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The area integral and its density for BMO and VMO functions

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0. Introduction

Let f be an integrable function in $\mathbf{R}^\nu$ and let u be its harmonic extension to $\mathbf{R}_+^{\nu+1}=\mathbf{R}^\nu\times\mathbf{R}_+$. The (conic) area integral of f at the point $\theta\in\mathbf{R}^\nu$ is defined by

$$A_a^2(\theta)=\int_{\Gamma_a(\theta)} y^{1-\nu}|\nabla u(x,y)|^2 dx\,dy$$

where $\Gamma_a(\theta)$ is the cone with vertex at θ and aperture a. That is,

$$\Gamma_a(\theta)=\{(x,y)\in\mathbf{R}_+^{\nu+1}:|x-\theta|<ay\}.$$

We also define the Littlewood–Paley square function of f, (the complete area integral of f), by

$$g_*^2(\theta)=\int_{\mathbf{R}_+^{\nu+1}} y\,p_\theta(x,y)|\nabla u(x,y)|^2 dx\,dy$$

where p_θ is the Poisson kernel of the half space. That is, if $z=(x,y)\in\mathbf{R}_+^{\nu+1}$ then

$$p_\theta(z)=\frac{C_\nu y}{(|x-\theta|^2+y^2)^{(\nu+1)/2}}$$

where C_ν is a constant depending on ν.

For $1<p<\infty$, the Littlewood–Paley theory asserts (see Torchinsky [16]) that the L^p-norm of A_a is equivalent to the L^p-norm of f (under some normalization of f at infinity) and the same is true for g_* provided $2\le p<\infty$. This result is no longer valid for $p=\infty$ as the example $f(x)=\chi_{(0,1)}(x)$ will show. However, a substitute for L^∞ in

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the Littlewood–Paley theory is the space of functions of bounded mean oscillation, BMO. It was proved in Meyer [12] that if f is in BMO then the Littlewood–Paley g_*^2 function is also in BMO unless it is identically equal to infinity. Kurtz [11] and Qian Tau [14] have shown more recently the same result for A_a and g_* (not their squares).

The aim of this paper is to extend these results in two directions. First we prove that A_a^2 and g_*^2, as well as their versions formed with more general kernels than the Poisson kernel, are in VMO if f is in VMO (see definitions below). Second, we extend the BMO and VMO results to the so called "maximal density of the area integral" first introduced by Gundy [9]. Finally we will also obtain some BMO and VMO results for the functional D^r (the density of the area integral) when f satisfies some very general hypotheses which include positivity and BMO. All our results concerning VMO are new, as are our results concerning the density of the area integral for BMO functions. We also prove similar results for the g-function, the radial area integral. Both the BMO and VMO results for g are new.

The space of functions of vanishing mean oscillation, VMO, and the density of the area integral do not seem to be as well known as BMO and the area integral. As a consequence we first say a few words about these two objects.

The space VMO is the closure in BMO of the space of all uniformly continuous functions (other equivalent definitions will be detailed in Section 1 below). In an appropriate sense VMO has the same relation to BMO that the space of uniformly continuous functions has to L^∞. For example, Sarason [15] proved in the case of the disc that the harmonic conjugate of the Poisson extension of a continuous function is the Poisson extension of a VMO function. It was also proved by Coifman and Weiss [6] that the Hardy space H^1 is the dual of a variant of VMO (Coifman and Weiss defined VMO as the closure of the continuous functions with compact support).

The density of the area integral D^r at a point $r\in\mathbf{R}$ is the equivalent notion in analysis of the Brownian local time and it can be defined as the area integral except that the measure $|\nabla u(z)|^2(dz)$ (which is $\Delta u^2(dz)$ due to the harmonicity of u) is replaced by the positive measure $\Delta|u-r|(dz)$. More precisely, for $r\in\mathbf{R}$ and $\theta\in\mathbf{R}^n$ we define

$$D_a^r(\theta)=\int_{\Gamma_a(\theta)} y^{1-\nu}\Delta|u-r|(dz),$$

where $z=(x,y)\in\mathbf{R}_+^{\nu+1}$.

We also define the maximal density by

$$D_a^*(\theta)=\sup_{r\in\mathbf{R}} D_a^r(\theta).$$

These operators were introduced by Gundy [9] (our definition of $D_a^r(\theta)$ here is from Gundy–Silverstein [10]), where it was also proved that

$$A_a^2(\theta)=\int_{\mathbf{R}} D_a^r(\theta)dr,$$

hence the name "density of the area integral". Since the measure $\Delta|u-r|(dz)$ is supported on the set $\{u=r\}$, $D_a^r(\theta)$ is a way to measure the oscillation of u around r in the cone $\Gamma_a(\theta)$. If $r=0$, it measures, in some sense, the positivity default of u in the cone since if $u>0$ in $\Gamma_a(\theta)$ then of course $D_a^0(\theta)=0$. It has been used in this manner by Brossard and Chevalier in [4] and [5]. The maximal density was used by Gundy [9] and by Gundy and Silverstein [10] to provide a different characterization of the H^p spaces. In Bañuelos and Moore [1] sharp good-λ inequalities are proved for the maximal density leading to laws of the iterated logarithm of the Kesten type.

We can now explain more precisely our results and how the paper is organized. In Section 1, we explain our notation and recall the main definitions of BMO and VMO. We devote Section 2 to prove the BMO and VMO results for several versions of the area integral. As we explained earlier, the BMO result is not new; it is a consequence of the H^1-BMO duality theorem and the fact that H^1 can be defined by convolution with other kernels besides the Poisson kernel (see Fefferman and Stein [7]). We shall, however, give a new proof of these results which is very elementary and does not use these two difficult theorems. Our proof, although elementary, will work for any kind of area integral including the most general ones, not defined with the gradient of the harmonic extension of the function f but with the convolution of f with a Littlewood–Paley function. Our arguments will allow us to obtain the same results for VMO, which as we said before, are all new. Another advantage of our method is that it provides the BMO and VMO results also for the "radial area function", that is, the usual g-function for which even the BMO result was not known and for which the duality argument does not work. To the best of our knowledge the only known result is the weaker result of Wang [17] which says that if f is in BMO, then g (and not its square) is in BMO. These new theorems are also presented in Section 2. In Sections 3 and 4, we prove similar results for the density of the area integral and the maximal density. We obtain two types of results: The first results in Section 3 say that if the Green potential ϕ_r of the measure $\Delta|u-r|(dz)$ is bounded for some r (and hence for all r), then the several versions of the density of the area integral D^r are in BMO, and if the potential goes to zero uniformly at the boundary then the different versions of D^r are in VMO. The hypothesis that ϕ_r is bounded has already been used by Brossard and Chevalier [5] in a different context and it is satisfied for example by a BMO or positive function.

The second results in Section 4 are about the maximal density or more precisely, about the different versions of the maximal density. Theorem 6 says that if f is in

BMO then so is D^* and Theorem 7 says the same for VMO. As we said earlier, our results for D^r and D^* are new even for BMO. These results are in the same line as the results of Gundy [9], Gundy and Silverstein [10], Bañuelos and Moore [1] and Moore [13]. In comparison with the results of Section 2, they provide one more confirmation that the D^*-functional is a good functional for theorems related to the Littlewood–Paley theory. Finally we mention that, mutatis mutandis, all the results we give in this paper are true for the ball of $\mathbf{R}^\nu$ (rather than the half-space) and many of the arguments are even simpler.

1. Notation and definitions

We shall denote by $z=(x,y)$ the generic point of $\mathbf{R}_+^{\nu+1}$ and by $B(x,r)$ the ball of $\mathbf{R}^\nu$ with center x and radius r. If f is a real function in $\mathbf{R}^\nu$ we will write $P_z(f)$ for the Poisson extension of f at the point z. That is,

$$P_z(f)=\int_{\mathbf{R}^\nu} p_\theta(z)f(\theta)d\theta$$

where $p_\theta(z)$ is as in the introduction. A function f is in BMO (bounded mean oscillation) if there is a constant C such that for all cubes Q in $\mathbf{R}^\nu$,

$$\int_Q |f(x)-f_Q|m_Q(dx)\le C \tag{1.1}$$

where m_Q is the normalized Lebesgue measure on Q and

$$f_Q=\int_Q f(x)m_Q(dx),$$

the mean of f on Q. The smallest constant C for which (1.1) holds is the BMO-norm of f which we shall write as $\|f\|_{C,*}$. It is well known (and easy to prove) that the measure m_Q can be replaced by the Poisson kernel and this provides an equivalent norm on BMO. More precisely, if we define

$$\|f\|_{p1,*}=\sup_{z\in\mathbf{R}_+^{\nu+1}} P_z(|f-P_z(f)|) \tag{1.2}$$

then there are constants C_1 and C_2 such that

$$C_1\|f\|_{C,*}\le\|f\|_{p1,*}\le C_2\|f\|_{C,*}. \tag{1.3}$$

Also it follows from the John–Nirenberg theorem, (see Torchinsky [16]), that we may replace the L^1-norm above by any L^p-norm and obtain an equivalent BMO-norm. In particular if we define

$$\alpha(z)=P_z\left((f-P_z(f))^2\right). \tag{1.4}$$

Then $\|f\|_{p2,*}=\sup_{z\in\mathbf{R}_+^{\nu+1}}\alpha(z)^{1/2}$ provides an equivalent norm for BMO. If we apply the Green formula (note that $\frac{1}{2}|\nabla u|^2=\Delta(u^2)$) we see that

$$\alpha(z)=\frac{1}{2}\int_{\mathbf{R}_+^{\nu+1}}\mathcal{G}(z,z')|\nabla u(z')|^2dz' \tag{1.5}$$

where $\mathcal{G}(z,z')$ is the Green function for $\mathbf{R}_+^{\nu+1}$. This last formulation is the main tool in our study of the relation between BMO and the area integral. It also provides the connection with Brownian motion. In a similar manner, if we define for $r\in\mathbf{R}$

$$\phi_r(z)=P_z\left(|(f-r)-|P_z(f-r)|\,|\right), \tag{1.6}$$

it can be shown that (see Brossard and Chevalier [5])

$$\|f\|_{p1,*}=\sup_{\substack{z\in\mathbf{R}_+^{\nu+1}\\ r\in\mathbf{R}}}\phi_r(z) \tag{1.7}$$

and as with $\alpha(z)$, Green's theorem implies that

$$\phi_r(z)=\int_{\mathbf{R}_+^{n+1}}\mathcal{G}(z,z')\Delta|u-r|(dz'). \tag{1.8}$$

Once again, (1.7) and (1.8) are the main tool in our study of BMO and the density of the area integral.

We shall now define VMO. For $\delta>0$ we let $\|f\|_{C,*\delta}$ be the supremum of the left hand side of (1.1) over all cubes Q which have edge length less than or equal to δ. The space VMO (vanishing mean oscillation) is the subspace of BMO consisting of those functions for which $\lim_{\delta\to 0}\|f\|_{C,*\delta}=0$. As before, we may define

$$\|f\|_{p2,*\delta}=\sup_{z\in\mathbf{R}^\nu\times(0,\delta)}\alpha(z)^{1/2} \tag{1.9}$$

and

$$\|f\|_{p1,*\delta}=\sup_{\substack{z\in\mathbf{R}^\nu\times(0,\delta)\\ r\in\mathbf{R}}}\phi_r(z). \tag{1.10}$$

It is easy to show that VMO is also the subspace of BMO consisting of those functions for which the quantities of (1.9) or (1.10) go to 0 as δ goes to 0 (see Garnett [8], Theorem 5.1).

We now define the more general variants of the area integral A_a and the Littlewood–Paley g_*. Let ϱ be a nonnegative integrable function in $\mathbf{R}^\nu$ satisfying the following:

(H) There exist two constants $C>0$ and $\varepsilon>0$ such that for all $x\in\mathbf{R}^\nu$, $\varrho(x)\le Cp_0(x,1)$ and such that for all $r\in\mathbf{R}$,

$$\frac{1}{r^\nu}\int_{\mathbf{R}^\nu\times B(0,r)}|\varrho(x-v)-\varrho(x)|\,dx\,dv\le Cr^\varepsilon.$$

This is a weaker condition than the following one which is easier to understand:

(H′) There exists a constant $C>0$ such that for all $x_0\in\mathbf{R}^\nu$, $\varrho(x_0)\le Cp_0(x_0,1)$ and

$$\int_{\mathbf{R}^\nu}|\varrho(x-x_0)-\varrho(x)|\,dx\le C|x_0|.$$

For example, if ϱ is a C^1 function such that both ϱ and $|\nabla\varrho|$ are majorized by $Cp_0(x,1)$ (in particular $p_0(x,1)$ itself or any C^1 function with compact support) then it clearly satisfies (H′) and hence (H). The function $\varrho(x)=\chi_{B(0,a)}(x)$ also satisfies (H′).

For $\theta\in\mathbf{R}^\nu$ and $z=(x,y)\in\mathbf{R}^{\nu+1}_+$, define

$$\varrho_\theta(z)=y^{-\nu}\varrho\left(\frac{x-\theta}{y}\right)$$

and

$$A^2_\varrho(\theta)=\int_{\mathbf{R}^{\nu+1}_+}y\,\varrho_\theta(z)|\nabla u(x,y)|^2\,dx\,dy.$$

Notice that if $\varrho(x)=\chi_{B(0,a)}(x)$ then $A^2_\varrho=A^2_a$ and if $\varrho(x)=p_0(x,1)$ then $A^2_\varrho=g^2_*$. Thus the classical area functions are particular cases of A^2_ϱ. We shall also need the following bilinear form version of A_ϱ defined by

$$A^2_\varrho[f,h](\theta)=\int_{\mathbf{R}^{\nu+1}_+}y\,\varrho_\theta(z)\langle\nabla u(z),\nabla v(z)\rangle\,dx\,dy$$

where u and v are the Poisson extension of f and h respectively and $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbf{R}^{\nu+1}$.

We have defined above the density D_a^r. The complete density (which plays the role of g_*) is defined by

$$D^r(\theta)=\int_{\mathbf{R}_+^{\nu+1}} y\,p_\theta(z)\Delta|u-r|(dz)$$

and the analogue of A_ϱ^2 is defined by

$$D_\varrho^r(\theta)=\int_{\mathbf{R}_+^{\nu+1}} y\,\varrho_\theta(z)\Delta|u-r|(dz)$$

where ϱ satisfies (H). The maximal density associated to $D_\varrho^r(\theta)$ is $D_\varrho^*(\theta)=\sup_{r\in\mathbf{R}} D_\varrho^r(\theta)$ with a similar definition for $D^*(\theta)$ (the case $\varrho(x)=p_0(x,1)$).

2. The area integral for BMO and VMO functions

The most convenient BMO-norm for the study of the area function is $\|\cdot\|_{p2,*}$. To simplify notation we shall write $\|\cdot\|_*$ instead of $\|\cdot\|_{p2,*}$ throughout this section. The aim of this section is to prove the following theorem and corollaries:

Theorem 1. *Let* BMO_0 *denote the space of* BMO *functions whose area function is not identically infinite. The mapping* $(f,h)\to A_\varrho[f,h]$ *is continuous from* $\mathrm{BMO}_0\times\mathrm{BMO}_0$ *into* BMO.

Corollary 1. *If* $f\in\mathrm{BMO}_0$, *then* A_a^2, g_*^2 *and* A_ϱ^2 *are in* BMO *with a* BMO-*norm majorized by a constant (independent of* f*) times the square of the* BMO-*norm of* f.

Corollary 2. *If* $f\in\mathrm{VMO}$ *and its area integral is not identically infinite, then* A_a^2, g_*^2 *and* A_ϱ^2 *are also in* VMO.

As mentioned earlier, the g_*^2-result in Corollary 1 was first proved by Meyer [12]. Of course, Corollary 1 implies that the square root of all the area integrals are also in BMO, a result proved by Kurtz [11] and Qian Tao [14].

The proof in Meyer [12] is based on the H^1-BMO duality. Indeed, if f is a BMO function, then $y|\nabla u(x,y)|^2dx\,dy$ is a Carleson measure. The Littlewood–Paley g_*^2 function is its balayage and by a duality argument it is easy to show that g_*^2 is in BMO; (see Torchinsky [16], p. 273). The same proof can be done with A_a^2 and A_ϱ^2 using the fact that H^1 does not depend on the kernel used to define it and that A_a^2 and A_ϱ^2 are the balayage of $y|\nabla u(x,y)|^2dx\,dy$ with respect to other kernels. The proof we shall give below is quite elementary and does not use these two difficult theorems; the H^1–BMO duality and the independence of H^1 upon the kernel.

Before proving Theorem 1, let us explain how Corollary 2 follows from it. Let $f_y(x)=P_z(f)$ where $z=(x,y)\in\mathbf{R}_+^{\nu+1}$. By Theorem 5.1 in Garnett [8], $f\in\mathrm{VMO}$ if and only if $f_y\to f$ in the BMO-norm as $y\to 0$. It is easy to see that the area integral of f_y, $A_\varrho(f_y)$ is uniformly continuous. Since $A_\varrho^2(f)-A_\varrho^2(f_y)=A_\varrho[f-f_y,f]+A[f,f-f_y]$, Theorem 1 implies that if $f\in\mathrm{VMO}$ then $A_\varrho^2(f)$ is the limit in the BMO-norm of $A_\varrho^2(f_y)$ as $y\to 0$. Thus $A_\varrho(f)\in\mathrm{VMO}$ whenever $f\in\mathrm{VMO}$ which proves the corollary.

Theorem 1 follows immediately from the following proposition.

Proposition 2.1. *Let f and $h\in\mathrm{BMO}_0$. Let Q_0 be the unit cube in $\mathbf{R}^n$ centered at* 0. *There exists a constant C_{Q_0} such that*

$$\int_{Q_0}|A_\varrho[f,h](\theta)-C_{Q_0}|m_{Q_0}(d\theta)\le C\|f\|_*\|h\|_*$$

where C is a constant independent of f and h.

The idea for the proof of Proposition 1 is to use a continuity argument to control the oscillation of the "upper part" (Lemma 2.3 below) and only an L^2-estimate for the "bottom part". We first start with a lemma (which is an amelioration of a classical lemma; we do not pretend it is new) to control $y|\nabla u|$.

Lemma 2.2. *If $z=(x,y)\in\mathbf{R}_+^{\nu+1}$ and $u(z)=P_z(f)$ is the harmonic extension of f, then*

$$y|\nabla u(z)|\le C\|f\|_*$$

where C is a constant independent of f and z.

Proof. If h is a harmonic function in the unit ball B of $\mathbf{R}^{\nu+1}$, then there exists a constant C depending only on ν such that

$$|\nabla h(0)|^2\le C\int_{\partial B}|h(\theta)-h(0)|^2d\sigma(\theta) \tag{2.1}$$

where $d\sigma$ is the normalized surface measure on ∂B. Let $h(z)=u(z_0+y_0z)$ whose $z_0=(x_0,y_0)$ is a fixed point in $\mathbf{R}_+^{\nu+1}$. Applying (2.1) to this h we obtain, (with B' the ball centered at z_0 and radius y_0),

$$y_0^2|\nabla u(z_0)|^2\le C\int_{\partial B'}|u(\theta)-u(z_0)|^2d\sigma(\theta)\le C\alpha(z_0)\le C\|f\|_{p2,*}^2=C\|f\|_*^2,$$

by the subharmonicity of the function $(u(z)-u(z_0))^2$ and the definition of $\|f\|_{p2,*}$.

We are now ready for the proof of the control of the upper part. Let us call $U(\theta)$ the part of the integral defining $A_\varrho[f,h]$ which is above 1. That is,

$$U(\theta)=\int_{\mathbf{R}^\nu\times(1,\infty)}y\varrho_\theta(z)\langle\nabla u(z),\nabla v(z)\rangle dx\,dy.$$

Lemma 2.3. *There exists a constant C such that if $U(0)<\infty$ then*

$$\int_{Q_0} |U(\theta)-U(0)| m_{Q_0}(d\theta) \le C\|f\|_*\|h\|_*.$$

Proof.

$$\begin{aligned} |U(\theta)-U(0)| &= \left|\int_{\mathbf{R}^\nu\times(1,\infty)} y\left(\varrho_\theta(z)-\varrho_0(z)\right)\langle\nabla u(z),\nabla v(z)\rangle dz\right| \\ &\le \int_{\mathbf{R}^\nu\times(1,\infty)} y|\varrho_\theta(z)-\varrho_0(z)|\,|\nabla u(z)|\,|\nabla v(z)|dz \\ &\le C\|f\|_*\|h\|_* \int_{\mathbf{R}^\nu\times(1,\infty)} |\varrho_\theta(z)-\varrho_0(z)|y^{-1}dz \end{aligned}$$

where the last inequality follows from Lemma 1. Integration in θ gives

$$\begin{aligned} &\int_{Q_0} |U(\theta)-U(0)| m_{Q_0}(d\theta) \\ &\qquad \le C\|v\|_*\|h\|_* \int_{Q_0}\left(\int_{\mathbf{R}^\nu\times(1,\infty)} |\varrho_\theta(z)-\varrho_0(z)|y^{-1}dz\right) m_{Q_0}(d\theta) \end{aligned}$$

and so we only need to estimate the last integral. For any $r>0$, let rQ_0 be the cube concentric with Q_0 and with side length r times the side length of Q_0. By Fubini's theorem and a change of variables,

$$\begin{aligned} &\int_{Q_0}\int_{\mathbf{R}^\nu\times(1,\infty)} |\varrho_\theta(z)-\varrho_0(z)|y^{-1}dz\,d\theta \\ &\qquad = \int_1^\infty y^{-(\nu+1)} \int_{\mathbf{R}^\nu}\int_{Q_0} \left|\varrho\left(\frac{x-\theta}{y}\right)-\varrho\left(\frac{x}{y}\right)\right| d\theta\,dx\,dy \\ &\qquad \le C\int_1^\infty y^{-(1+\varepsilon)}dy < \infty \end{aligned}$$

where the last inequality follows from hypotheses (H) applied with $r=1/y$. This completes the proof.

For the control of the bottom part we will use the following

Lemma 2.4. *If ϱ satisfies hypothesis* (H) *and $z=(x,y)\in\mathbf{R}^\nu\times(0,1)$, then*

$$\int_{\mathbf{R}^\nu} \varrho_\theta(z)p_\theta(0,1)d\theta \le Cy^{-1}\mathcal{G}\left(z,(0,1)\right).$$

Proof. Fix $z_0=(x_0,y_0)\in\mathbf{R}^\nu\times(0,1)$. By (H) and the semigroup property of the Poisson kernel,

$$\int_{\mathbf{R}^\nu}\varrho_\theta(z_0)p_\theta(0,1)d\theta\le C\int_{\mathbf{R}^\nu}p_\theta(z_0)p_\theta(0,1)d\theta=Cp_0(x_0,y_0+1).$$

Then we only need to prove that

$$y_0p_0(x_0,y_0+1)=y_0p_{x_0}(0,y_0+1)\le C\mathcal{G}\left(z_0,(0,1)\right) \tag{2.2}$$

for a suitable constant C independent of z_0. To do this define the harmonic function in $\mathbf{R}^\nu\times(y_0,\infty)$ by

$$H(z)=C\mathcal{G}(z_0,z)-y_0p_{x_0}(x,y_0+y).$$

We wish to prove that if C is a suitable constant, (independent of z_0), $H(0,1)\ge0$. Since $\lim_{|z|\to\infty}H(z)=0$, by the maximum principle, it suffices to prove that the inequality holds when $z\in\mathbf{R}^\nu\times\{y_0\}$, that is, when $y=y_0$. However, since we have explicit formulas for both $\mathcal{G}$ and p_θ, it is easy and elementary to check that

$$\inf_{x\in\mathbf{R}^\nu}\left(\frac{\mathcal{G}((x_0,y_0),(x,y_0))}{y_0p_{x_0}(x,2y_0)}\right)=\inf_{x\in\mathbf{R}^\nu}\left(\frac{\mathcal{G}((0,1),(x,1))}{p_0(x,2)}\right)>0$$

and (2.1), and hence the lemma, follows.

We are now ready to estimate the bottom part. Define

$$B(\theta)=\int_{\mathbf{R}^\nu\times(0,1)}y\varrho_\theta(z)\langle\nabla u(z),\nabla v(z)\rangle dz.$$

Lemma 2.5. *There exists a constant C which depends only on ν and ϱ, such that*

$$\int_{Q_0}|B(\theta)|m_{Q_0}(d\theta)\le C\|f\|_*\|h\|_*.$$

Proof. By Fubini's theorem, Lemma 2.4, and the fact that the measure m_{Q_0} can be majorized by $p_\theta(0,1)$, we have

$$\begin{aligned}\int_{Q_0}|B(\theta)|m_{Q_0}(d\theta)&\le C\int_{\mathbf{R}^\nu}|B(\theta)|p_\theta(0,1)d\theta\\&\le C\int_{\mathbf{R}^\nu\times(0,1)}\left(\int_{\mathbf{R}^\nu}\varrho_\theta(z)p_\theta(0,1)d\theta\right)y|\langle\nabla u(z),\nabla v(z)\rangle|dz\\&\le C\int_{\mathbf{R}^\nu\times(0,1)}\mathcal{G}(z,(0,1))|\langle\nabla u(z),\nabla v(z)\rangle|dz\\&\le C\left(\int_{\mathbf{R}^\nu\times(0,1)}\mathcal{G}\left(z,(0,1)\right)|\nabla u(z)|^2dz\right)^{1/2}\\&\qquad\times\left(\int_{\mathbf{R}^\nu\times(0,1)}\mathcal{G}\left(z,(0,1)\right)|\nabla v(z)|^2dz\right)^{1/2}\\&\le C\|f\|_*\|h\|_*\end{aligned}$$

by (1.4) and (1.5).

We are now ready to finish the proof of Proposition 2.1. First, if for some point θ the area integral is finite, Lemmas 2.3 and 2.5 imply that it is finite a.e. in the unit cube centered at θ and hence almost everywhere in $\mathbf{R}^\nu$. Let $C_{Q_0}=U(0)$. Since $A_\varrho[f,h](\theta)=U(\theta)+B(\theta)$ we have

$$\int_{Q_0} |A_\varrho[f,h](\theta)-C_Q|m_{Q_0}(d\theta)\le \int_{Q_0} |U(\theta)-U(0)|m_{Q_0}(d\theta)+\int_{Q_0} |B(\theta)|m_{Q_0}(d\theta)$$
$$\le C\|f\|_*\|h\|_*,$$

by Lemmas 2.3 and 2.5.

Proposition 1.1 and a scaling argument (apply the proposition to the function $f\circ\sigma$ where σ is a homothety which sends Q_0 onto Q) show that for any cube Q there is a constant C_Q such that

$$\int_Q |A_\varrho[f,h](\theta)-C_Q|m_Q(d\theta)\le C\|f\|_*\|h\|_*$$

where C is the constant of Proposition 1.1. Since

$$\int_Q |A_\varrho[f,h](\theta)-(A_\varrho[f,h])_Q|m_Q(d\theta)\le 2\int_Q |A_\varrho[f,h](\theta)-C_Q|m_Q(d\theta).$$

Theorem 1 follows.

Next, we derive the corresponding results for the Littlewood–Paley g-function. This is the radial area function defined by

$$g^2(\theta)=\int_0^\infty y|\nabla u(\theta,y)|^2dy$$

whereas before u is the harmonic extension of f. As before we define

$$g[f,h](\theta)=\int_0^\infty y\langle\nabla u(\theta,y),\nabla v(\theta,y)\rangle dy$$

where u and v are the harmonic extensions of f and h respectively. We have

Theorem 2. *The mapping* $(f,h)\to g[f,h]$ *is continuous from* $\mathrm{BMO}_0\times\mathrm{BMO}_0$ *into* BMO.

Corollary 3. *Suppose* $f\in\mathrm{BMO}_0$. *Then* $g^2\in\mathrm{BMO}$ *with* BMO-*norm majorized by a constant (independent of* f*) times the square of the norm of* f.

Corollary 4. *If* $f \in \mathrm{VMO}$ *and its* g*-function is not identically infinite, then* $g^2 \in \mathrm{VMO}$.

The proof of Theorem 2 is essentially contained in the proof of Theorem 1. As before define the upper part and bottom part of $g[f,h]$ by

$$U(\theta)=\int_1^\infty y\langle\nabla u(\theta,y),\nabla v(\theta,y)\rangle dy$$

and

$$B(\theta)=\int_0^1 y\langle\nabla u(\theta,y),\nabla v(\theta,y)\rangle dy.$$

We have the following analogues of Lemmas 2.3 and 2.5 from which Theorem 2 follows.

Lemma 2.6. *There exists a constant* C *such that if* $U(0)<\infty$, *then*

$$\int_{Q_0}|U(\theta)-U(0)|m_{Q_0}(d\theta)\le C\|f\|_*\|h\|_*.$$

Lemma 2.7. *There is a constant* C *such that*

$$\int_{Q_0}|B(\theta)|m_{Q_0}(d\theta)\le C\|f\|_*\|h\|_*.$$

Proof of Lemma 2.6. First, we notice that if we denote by $D^n u$ any derivative of u of total order n, then by the proof of Lemma 2.2,

$$y^n|D^n u(z)|\le C\|f\|_*, \tag{2.3}$$

where C is a constant independent of z and f; (simply notice that (2.1) holds if we replace $\nabla h(0)$ by $D^n h(0)$). Using (2.3) with $n=2$ and the elementary identity

$|\langle a,b\rangle-\langle a_0,b_0\rangle|=|\langle a-a_0,b\rangle+\langle a_0,b-b_0\rangle|$, we have for $\theta\in Q_0$ that

$$\begin{aligned}
|U(\theta)-U(0)| &\le \int_1^\infty y|\langle\nabla u(\theta,y),\nabla v(\theta,y)\rangle-\langle\nabla u(0,y),\nabla v(0,y)\rangle|dy\\
&\le \int_1^\infty y|\langle\nabla u(\theta,y)-\nabla u(0,y),\nabla v(\theta,y)\rangle|dy\\
&\quad+\int_1^\infty y|\langle\nabla u(0,y),\nabla v(\theta,y)-\nabla v(0,y)\rangle|dy\\
&\le \int_1^\infty y|\nabla v(\theta,y)|\,|\nabla u(\theta,y)-\nabla u(0,y)|dy\\
&\quad+\int_1^\infty y|\nabla u(0,y)|\,|\nabla v(\theta,y)-\nabla v(0,y)|dy\\
&\le C\|h\|_*\int_1^\infty |\nabla u(\theta,y)-\nabla u(0,y)|dy\\
&\quad+C\|f\|_*\int_1^\infty |\nabla v(\theta,y)-\nabla v(0,y)|dy\\
&\le C\|f\|_*\|h\|_*\int_1^\infty \frac{dy}{y^2}\le C\|f\|_*\|h\|_*.
\end{aligned}$$

Integrating we find that

$$\int_{Q_0}|U(\theta)-U(0)|m_{Q_0}(d\theta)\le C\|f\|_*\|h\|_*$$

which is the assertion of Lemma 2.6.

We now recall that since the partial derivatives of u are also harmonic, we have

$$\frac{\partial u}{\partial x_i}(\theta,2y)=\int_{\mathbf{R}^\nu}p_\theta(x,y)\frac{\partial u}{\partial x_i}(x,y)dx.$$

If we first apply Jensen's inequality and then sum we find that

$$|\nabla u(\theta,2y)|^2\le\int_{\mathbf{R}^\nu}p_\theta(x,y)|\nabla u(x,y)|^2dx.$$

Multiplying both sides by y and integrating we get that

$$\int_0^{1/2}y|\nabla u(\theta,2y)|^2dy\le\int_0^{1/2}\int_{\mathbf{R}^\nu}yp_\theta(z)|\nabla u(z)|^2dz$$

which after changing variables gives

$$\int_0^1 y|\nabla u(\theta,y)|^2dy\le 2\int_0^1\int_{\mathbf{R}^\nu}yp_\theta(z)|\nabla u(z)|^2dz. \tag{2.4}$$

Lemma 2.7 now follows from (2.4) and Lemma 2.5. Thus we have proved Theorem 2.

We shall conclude this section with some remarks concerning other types of area integrals usually used in Littlewood–Paley theory. Let ψ be a Littlewood–Paley function in $\mathbf{R}^\nu$. That is, an integrable function satisfying

$$(\mathrm{H}_{L-P})\begin{cases} \text{(a)} & \displaystyle\int_{\mathbf{R}^\nu} \psi(x)dx=0, \\ \text{(b)} & \text{for all } x\in\mathbf{R}^\nu, |\psi(x)|\le C(1+|x|)^{1-\alpha}p_x(0,1) \\ & \quad \text{where } \alpha>0 \text{ is fixed and} \\ \text{(c)} & \displaystyle\int_{\mathbf{R}^\nu} |\psi(x+y)-\psi(x)|dx\le C|y|^\gamma, y\in\mathbf{R}^\nu \text{ where } \gamma>0 \text{ is also fixed.} \end{cases}$$

The condition (b) is generally written as $|\psi(x)|\le C(1+|x|)^{-(\nu+\alpha)}$. We prefer to write it this way; it implies that $\psi_y(x-\theta)\le C\,(1+(|x-\theta|)/y)^{1-\alpha}\,p_\theta(x,y)$ which facilitates some of our arguments.

We define the more general area integral. For $y\in\mathbf{R}_+$, let $\psi_y(x)=y^{-\nu}\psi(y^{-1}x)$ and replace $y\nabla u(x,y)$ by $F_\psi(x,y)=\psi_y*f(x)$. We define the general area integral by

$$S^2_{\varrho,\psi}(\theta)=\int_{\mathbf{R}^{\nu+1}_+} y^{-1}\varrho_\theta(x,y)|F_\psi(x,y)|^2dx\,dy$$

and its associated bilinear form by

$$S_{\varrho,\psi}[f,h](\theta)=\int_{\mathbf{R}^{\nu+1}_+} y^{-1}\varrho_\theta(z)F_\psi(z)H_\psi(z)dz$$

where $H_\psi(z)=\psi_y*h(x)$.

Then Theorem 1 as well as its corollaries remain true for $S^2_{\varrho,\psi}$ and $S_{\varrho,\psi}[f,h]$. More precisely we have

Theorem 3. *The map $(f,h)\to S_{\varrho,\psi}[f,h]$ is continuous from* $\mathrm{BMO}_0\times\mathrm{BMO}_0$ *into* BMO.

Corollary 5. *Suppose $f\in\mathrm{BMO}_0$. Then $S^2_{\varrho,\psi}\in\mathrm{BMO}$ with a* BMO *norm majorized by a constant (independent of f) times the square of that of f.*

Corollary 6. *If $f\in\mathrm{VMO}$ and its area integral is not identically infinite, then $S^2_{\varrho,\psi}\in\mathrm{VMO}$.*

The proofs are essentially the same as above except that Lemmas 2.2 and 2.5 have to be replaced by the following two lemmas whose proofs will be very briefly indicated.

Lemma 2.8. *Suppose ψ satisfies* (H_{L-P}). *Then* $|F_\psi(x,y)|\le C\|f\|_*$.

Lemma 2.9. *Suppose that ϱ satisfies* (H) *and ψ satisfies* (H_{L-P}). *Define*

$$B(\theta)=\int_{\mathbf{R}^\nu\times(0,1)} y^{-1}\varrho_\theta(z)|F_\psi(z)|^2dz.$$

There exists a constant C (which depends only on ν, ϱ, and ψ) such that

$$\int_{Q_0} B(\theta)m_{Q_0}(d\theta)\le C\|f\|_*^2.$$

Proof of Lemma 2.8. By our assumption (H_{L-P}) and the remark following it we have that for any constant C_0,

$$\begin{aligned}|F_\psi(x,y)|&=\left|\int_{\mathbf{R}^\nu}(f(\theta)-C_0)\psi_y(x-\theta)d\theta\right|\\&\le C\int_{\mathbf{R}^\nu}|f(\theta)-C_0|\left(1+\frac{|x-\theta|}{y}\right)^{1-\alpha}p_\theta(z)d\theta\\&\le C\left(\int_{\mathbf{R}^\nu}|f(\theta)-C_0|^pp_\theta(z)d\theta\right)^{1/p}\left(\int_{\mathbf{R}^\nu}\left(1+\frac{|x-\theta|}{y}\right)^{(1-\alpha)q}p_\theta(z)d\theta\right)^{1/q}\end{aligned}$$

where $1/p+1/q=1$, p and $q>1$. The first term is majorized by the BMO-norm if we take $C_0=P_z(f)$, (by (1.4) and p replacing 2). The second term is bounded as soon as $(1-\alpha)q<1$. This completes the proof.

Proof of Lemma 2.9. As in the proof of Lemma 2.5,

$$\int_{Q_0} B(\theta)m_{Q_0}(d\theta)\le C\int_0^1\int_{\mathbf{R}^\nu}p_0(x,y+1)|\psi_y*f(x)|^2\frac{dx\,dy}{y}. \tag{2.5}$$

Next, we explain how to bound the last integral by the square of the BMO-norm of f. First, by the Harnack inequality, $p_0(x,y+1)\le Cp_0(x,1)$. If we let

$$I(r)=\frac{C_\nu}{(r^2+1)^{(\nu+1)/2}}$$

we find that

$$p_0(x,1)=-\int_{|x|}^\infty I'(r)dr=C_\nu\int_{|x|}^\infty\frac{r\,dr}{(r^2+1)^{(\nu+3)/2}}$$

and the expression in (2.5) is dominated by, (after applying Fubini's theorem),

$$\begin{aligned}(2.6)\qquad &\int_0^\infty \int_0^1 \int_{B(0,r)} |\psi_y * f(x)|^2 \frac{dx\,dy}{y}(-I'(r))dr \\ &\qquad = C_\nu \int_0^\infty \int_0^1 \int_{B(0,r)} |\psi_y * f(x)|^2 \frac{dx\,dy}{y} \frac{r}{(r^2+1)^{(\nu+3)/2}} dr.\end{aligned}$$

Since $|\psi_y * f(x)|^2 \dfrac{dx\,dy}{y}$ is a Carleson measure with Carleson norm smaller than or equal to $C\|f\|_*^2$, (see Torchinsky [16], p. 373), we have that the right hand side is dominated by

$$\left(C_\nu \int_0^\infty \frac{(1+r)^\nu r}{(1+r^2)^{(\nu+3)/2}} dr\right) \|f\|_*^2 \le C_\nu \|f\|_*^2$$

and the proposition is proved.

3. The density of the area integral for BMO, VMO and functions bounded below

To deal with the density of the area integral, the BMO norm which is most convenient is $\|\cdot\|_{p1,*}$ which is defined in (1.7). To simplify notation, we shall again write throughout this section $\|\cdot\|_*$ instead of $\|\cdot\|_{p1,*}$.

We shall also deal in this section with versions of the density of the area integral for which the truncation is not too rough, that is, we shall assume that ϱ satisfies the hypothesis

(H″) $\quad \varrho$ is C^1 with ϱ and $|\nabla \varrho|$ both majorized by $Cp_0(x,1)$.

In particular, $p_0(x,1)$ which gives D and any C^1 function of compact support which gives the version first studied by Gundy and Silverstein [10] and Bañuelos and Moore [1], satisfy hypothesis (H″).

Theorem 4. *If f is such that ϕ_0, (the Green potential of $\Delta|u|$ as defined in (1.8)), is bounded, (in particular, if $f \in$ BMO or if f is positive or bounded below), then D_ϱ^r, D^r belong to* BMO.

Theorem 5. *Let r be a fixed real number. If f is such that $\phi_r(x,y)$ is bounded and goes to 0 uniformly in x as $y \downarrow 0$, (in particular if $f \in$ VMO), then D_ϱ^r, D^r belong to* VMO.

To prove these theorems we again use a continuity argument to control the "upper part" of D_ϱ^r and this time an L^1-argument to control the "bottom part".

Define

$$U^r(\theta)=\int_{\mathbf{R}^\nu\times(1,\infty)} y\,\varrho_\theta(z)\Delta|u-r|(dz).$$

Lemma 3.1. *There exists a constant C such that if $U^r(0)<\infty$ then for all $\theta\in Q_0$ we have*

$$|U^r(\theta)-U^r(0)|\le C\int_1^\infty t^{-2}\phi_r(0,t)\,dt.$$

Proof.

$$\begin{aligned}|U^r(\theta)-U^r(0)|&=\left|\int_{\mathbf{R}^\nu\times(1,\infty)} y\,(\varrho_\theta(z)-\varrho_0(z))\,\Delta|u-r|(dz)\right|\\&\le\int_{\mathbf{R}^\nu\times(1,\infty)} y\,|\varrho_\theta(z)-\varrho_0(z)|\,\Delta|u-r|(dz).\end{aligned}$$

But if $\theta\in Q_0$,

$$\begin{aligned}|\varrho_\theta(z)-\varrho_0(z)|&\le\frac{|\theta|}{y}\sup_{\lambda\in[0,1]}\frac{1}{y^\nu}\left|\nabla\varrho\left(\frac{x}{y}-\lambda\frac{\theta}{y}\right)\right|\\&\le\frac{1}{y}\sup_{\lambda\in[0,1]}p_0(x-\lambda\theta,y)\le\frac{C}{y}\,p_0(x,y)\\&\le C\int_y^{2y}p_0(x,t)\,\frac{dt}{t^2}\le C\int_y^\infty p_0(x,t)\,\frac{dt}{t^2}.\end{aligned}$$

The first inequality above follows from the mean value theorem the second by hypothesis (H$''$) and the third and fourth by the Harnack inequality. And so we obtain,

$$\begin{aligned}|U^r(\theta)-U^r(0)|&\le C\int_1^\infty t^{-2}\int_{\mathbf{R}^\nu\times(1,t)} y\,p_0(x,t)\Delta|u-r|(dz)dt\\&\le C\int_1^\infty t^{-2}\int_{\mathbf{R}^\nu\times(0,t)} y\,p_0(x,t)\Delta|u-r|(dz)dt\\&\le C\int_1^\infty t^{-2}\int_{\mathbf{R}^\nu\times(0,t)} \mathcal{G}(z,(0,t))\Delta|u-r|(dz)dt\\&=C\int_1^\infty t^{-2}\phi_r(0,t)dt\end{aligned}$$

where the last inequality follows from the fact that for $y<t$, $yp_0(x,t)\le Cp_0(x,y+t)$ (by the Harnack inequality) which in turn is less than or equal to $C\mathcal{G}(z,(0,t))$ by inequality (2.1) used with $z_0=z/t$ and scaling.

To control the bottom part of $D^r_\varrho(\theta)$ define

$$B(\theta)=\int_{\mathbf{R}^\nu\times(0,1)} y\varrho_\theta(z)\Delta|u-r|(dz).$$

We have

Lemma 3.2. *There exists a constant C (which depends only on ν and ϱ) such that*

$$\int_{Q_0} B^r(\theta)m_{Q_0}(d\theta)\le C\phi_r(0,1).$$

Proof. Applying Fubini's theorem and Lemma 2.4 we get

$$\begin{aligned}\int_{Q_0} B^r(\theta)m_{Q_0}(d\theta)&\le C\int_{\mathbf{R}^\nu} B^r(\theta)p_\theta(0,1)d\theta\\&=C\int_{\mathbf{R}^\nu\times(0,1)}\left(\int_{\mathbf{R}^\nu}\varrho_\theta(z)p_\theta(0,1)d\theta\right)y\Delta|u-r|(dz)\\&\le C\int_{\mathbf{R}^\nu\times(0,1)}\mathcal{G}(z,(0,1))\Delta|u-r|(dz)\\&\le C\phi_r(0,1).\end{aligned}$$

The next proposition is an immediate consequence of Lemmas 3.1 and 3.2.

Proposition 3.3. *Suppose f belongs to* BMO_0. *Then we can find a constant C_{Q_0} such that*

$$\int_{Q_0}\left|D^r_\varrho(\theta)-C_{Q_0}\right|m_{Q_0}(d\theta)\le C\phi_r(0,1)+C\int_1^\infty t^{-2}\phi_r(0,t)dt$$

where C is a constant independent of f and g.

Proof. Take $C_{Q_0}=U^r(0)$. Then by the above lemmas,

$$\begin{aligned}\int_{Q_0}\left|D^r_\varrho(\theta)-C_{Q_0}\right|m_{Q_0}(d\theta)&\le\int_{Q_0}|U^r(\theta)-U^r(0)|m_{Q_0}(d\theta)+\int_{Q_0}B^r(\theta)m_{Q_0}(d\theta)\\&\le C\int_1^\infty t^{-2}\phi_r(0,t)dt+C\phi_r(0,1)\end{aligned}$$

and the proposition is proved.

To prove Theorems 4 and 5 we remark that if Q is any cube centered at x_Q and with length l_Q, then by scaling (as in the proof of Theorem 1),

$$\int_Q\left|D^r_\varrho(\theta)-C_Q\right|m_Q(d\theta)\le C\int_1^\infty t^{-2}\phi_r(x_Q,l_Qt)\,dt+C\phi_r(x_Q,l_Q). \tag{3.1}$$

Thus if ϕ is bounded we immediately get that $D^r_\varrho\in\mathrm{BMO}$ which proves Theorem 4. Theorem 5 also follows from (3.1) by the Lebesgue dominated convergence theorem and our remark following (1.10).

4. The maximal density of the area integral for BMO and VMO functions

In this section we shall prove that the maximal density of the area integral is a good functional for BMO and VMO and this completes the results of Gundy [9] who showed that this functional is in L^p if f is in H^p. More precisely, we prove

Theorem 6. *If $f\in$BMO and D^* is not identically infinite, then D^*_ϱ, D^* belong to* BMO *with the* BMO*-norm majorized by a constant (independent of f) times the* BMO*-norm of f.*

Theorem 7. *If $f\in$VMO and D^* is not identically infinite, then D^*_ϱ, D^* belong to* VMO.

As before, we control the "bottom part" of D^*_ϱ and its "top part". We remind the reader that ϱ satisfies hypothesis (H$''$) defined in §3. Let us set

$$B^*(\theta)=\sup_{r\in\mathbf{R}}\int_{\mathbf{R}^\nu\times(0,1)} y\,\varrho_\theta(z)\Delta|u-r|(dz).$$

Lemma 4.1. *There exists a constant C (which depends only on ν and ϱ) such that:*

$$\int_{Q_0} B^*(\theta)m_{Q_0}(d\theta)\le C\|f\|_{p2,*1}+C\|f\|_{p1,*1}$$

where $\|f\|_{p1,*1}$ and $\|f\|_{p2,*1}$ are as defined in (1.9) *and* (1.10). *As before, Q_0 is the unit cube in $\mathbf{R}^\nu$ centered at the origin.*

Proof. Let

$$B_1^*(\theta)=\sup_{r\in\mathbf{R}}\int_{(\mathbf{R}^\nu\setminus 2Q_0)\times(0,1)} y\,p_\theta(z)\Delta|u-r|(dz)$$

and

$$B_2^*(\theta)=\sup_{r\in\mathbf{R}}\int_{2Q_0\times(0,1)} y\,p_\theta(z)\Delta|u-r|(dz).$$

By our assumption (H) we have $B^*(\theta)\le B_1^*(\theta)+B_2^*(\theta)$. If $\theta\in Q_0$, $x\in\mathbf{R}^\nu\setminus 2Q_0$ and $y\le 1$, then $p_\theta(z)\le Cp_0(z)$ and $y\,p_0(z)\le\mathcal{G}(z,(0,1))$. Thus

$$\begin{aligned}B_1^*(\theta)&\le C\sup_{r\in\mathbf{R}}\int_{(\mathbf{R}^\nu\setminus 2Q_0)\times(0,1)}\mathcal{G}(z,(0,1))\Delta|u-r|(dz)\\&\le C\sup_{r\in\mathbf{R}}\phi_r(0,1)\le C\|f\|_{p1,*1}\end{aligned}\tag{4.1}$$

and so we have the correct bound for B_1^*.

For $B_2^*(\theta)$ we use the Barlow–Yor [2] L^2-estimate. Let B_t, $t<\tau$ be Brownian motion in $\mathbf{R}_+^{\nu+1}$ until its exit time τ. Then $u(B_t)$, $t<\tau$, is a local martingale. Let $\{L_t^r; r\in\mathbf{R}, t<\tau\}$ be its local time. Then, (by Brossard [3]),

$$E_{z_0}^\theta[L_\tau^r] = \frac{1}{p_\theta(z_0)} \int_{\mathbf{R}_+^{\nu+1}} \mathcal{G}(z_0,z)p_\theta(z)\Delta|u-r|(dz) \tag{4.2}$$

where $E_{z_0}^\theta$ denotes the expectation of Brownian motion starting at z_0 and conditioned to exit at θ; (the Doob $-h$ process associated with $h(z)=p_\theta(z)$).

From (4.1) and the fact that for $z=(x,y)\in 2Q_0\times(0,1)$ and $z_0=(0,1)$, $\mathcal{G}(z_0,z)\geq Cy$, we have for $\theta\in Q_0$,

$$B_2^*(\theta)\leq C\sup_{r\in\mathbf{R}} E_{z_0}^\theta[L_\tau^r]\leq CE_{z_0}^\theta[L_\tau^*].$$

Integrating we obtain,

(4.3)
$$\begin{aligned}\int_{Q_0} B_2^*(\theta)m_{Q_0}(d\theta) &\leq C\int_{\mathbf{R}^\nu} B_2^*(\theta)p_\theta(0,1)d\theta \leq C\int_{\mathbf{R}^\nu} E_{z_0}^\theta[L_\tau^*]p_\theta(0,1)d\theta\\ &= CE_{z_0}[L_\tau^*]\leq C\left(E_{z_0}[L_\tau^*]^2\right)^{1/2}\leq C\left(E_{z_0}|u(B_\tau)-u(z_0)|^2\right)^{1/2}\\ &= C(\alpha(z_0))^{1/2}\leq C\|f\|_{p2,*1}.\end{aligned}$$

In the first equality above we used the fact that $p_\theta(z_0)d\theta$ is the law of B_τ given that $B_0=z_0$. We also used the result of Barlow and Yor [2] to bound the L^2-norm of L^* by the L^2-norm of $u(B_\tau)-u(z_0)$. Lemma 4.1 now follows from (4.1) and (4.3).

Theorems 6 and 7 follow from the following proposition as before.

Proposition 4.2. *Suppose $f\in\mathrm{BMO}_0$. There exists a constant C_{Q_0} such that*

$$\int_{Q_0} |D_\varrho^*(\theta)-C_{Q_0}|\, m_{Q_0}(d\theta)\leq C\|f\|_{p1,*1}+C\|f\|_{p2,*1}+C\int_1^\infty t^{-2}\|f\|_{p1,*t}dt$$

where C is a constant independent of f.

Proof. By Lemma 3.1 and (1.10) we have

$$|U^r(\theta)-U^r(0)|\leq C\int_1^\infty t^{-2}\phi_r(0,t)\,dt\leq C\int_1^\infty t^{-2}\|f\|_{p1,*t}\,dt$$

and taking supremum over r we obtain, with $U^*(\theta)=\sup_{r\in\mathbf{R}} U^r(\theta)$,

$$|U^*(\theta)-U^*(0)|\leq C\int_1^\infty t^{-2}\|f\|_{p1,*t}dt.$$

To prove the proposition, take $C_{Q_0}=U^*(0)$ and write

$$\begin{aligned}|D_\varrho^*(\theta)-C_{Q_0}|\le|D_\varrho^*(\theta)-U^*(\theta)|+|U^*(\theta)-C_{Q_0}|&\le B^*(\theta)+|U^*(\theta)-U^*(0)|\\&\le B^*(\theta)+C\int_1^\infty t^{-2}\|f\|_{p1,*t}\,dt.\end{aligned}$$

Integrating over Q_0 and using Lemma 4.1 gives the proposition.

For a general cube Q centered at x_Q and length l_Q, scaling again as before gives

$$\int_Q\left|D_\varrho^*(\theta)-C_Q\right|m_Q(d\theta)\le C\|f\|_{p1,*l_Q}+C\|f\|_{p2,*l_Q}+C\int_1^\infty t^{-2}\|f\|_{p1,*l_Qt}dt$$

and Theorems 6 and 7 follow from this.

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References

1. BAÑUELOS, R. and MOORE, C., Distribution function inequalities for the density of the area integral, *Ann. Inst. Fourier (Grenoble)* **41** (1991), 137–171.
2. BARLOW, M. and YOR, M., (Semi)Martingale inequalities and local times, *Z. Wahrsch. Verw. Gebiete* **55** (1981), 237–254.
3. BROSSARD, J., Densité de l'intégrale d'aire dans $\mathbf{R}_+^{n+1}$ et limites non tangentielles, *Invent. Math.* **93** (1988), 297–308.
4. BROSSARD, J. and CHEVALIER, L., Classe $L\log L$ et densité de l'intégrale d'aire dans $\mathbf{R}_+^{n+1}$, *Ann. of Math.* **128** (1988), 603–618.
5. BROSSARD, J. and CHEVALIER, L., Problème de Fatou ponctuel et dérivabilité des mesures, *Acta Math.* **164** (1990), 237–263.
6. COIFMAN, R. and WEISS, G., Extension of Hardy spaces and their uses in analysis, *Bull. Amer. Math. Soc.* **83** (1977), 569–645.
7. FEFFERMAN, C. and STEIN, E. M., H^p spaces of several variables, *Acta Math.* **129** (1972), 137–193.
8. GARNETT, J., *Bounded Analytic Functions*, Academic Press, New York, 1981.
9. GUNDY, R. F., The density of the area integral, in *Conference on Harmonic Analysis in Honor of Antoni Zygmund* (Beckner, W., Calderón, A. P., Fefferman, R. and Jones, P., eds.), pp. 138–149, Wadsworth, Belmont, Calif., 1983.
10. GUNDY, R. F. and SILVERSTEIN, M. L., The density of the area integral in $\mathbf{R}_+^{n+1}$, *Ann. Inst. Fourier (Grenoble)* **35** (1985), 215–224.
11. KURTZ, D., Littlewood–Paley operators on BMO, *Proc. Amer. Math. Soc.* **99** (1987), 657–666.

12. MEYER, P. A., Le dual de $H^1(\mathbf{R}^\nu)$: démonstrations probabilistes, in *Séminaire de Probabilités XI, Lecture Notes in Math.* **581**, pp. 132–195, Springer-Verlag, Berlin–Heidelberg–New York, 1977.
13. MOORE, C. N., Some inequalities for the density of the area integral, in *Proceeding of University of Chicago Conference on Partial Differential Equations with Minimal Smoothness and Applications*, Inst. Math. Appl. Publ. **42**, to appear.
14. QIAN, T., On BMO boundedness of a class of operators (Chinese), *J. Math. Res. Exposition* **7** (1987), 331–334, (see Math. Reviews, 89d:42029).
15. SARASON, D., Functions of vanishing mean oscillation, *Trans. Amer. Math. Soc.* **209** (1975), 391–405.
16. TORCHINSKY, A., *Real-Variable Methods in Harmonic Analysis*, Academic Press, San Diego, Calif., 1986.
17. WANG, S., Some properties of Littlewood–Paley's g–function, *Sci. Sinica Ser. A* **28** (1985), 252–262.

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