K})\cong(\mathbf{P}^4,\mathcal{O}_{\mathbf{P}^4}(1))$, $L\cong\mathcal{O}_{\mathbf{P}^4}(3)$;*

(2.1.2) *$\tau=3$, X is a Gorenstein Fano 4-fold with $-K_X\approx 3\mathcal{K}$ and $-4K_X\approx 6L$;*

(2.1.3) *$\tau=3$, there exists a holomorphic map $\psi\colon X\to C$, where C is a smooth curve, $4K_X+6L\approx\psi^*H$ for some ample line bundle H on C. Furthermore the general fiber of ψ is $(Q,\mathcal{O}_Q(2))$, Q a hyperquadric in $\mathbf{P}^4$. Note that $K_X+3\mathcal{K}\approx\psi^*H$, i.e. $(X,\mathcal{K})$ is a quadric fibration over C;*

(2.1.4) *$\tau=3$, there exists a holomorphic map $\psi\colon X\to S$, where S is a smooth surface, $4K_X+6L\approx\psi^*H$ for some ample line bundle H on S. Furthermore the general fiber of ψ is $(\mathbf{P}^2,\mathcal{O}_{\mathbf{P}^2}(2))$. Note that $K_X+3\mathcal{K}\approx\psi^*H$, i.e. $(X,\mathcal{K})$ is a scroll over S.*

By the above we can assume that $K_X+3\mathcal{K}$ is nef and big. We can prove now the following result.

(2.2) **Theorem–Definition** (Structure theorem for the $2\frac{1}{2}$ reduction). *Let $(X^\wedge,L^\wedge)$, $(X,\mathcal{K})$, $\mathcal{K}\approx K_X+2L$ be as in (2.1). Assume that $K_X+3\mathcal{K}$ is nef and big. Then there exists a normal variety Z and a morphism with connected fibers $f\colon X\to Z$ such that*

(2.2.1) *f expresses X as the blowing up of Z along a finite set $B\subset\mathrm{reg}(Z)$;*

(2.2.2) *$K_X+3\mathcal{K}\approx f^*(K_Z+3\mathcal{K}(Z))$ where $\mathcal{K}(Z):=(f_*\mathcal{K})^{**}$ is an ample line bundle and $K_Z+3\mathcal{K}(Z)$ is ample;*

(2.2.3) *$\mathcal{K}(Z)\approx K_Z+2\widetilde{L}$ where $\widetilde{L}:=(f_*L)^{**}$ is a 2-Cartier divisor on Z.*

Note that $(Z,\mathcal{K}(Z))$ is the first reduction of $(X,\mathcal{K})$. We call $(Z,\mathcal{K}(Z))$ the $2\frac{1}{2}$ reduction of $(X^\wedge,L^\wedge)$.

Proof. If $K_X+3\mathcal{K}$ is ample, then by taking $Z:=X$ and f as the identity map one has the result. Therefore we can assume without loss of generality that $K_X+3\mathcal{K}$ is nef and big but not ample. Let τ be the nef value of $(X,\mathcal{K})$. In this case, by Lemma (0.2.3), we have $\tau=3$. Let $\phi\colon X\to W$ be the nef value morphism of

$(X,\mathcal{K})$. Then ϕ is a birational morphism which contracts some curve. From standard results of Mori's theory (see e.g. Beltrametti and Sommese [2], Lemma (0.4.3)) we know that there exists an extremal ray, R, such that $(K_X+3\mathcal{K})\cdot R=0$. Let $\varrho=\text{cont}_R: X\to Y$ be the contraction of R. Then ϕ factors through $\varrho, \phi=\alpha\circ\varrho$. Let E be the locus of R. Since $K_X+3\mathcal{K}$ is nef and big and $K_X+3\mathcal{K}\approx\phi^*(H)=\varrho^*(\alpha^*H)$ for some ample line bundle H on W, Proposition (1.5) of Beltrametti and Sommese [2] applies to give $(E,\mathcal{K}_E)\cong(\mathbf{P}^3,\mathcal{O}_{\mathbf{P}^3}(1))$, where $\mathcal{K}_E$ denotes the restriction of $\mathcal{K}$ to E. Hence in particular $\varrho(E)$ is a point and $K_{X|E}\approx-3\mathcal{K}_E$. Let L_E be the restriction of L to E. Then $L_E\cong\mathcal{O}_{\mathbf{P}^3}(a)$ for some integer a. From

$$\mathcal{O}_{\mathbf{P}^3}(1)\approx\mathcal{K}_E\approx K_{X|E}+2L_E\approx-3\mathcal{K}_E+2L_E\approx\mathcal{O}_{\mathbf{P}^3}(2a-3)$$

we find $a=2$, i.e. $L_E\cong\mathcal{O}_{\mathbf{P}^3}(2)$.

(2.2.4) **Claim.** *X is factorial along $E\cong\mathbf{P}^3$.*

Proof (of the Claim). Let (X',L') be the first reduction of $(X^\wedge,L^\wedge)$ and let $\varphi: X'\to X$ be the 2nd reduction map as in (0.3). Assume that there exists a point $x\in E$ such that X is not factorial at x. Then by Beltrametti, Fania, and Sommese [1], (0.2) (see in particular (0.2.2)) we know that

$$\varphi^{-1}(x)\cong\mathbf{P}^3;\quad \mathcal{N}_{\mathbf{P}^3}^{X'}\cong\mathcal{O}_{\mathbf{P}^3}(-2). \tag{2.2.5}$$

We also know that

$$L'\approx\varphi^*L-\varphi^{-1}(\mathcal{B}_2)-\varphi^{-1}(\mathcal{B}_1)/2, \tag{2.2.6}$$

where $\mathcal{B}_1:=$ set of points $x\in X$ which satisfy the conditions (2.2.5) and $\mathcal{B}_2:=\mathcal{B}-\mathcal{B}_1$ where $\mathcal{B}$ is the algebraic subset of X, with $\dim\mathcal{B}\le1$, such that φ is an isomorphism outside of $\mathcal{B}$. Let l be a line in $E\cong\mathbf{P}^3$ containing x and no other singular points of X. Let l' be the proper transform of l under φ. Note that l' intersects transversely $\varphi^{-1}(x)\cong\mathbf{P}^3$ at a single point. Indeed if either l' intersects $\varphi^{-1}(x)$ in more than one point or l' is tangent to $\varphi^{-1}(x)$ at a given point, then the restriction, $\varphi_{l'}$ of φ to l' would not be a biholomorphism and l would be singular. Thus $l'\cdot\varphi^{-1}(x)=1$ and hence $\varphi^{-1}(\mathcal{B}_1)\cdot l'=1$. Note also that $L\cdot l=2$ since $L_E\cong\mathcal{O}_{\mathbf{P}^3}(2)$. Therefore from (2.2.6) we get

$$2L'\cdot l'=2\varphi^*L\cdot l'-\varphi^{-1}(\mathcal{B}_1)\cdot l'-2\varphi^{-1}(\mathcal{B}_2)\cdot l'$$

or

$$2L'\cdot l'=2L\cdot l-\varphi^{-1}(\mathcal{B}_1)\cdot l'-2\varphi^{-1}(\mathcal{B}_2)\cdot l'=3-2\varphi^{-1}(\mathcal{B}_2)\cdot l'.$$

This leads to a parity contradiction.

Note that since $\dim X=4$ and ϕ is the morphism given by $|N(K_X+3\mathcal{K})|$ for $N\gg 0$, ϕ is in fact the 1st reduction map for the pair $(X,\mathcal{K})$. Then, in view of the claim above, the proof of Theorem (3.1) of Beltrametti and Sommese [2] applies with no changes to say that $\phi: X\to W$ is the simultaneous contraction to distinct smooth points of divisors $E_i\cong\mathbf{P}^3$ such that $E_i\subset\operatorname{reg}(X)$, $\mathcal{O}_{E_i}(E_i)\cong\mathcal{O}_{\mathbf{P}^3}(-1)$ for $i=1,\dots,t$, where $\mathcal{K}_{E_i}$ denotes the restriction of $\mathcal{K}$ to E_i. In particular W has the same singularities as X.

Now, let $\mathcal{K}':=(\phi_*\mathcal{K})^{**}$, the double dual. Since $\mathcal{K}$ is ample and ϕ is an isomorphism outside of a finite set of points, $\mathcal{K}'$ is ample. The fact that $K_X+3\mathcal{K}\approx\phi^*(K_W+3\mathcal{K}')$ simply follows by noting that ϕ expresses X as the blowing up of W along a finite set, B, of points with $B\subset\operatorname{reg}(W)$. Hence in particular $K_W+3\mathcal{K}'$ is ample since $K_X+3\mathcal{K}\approx\phi^*H$ for some ample line bundle H on W. Thus we have shown (2.2.1) and (2.2.2) of the theorem by taking $Z:=W$, $f:=\phi$ and $\mathcal{K}(Z):=\mathcal{K}'$.

The decomposition $\mathcal{K}(Z)\approx K_Z+2\widetilde{L}$, where $\widetilde{L}:=(f_*L)^{88}$, follows immediately from $\mathcal{K}\approx K_X+2L$ and the fact that f is an isomorphism outside of a finite number of points. Note that $\widetilde{L}$ is a 2-Cartier divisor on Z since L is a 2-Cartier divisor on X (see (0.3.2.1)). □

The following is the main result of this section.

(2.3) **Theorem.** *Let $(X^\wedge,L^\wedge)$, $(X,\mathcal{K})$, $\mathcal{K}\approx K_X+2L$, $(Z,\mathcal{K}(Z))$, $f: X\to Z$ be as in* (2.2). *Then $K_Z+2\mathcal{K}(Z)$ is nef.*

Proof. Let $\tau(Z)$ be the nef value of $(Z,\mathcal{K}(Z))$. By (2.2.2) we know that $K_Z+3\mathcal{K}(Z)$ is ample so we have $\tau(Z)<3$ by (0.2.3). Since we have to show $\tau(Z)\le 2$, let us assume that $2<\tau(Z)<3$. By the Kawamata rationality theorem (0.2.1) we know that $\tau(Z)=u/v$ for some coprime integers u,v such that $u\le n+1=5$. We also have $2<u/v<3$. This clearly shows that $v\ge 2$ as well as $v\le 2$, otherwise we contradict the bound $u\le 5$. Then the only possible case is $(u,v)=(5,2)$. Since $\tau(Z)$ coincides with the f-nef value of $(Z,\mathcal{K}(Z))$ (see (0.2.2)) we have in this case

$$u=5\le\max_{z\in Z}\{\dim f^{-1}(z)\}+1.$$

Therefore, since $n=4$, the morphism f contracts Z to a point, so that $2K_Z+5\mathcal{K}(Z)\approx\mathcal{O}_Z$. By Lemma (0.11) of Beltrametti and Sommese [2], it thus follows that $K_Z\approx -5M$, $\mathcal{K}(Z)\approx 2M$ for some ample line bundle M on Z. By the Kobayashi–Ochiai result (see e.g. Beltrametti and Sommese [2], (1.3)) we conclude that $(Z,M)\cong(\mathbf{P}^4,\mathcal{O}_{\mathbf{P}^4}(1))$, $\mathcal{K}(Z)\cong\mathcal{O}_{\mathbf{P}^4}(2)$. Let $\widetilde{L}:=(f_*L)^{**}$ and recall that $\mathcal{K}(Z)\approx K_Z+2\widetilde{L}$. Then $\widetilde{L}\cong\mathcal{O}_{\mathbf{P}^4}(a)$ for some integer a. Thus the condition $\mathcal{K}(Z)\approx\mathcal{O}_{\mathbf{P}^4}(2)$ leads to the numerical contradiction $2=-5+2a$. □

(2.4) *Example* (of non-trivial $2\frac{1}{2}$ reduction). Let $\mathcal{L}$ be a very ample line bundle on a 4-fold X with $(Z,\mathcal{L})$, $\pi\colon X\to Z$, a non-trivial first reduction, π not an isomorphism. Then $K_X+3\mathcal{L}\approx\pi^*(K_Z+3\mathcal{L})$ and $K_Z+3\mathcal{L}'$ is ample. Let $L:=2\mathcal{L}$, $\mathcal{K}:=K_X+2L$, $L':=(\pi_*L)^{**}$, $\mathcal{L}':=(\pi_*\mathcal{L})^{**}$, $\mathcal{K}(Z):=K_Z+2L'=K_Z+4\mathcal{L}'$. If $E\cong\mathbf{P}^3$ is contracted to a point under π we have

$$K_{X|E}\approx -3\mathcal{L}_E\approx\mathcal{O}_{\mathbf{P}^3}(-3);\quad \mathcal{K}_E\approx K_{X|E}+4\mathcal{L}_E\approx\mathcal{O}_{\mathbf{P}^3}(1);\quad L_E\approx\mathcal{O}_{\mathbf{P}^3}(2).$$

Moreover,

$$K_X+3\mathcal{K}=4K_X+12\mathcal{L}=4(K_X+3\mathcal{L})\approx\pi^*(4K_Z+12\mathcal{L}')=\pi^*\big(K_Z+3\mathcal{K}(Z)\big).$$

Thus we conclude that $(Z,\mathcal{K}(Z))$ is the first reduction of $(X,\mathcal{K})$, i.e. the $2\frac{1}{2}$ reduction of (X,L). The nef value, $\tau(Z)$, of $(Z,\mathcal{K}(Z))$ is $\tau(Z)<3$ since $K_Z+3\mathcal{K}(Z)=4(K_Z+3\mathcal{L}')$ is ample and the nef value, τ, of $(X,\mathcal{K})$ is $\tau=3$ since $K_X+3\mathcal{K}$ is nef and big but not ample (because X is not isomorphic to Z).

(2.5) **Proposition.** *Let $(X^\wedge,L^\wedge)$, $(X,\mathcal{K})$, $\mathcal{K}\approx K_X+2L$ be as in* (2.1). *Assume $K_X+3\mathcal{K}$ is nef and big and let $(Z,\mathcal{K}(Z))$, $f\colon X\to Z$, be the $2\frac{1}{2}$ reduction of $(X^\wedge,L^\wedge)$. Let $\widetilde{L}:=(f_*L)^{**}$. Let ϕ be the nef value morphism of $(Z,\mathcal{K}(Z))$. Then $K_Z+2\mathcal{K}(Z)$ is nef and it is also big unless either:*

(2.5.1) *$(Z,\mathcal{K}(Z))$ is a Mukai 4-fold, $K_Z\approx-2\mathcal{K}(Z)$ (if $\widetilde{L}$ is a line bundle, then $(Z,\mathcal{K}(Z))\cong(Q,\mathcal{O}_Q(2))$, Q a hyperquadric in $\mathbf{P}^5$), or*

(2.5.2) *$\phi\colon Z\to C$ is a surjective morphism onto a smooth curve C with general fiber $F\cong\mathbf{P}^3$, the restriction of $\mathcal{K}(Z)$ to F isomorphic to $\mathcal{O}_{\mathbf{P}^3}(2)$ and $\widetilde{L}_F\cong\mathcal{O}_{\mathbf{P}^3}(3)$.*

Proof. By (2.3) we know that $K_Z+\mathcal{K}(Z)$ is nef. Let $\tau:=\tau(Z)$ be the nef value of $(Z,\mathcal{K}(Z))$. If $K_Z+2\mathcal{K}(Z)$ is not big, then not ample, we have $\tau=2$. Hence in particular ϕ is the morphism given by $|N(K_Z+2\mathcal{K}(Z))|$ for $N\gg0$.

If $\dim\phi(Z)=0$, we are in case (2.5.1). Note that $\widetilde{L}$ is ample in this case since

$$\mathcal{O}_Z\approx K_Z+2\mathcal{K}(Z)\approx 3K_Z+4\widetilde{L}\tag{2.5.3}$$

and $-K_Z\approx2\mathcal{K}(Z)$ is ample. Note also that if $\widetilde{L}$ is a line bundle, by (2.5.3) and by Lemma (0.11) of Beltrametti and Sommese [2], we get $K_Z\approx-4M$, $\widetilde{L}\approx3M$ for some ample line bundle M on Z. By the Kobayashi–Ochiai result (see e.g. [2], (1.3)) this implies that $(Z,M)\cong(Q,\mathcal{O}_Q(1))$, Q a hyperquadric in $\mathbf{P}^5$. Then $\mathcal{K}(Z)\approx K_Z+2\widetilde{L}\approx\mathcal{O}_Z(2)$.

If $\dim\phi(Z)>0$, let F be a general fiber of ϕ. Note that F is smooth since Z has only isolated singularities. Let $\mathcal{K}(Z)_F$, $\widetilde{L}_F$ be the restriction of $\mathcal{K}(Z)$, $\widetilde{L}$ to F respectively. We have $K_F+2\mathcal{K}(Z)_F\approx3K_F+4\widetilde{L}_F\sim\mathcal{O}_F$. Note that $\widetilde{L}_F$ is an ample line bundle on F since $-K_F \approx 2\mathcal{K}(Z)_F$ is ample. Therefore there exists an ample line bundle A on F such that $K_F \approx -4A$, $\widetilde{L}_F \approx 3A$. This implies that $\dim F \geq 3$ or $\dim F = 3$ (see again [2], (0.11)). Hence by the Kobayashi–Ochiai result we get $(F, A) \cong (\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1))$, $\widetilde{L}_F \cong \mathcal{O}_{\mathbf{P}^3}(3)$ and $\mathcal{K}(Z)_F \approx (K_Z + 2\widetilde{L})_F \approx \mathcal{O}_{\mathbf{P}^3}(2)$. $\square$

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