LINEAR DIFFERENTIAL EQUATIONS WITH ALMOST PERIODIC COEFFICIENTS.

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§ 1. Introduction.

1.1. We shall deal with the system of differential equations

$$\begin{aligned} \frac{d\,\xi_1(t)}{dt} &= \alpha_{11}(t)\,\xi_1(t) + \cdots + \alpha_{1n}(t)\,\xi_n(t) + \beta_1(t) \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \frac{d\,\xi_n(t)}{dt} &= \alpha_{n1}(t)\,\xi_1(t) + \cdots + \alpha_{nn}(t)\,\xi_n(t) + \beta_n(t); \end{aligned} \tag{1.11}$$

in which the functions $a_{\nu\nu}(t)$ and β_μ are real or complex a. p.[2] functions of the real variable t, and the $\beta_\mu(t)$ may or may not be identically zero. We shall seek to determine conditions under which the solutions of (1.11) are of a rather general type involving a. p. functions. Before characterizing this type of solution more explicitly, we shall introduce a shorter vector terminology.

1.2. Troughout this paper we shall use the letters x, y, z, and b to denote n-dimensional vectors (or matrices of n rows and one column) having the components $\xi_1, \ldots, \xi_n$; $\eta_1, \ldots, \eta_n$; $\zeta_1, \ldots, \zeta_n$; and $\beta_1, \ldots, \beta_n$. The vector $\frac{d}{dt}\xi_1(t), \ldots, \frac{d}{dt}\xi_n(t)$ will be denoted by $D[x]$; the n-by-n matrix whose elements are $\alpha_{\mu\nu}$ will be denoted by A, and the matrix product of A and x will be denoted by $A \,.\, x$. Hence in this terminology (1.11) becomes

$$D[x(t)] = A(t) \cdot x(t) + b(t). \tag{1.21}$$

We shall also define a norm for vectors, namely $\|x\| = |x_1| + \cdots + |x_n|$.

[1] This paper was written while the author was a National Research Fellow.

[2] a. p. = almost periodic (in Bohr's sense).

1.3. If we consider the known facts in the case where $A(t)$ and $b(t)$ are actually periodic with a common period P, we find that all of the solutions of (1.21) are of the form

$$x(t) = \sum_{\nu=1}^{p} e^{\lambda_\nu t} r_\nu \, y^{(\nu)}(t), \tag{1.31}$$

where p is some positive integer, the λ_ν are complex numbers, the r_ν are non-negative integers, and the $y^{(\nu)}(t)$ are a. p. vector functions.[1] It therefore seems very natural to ask whether in the general case (when $A(t)$ and $b(t)$ are a. p.) the solutions will all be of the form (1, 31) with the $y^{(\nu)}(t)$ a. p. instead of periodic. Unfortunately such is not the case, as can be readily shown by examples. However, the analogy with the periodic case makes (1.31) seem a natural type of solution, and we are therefore going to seek for conditions under which the solutions will all be of this type. It is clear that in the general a. p. case we can without loss of generality assume the λ_ν to be real, for if they have an imaginary part the exponential breaks up into two factors of which the one with the imaginary part can be absorbed into the a. p. vector functions $y^{(\nu)}(t)$.

1.4. *Definition.* A vector function $x(t)$ will be said to be of the a. p. type if there exist a positive integer p, real numbers $\lambda_1, \ldots, \lambda_p$, non-negative integers $r_1, \ldots, r_p$, and a. p. vector functions $y^{(1)}(t), \ldots, y^{(n)}(t)$ such that (1.31) holds identically in t. The least common module of $y^{(1)}(t), \ldots, y^{(n)}(t)$ will be called the module of $x(t)$.

1.5. It is the aim of this paper to obtain necessary and sufficient conditions that all of the solutions of (1.21) be of the a. p. type and to obtain sufficient conditions that a particular solution be of the a. p. type.

1.6. The vector $b(t)$ may without loss of generality be taken to be identically zero and equation (1.21) replaced by

$$D[x(t)] = A(t) \cdot x(t). \tag{1.61}$$

For consider the system

$$\begin{cases} D[x(t)] = A(t) \cdot x(t) + \xi_{n+1}(t)\, b(t) \\ \dfrac{d}{dt} \xi_{n+1}(t) = 0; \end{cases} \tag{1.62}$$

[1] S. Bochner, Abstrakte fastperiodische Funktionen, Acta mathematica, vol. 61 (1933), 149—184.

where $\xi_{n+1}(t)$ is a scalar function; and let $x^*(t)$ be the $n+1$ dimensional vector consisting of $x(t)$ and $\xi_{n+1}(t)$; so that its components are $\xi_1(t), \xi_2(t), \ldots, \xi_{n+1}(t)$. If $x(t)$ is a solution of (1.21), then for every constant C, $x^*(t) = \{Cx(t), C\}$ is a solution of (1.62); and every solution $x^*(t)$ can be written in this form for some $x(t)$ if $\xi_{n+1}(t) \neq 0$. But (1.62) is homogeneous in the $\xi_\nu(t)$ and is really in the form (1.61). Thus an equation of the form (1.21) can be reduced to one of the form (1.61), and theorems proved concerning the homogeneous equation can readily be re-phrased so as to apply to the non-homogeneous one.

§ 2. Decomposition of a Solution of the a. p. Type.

2.1. In our search for necessary and sufficient conditions that all the solutions be of the a. p. type, we shall first study the properties of solutions which are of the a. p. type and thus obtain necessary conditions. Consequently it will usually be assumed in sections 2 and 3 not only that $A(t)$ is a. p. but also that one or more solutions of (1.61) is of the a. p. type.

In particular we shall be interested in the assymptotic behavior of solutions as $t \to +\infty$ or as $t \to -\infty$, and therefore introduce the following notation. If there exists a positive constant C such that $\|x(t)\| < Cf(t)$ for all sufficiently great t, we say that

$$x(t) = O[f(t)] \text{ at } +\infty;$$

and if there exist two positive constants C_1 and C_2 such that $C_1 f(t) < \|x(t)\| < C_2 f(t)$ for all sufficiently great t, we say that

$$x(t) = O^*[f(t)] \text{ at } +\infty.$$

The meaning to be attached to the statement that $x(t)$ equals $O[f(t)]$ or $O^*[f(t)]$ at $-\infty$ or at $\pm\infty$ is obvious.

It is obvious that if a solution $x(t)$ of (1.61) is of the a. p. type and is expressed in the form (1.31) with real λ_ν, a. p. $y^{(\nu)}(t)$, and non-negative integers r_ν, then

$$x(t) = O[e^{\lambda t} t^r] \text{ at } +\infty, \tag{2.11}$$

where λ is the greatest value of λ_ν occuring in its expression of the form (1.31), and r is the greatest value of r_ν for values of ν such that $\lambda_\nu = \lambda$. If λ is the least of the λ_ν instead of the greatest, (2.11) holds at $-\infty$, and if λ_ν is a constant independent of ν, (2.11) holds at $\pm\infty$. This suggests the

Definition. Let $A(t)$ be any continuous matrix function. Then a solution $x(t)$ of (1.61) will be called a primary solution of order λ and degree r if $x(t) = O[e^{\lambda t} t^r]$ at $\pm\infty$.

In this section it will be shown that if $A(t)$ is a. p., any solution $x(t)$ of the a. p. type can be decomposed into the sum of a finite number of primary solutions.

2.2. Let $A(t)$ be an a. p. matrix function, and let $x(t)$ be a solution (1.61) of the a. p. type. Then $x(t)$ can be written in the form

$$(2.21)\quad x(t) \equiv e^{\lambda_1 t}[y_{1,0}(t) + t y_{1,1}(t) + \cdots + t^{r_1} y_{1,r_1}(t)] + \cdots$$
$$+ e^{\lambda_p}[y_{p,0}(t) + t y_{p,1}(t) + \cdots + t^{r_p} y_{p,r_p}(t)],$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_p$, the $y_{\mu,\nu}(t)$ are a. p., and none of the functions $y_{1,r_1}(t), \ldots, y_{p,r_p}(t)$ is identically zero. We shall first show that when $x(t)$ is written in this way each of the terms in the above expression (i. e., an entire bracketed expression with its exponential multiplier) is itself a solution. To simplify the notation, we shall drop the subscripts from λ_1 and r_1 and let

$$(2.22)\qquad y_\mu(t) = (r-\mu)!\, y_{1,r-\mu}(t) \qquad (\mu = 0, 1, \ldots, r)$$

$$(2.23)\qquad x_\mu(t) = e^{\lambda t} \sum_{\nu=0}^{\mu} \frac{t^\nu}{\nu!} y_{\mu-\nu}(t) \qquad (\mu = 0, 1, \ldots, r).$$

(The reason for introducing the factorials will appear shortly.) Here $x_r(t)$ is identical with the first term of (2.21), and we shall let $x^*(t)$ denote the sum of all the other terms. It follows that $x(t) = x_r(t) + x^*(t)$; and that for any s,

$$(2.24)\qquad \lim_{t\to\infty} e^{-\lambda t} t^s x^*(t) = 0.$$

2.3. We shall now try, by using (2.24), to perform a transformation on $x(t)$ which will still leave it a solution of (1.61) but will get rid of $x^*(t)$. To do this, we find a sequence of positive numbers $h_1, h_2, \ldots$ whose limits is infinity and for which $\lim_{i\to\infty} A(t+h_i) = A(t)$ uniformly in t and $\lim_{i\to\infty} y_\mu(t+h_i) = y_\mu(t)$ uniformly in t for each $\mu \leqq r$. Such a sequence exists since for each i we can choose h_i so as to be greater than i and at the same time a $1/i$-translation number for $A(t)$ and every $y_\mu(t)$. Having chosen the sequence, and remembering (2.23) and (2.24), we have

$$r!\lim_{i\to\infty}[e^{-\lambda h_i}h_i^{-r}x(t+h_i)] = e^{\lambda t}y_0(t) = x_0(t).$$

But $r!\,e^{-\lambda h_i}h_i^{-r}x(t+h_i)$ is a solution of

$$D[z(t)] = A(t+h_i)\cdot z(t), \tag{2.31}$$

and therefore the limit $x_0(t)$ of the sequence is a solution of the limiting system (1.61). Hence our method is partially successful, as we have transformed $x(t)$ in such a way that all the terms of (2.21) except the first have vanished. However, the first term $x_r(t)$ has become $x_0(t)$, and we must therefore find some way of getting back to $x_r(t)$.

2.4. If our new solution $x_0(t)\equiv e^{\lambda t}y_0(t)$ had contained an extra factor t^r, we would have been able to subtract out the last term of $x_r(t)$ and obtain a new solution to which we could have applied the same process. Thus we would have obtained the result that each term of $x_r(t)$ was a solution, and hence that $x_r(t)$ was a solution. However, since the factor t^r is missing, the last term of $x_r(t)$ is not a solution unless $r=0$. As a matter of fact none of the terms of $x_r(t)$ is a solution when $r>0$, and hence our process must be modified; and we will find that as we repeat the modified process we get successively $x_1(t)$, $x_2(t)$, ... and finally $x_r(t)$ instead of obtaining monominal terms from $x_r(t)$. Before passing to the complete induction proof that all of the $x_\mu(t)$ are solutions, we will carry through one more step of the process in order to see what kind of modifications enter.

2.5. We again deal with $x(t+h_i)$, and again the only significant term is $x_r(t+h_i)$, which may be written in the expanded form

$$x_r(t+h_i) = e^{\lambda(t+h_i)}\sum_{\nu=0}^{r} y_{r-\nu}(t+h_i)\sum_{\varrho=0}^{\nu}\frac{t^{\nu-\varrho}h_i^{\varrho}}{(\nu-\varrho)!\,\varrho!} \tag{2.51}$$

$$\equiv e^{\lambda(t+h_i)}\frac{1}{r!}y_0(t+h_i)h_i^r + e^{\lambda(t+h_i)}\frac{1}{(r-1)!}[y_1(t+h_i)+t\,y_0(t+h_i)]h_i^{r-1}$$

$$+\cdots(\text{terms in } h_i^{r-2},\ h_i^{r-3},\ \text{etc.}).$$

But the first term of this expression is $\frac{h_i^r}{r!}x_0(t+h_i)$, and $x(t+h_i)-\frac{h_i^r}{r!}x_0(t+h_i)$ is a solution of (2.31). Thus

$$x_1(t) = e^{\lambda t}[y_1(t) + t y_0(t)] = (r-1)! \lim_{i\to\infty} \frac{x(t+h_i) - \frac{h_i^r}{r!} x_0(t+h_i)}{e^{\lambda h_i} h_i^{r-1}}$$

is a solution of (1.61).

2.6. From the way in which the second step was carried out, we see that the subtraction of a multiple of the new solution is carried out after the argument of the solution has been translated, and that the multiplier to be used is a function of the h_i. In succeeding steps a further complication arises because a combination of all the solutions already obtained has to be subtracted, and further summations have to be used. We now pass to the general case.

Since we shall need to pass from the $y_\mu(t)$ to the $x_\mu(t)$ as well as from the $x_\mu(t)$ to the $y_\mu(t)$, we first note that the r equations (2.23) have a unique solution if we regard the n functions $y_\mu(t)$ as unknowns; and it can readily be verified that the expressions

$$\text{(2.61)} \qquad y_\mu(t) = e^{-\lambda t} \sum_{\nu=0}^{\mu} \frac{(-t)^\nu}{\nu!} x_{\mu-\nu}(t) \qquad (\mu = 0, \ldots, r)$$

satisfy (2.23), and hence that (2.61) holds.

2.7. We now assume that $x_\mu(t)$ is a solution of (1.61) for all μ less than a certain integer q which does not exceed r. As before, we consider $x_r(t+h_i)$ and obtain on interchanging the order of summation in (2.51),

$$x_r(t+h_i) = e^{\lambda(t+h_i)} \sum_{\varrho=0}^{r} \frac{h_i^\varrho}{\varrho!} \sum_{\nu=\varrho}^{r} \frac{t^{\nu-\varrho}}{(\nu-\varrho)!} y_{r-\nu}(t+h_i) = \sum_{\varrho=0}^{r} \frac{h_i^\varrho}{\varrho!} \tilde{x}_{r-\varrho}(t;\, h_i),$$

where

$$\text{(2.71)} \qquad \tilde{x}_\mu(t;\, h) \equiv e^{\lambda(t+h)} \sum_{\nu=0}^{\mu} \frac{t^\nu}{\nu!} y_{\mu-\nu}(t+h), \qquad (\mu = 0, \ldots, r).$$

Substituting from (2.61) in (2.71), we obtain

$$\tilde{x}_\mu(t;\, h) = \sum_{\nu=0}^{\mu} \frac{t^\nu}{\nu!} \sum_{\varrho=0}^{\mu-\nu} \frac{(-t-h)^\varrho}{\varrho!} x_{\mu-\nu-\varrho}(t+h) = \sum_{\sigma=0}^{\mu} x_{\mu-\sigma}(t+h) \sum_{\nu=0}^{\sigma} \frac{t^\nu(-t-h)^{\sigma-\nu}}{\nu!\,(\sigma-\nu)!}$$

$$= \sum_{\sigma=0}^{\mu} \frac{(-h)^\sigma}{\sigma!} x_{\mu-\sigma}(t+h)$$

and it follows that if $\mu < q$, $\tilde{x}_\mu(t, h_i)$ is a solution of (2.31). Thus

$$X(t, h_i) \equiv x(t+h_i) - \sum_{\varrho=r-q+1}^{r} \frac{h_i^{\varrho}}{\varrho!} \tilde{x}_{r-\varrho}(t; h_i)$$

$$\equiv x^*(t+h_i) + e^{\lambda(t+h_i)} \sum_{\varrho=0}^{r-q} \frac{h_i^{\varrho}}{\varrho!} \sum_{\nu=\varrho}^{r} \frac{t^{\nu-\varrho}}{(\nu-\varrho)!} y_{r-\nu}(t+h_i)$$

is also a solution of (2. 31), and

$$\lim_{i\to\infty} \frac{(r-q)!\, X(t, h_i)}{e^{\lambda h_i} h_i^{r-q}} = e^{\lambda t} \sum_{\nu=r-q}^{r} \frac{t^{\nu-r+q}}{(\nu-r+q)!} y_{r-\nu}(t) = x_q(t)$$

is a solution of (1. 61). We now know that $x_\mu(t)$ satisfies (1. 61) for all $\mu < q+1$, and the induction is complete.

2. 8. Finally, since $x_r(t)$ is a solution, so is $x^*(t)$; and the same argument can be applied to it. Thus each group of terms in $x(t)$ having the same exponential factors forms a solution. Moreover the expansion (2. 21) of $x(t)$ is unique, for $x_0(t), \ldots, x_r(t)$ have been given in terms of $x(t)$ by a uniquely defined process, the $y_{1,0}(t), \ldots, y_{1,r_1}(t)$ are uniquely defined in terms of the $x_\mu(t)$, and the $y_{2,0}(t), \ldots, y_{2,r_2}(t)$, etc. are uniquely defined afterwards in turn. These results will be summed up in (2. 9).

2. 9. In order to state the results of (2. 2—2. 8) more concisely, we first introduce by means of the following definition a terminology for some of the concepts which have arisen.

Definition. — Let $A(t)$ be any continuous square matrix function and let $x_0(t) \not\equiv 0$, $x_1(t)$, $x_2(t)$, $\ldots$, $x_r(t)$ be solutions of (1. 61) and λ a real number such that for all $\mu \leqq r$,

$$y_\mu(t) \equiv e^{-\lambda t} \sum_{\nu=0}^{\mu} \frac{(-t)^\nu}{\nu!} x_{\mu-\nu}(t) \tag{2.91}$$

is bounded for all t. Then $y_r(t)$ will be called a pseudo-solution of (1. 61) of order λ and degree r. Moreover for $\mu \leqq r$, $x_\mu(t)$ and $y_\mu(t)$ will be called respectively its generator and its minor of degree μ, and $x_r(t)$ will be called its leader. Finally, a solution $x(t)$ of (1. 61) will be called a satisfactory solution of order λ and degree r if it is the leader of a pseudo-solution $y(t)$ of order λ and degree r, and $y(t)$ will be called its associated pseudo-solution.

If we express the quantity $e^{\lambda t} \sum_{\nu=0}^{\mu} \frac{t^\nu}{\nu!} y_{\mu-\nu}(t)$ in terms of $x_0(t), \ldots, x_\mu(t)$ by means of (2. 91), we obtain $x_\mu(t)$ as a result. Thus we have

Lemma 1. Let $A(t)$ be any continuous square matrix function. Then if $y_r(t)$ is a pseudo-solution of order λ and degree r and has the minors $y_0(t), \ldots, y_r(t)$ and generators $x_0(t), \ldots, x_r(t)$, it follows that for each $\mu \leqq r$,

$$x_\mu(t) \equiv e^{\lambda t} \sum_{\nu=0}^{\mu} \frac{t^\nu}{\nu!} y_{\mu-\nu}(t). \tag{2.92}$$

Hence every satisfactory solution is a primary solution.

Definition. Let $A(t)$ be any continuous square matrix function. Then a solution $x(t)$ which is the sum of a finite number of satisfactory solutions $x^{(1)}(t), \ldots, x^{(p)}(t)$ having distinct orders $\lambda_1, \ldots, \lambda_p$ is called a decomposable solution. If λ is the greatest of $\lambda_1, \ldots, \lambda_p$, $x(t)$ is called a decomposable solution of order λ.

Now in terms of the above definitions, we sum up the results of (2.2—2.8).

Theorem I. Let $A(t)$ be an a. p. square matrix function and let $x(t)$ be a solution of (1.61) of the a. p. type. Then $x(t)$ can be expressed in one and only one way in the form

$$x(t) \equiv \sum_{\sigma=1}^{p} e^{\lambda_\sigma t} \sum_{\nu=0}^{r_\sigma} \frac{t^\nu}{\nu!} y_{\sigma, r_\sigma - \nu}(t), \tag{2.93}$$

where the λ_σ are real and distinct, the $y_{\sigma,\nu}(t)$ are a. p., and none of the $y_{\sigma,0}(t)$ are identically zero. Moreover for each $\sigma \leqq p$, $y_{\sigma, r_\sigma}(t)$ is a pseudo-solution of (1.61) of order λ_σ and degree r_σ having the minors $y_{\sigma,0}(t), \ldots, y_{\sigma, r_\sigma}(t)$. Finally, $x(t)$ is the sum of the leaders of $y_{1, r_1}(t), \ldots, y_{p, r_p}(t)$, and hence $x(t)$ is decomposable.

§ 3. The Necessity of Condition I.

3.1. In this section we shall show that if $A(t)$ is a. p. and all the solutions of (1,61) are of the a. p. type, then for each non-trivial solution $x(t)$ there exist a real number λ and a non-negative integer r such that $x(t) = O^*(e^{\lambda t} t^r)$ at $+\infty$. In fact, we shall show that a similar statement holds for certain combinations of solutions, so that the following condition holds.

Condition I. The system (1.61) will be said to satisfy Condition I if to every finite set of solutions $x^{(0)}(t), x^{(1)}(t), \ldots, x^{(p)}(t)$ (not all identically zero) there correspond a real number λ and a non-negative integer r such that

$$\sum_{\nu=0}^{p} t^{\nu} x^{(\nu)}(t) = O^*(e^{\lambda t} t^r) \quad \text{at} \quad + \infty.$$

Since Theorem I enables us (under the above hypothesis) to express solutions in terms of pseudo-solutions, we will determine the asymptotic behavior of pseudo-solutions as a preliminary to determining the asymptotic behavior of solutions and combinations of solutions. We therefore develop in this section certain properties of pseudo-solutions — first some general properties and later asymptotic properties.

3.2. We begin with a general uniqueness lemma.

Lemma 2. Let $A(t)$ be any continuous matrix function. Then every pseudo-solution $y(t)$ of (1.61) has a unique order and degree and a unique set of minors and generators.

For suppose that $y(t)$ has the set of generators $x_0(t), \ldots, x_r(t)$ and the corresponding order λ and degree r, and at the same time $y(t)$ has the set of generators $x'_0(t), \ldots, x'_{r'}(t)$ and the corresponding order λ' and degree r'. Then if t_0 is any constant value of t, the two solutions

$$e^{-\lambda t_0} \sum_{\nu=0}^{r} \frac{(-t_0)^{\nu}}{\nu!} x_{r-\nu}(t) \quad \text{and} \quad e^{-\lambda' t_0} \sum_{\nu=0}^{r'} \frac{(-t_0)^{\nu}}{\nu!} x'_{r'-\nu}(t)$$

are both equal to $y(t_0)$ when $t = t_0$, and hence are equal for all t. Putting $t = 0$ and varying t_0, we obtain a linear relation between the functions

$$e^{-\lambda t_0},\ e^{-\lambda t_0} t_0,\ \ldots,\ e^{-\lambda t_0} t_0^{r};\quad e^{-\lambda' t_0},\ e^{-\lambda' t_0} t_0,\ \ldots,\ e^{-\lambda' t_0} t_0^{r'}$$

with constant vector coefficients not all of which are zero (since $x_0(t)$ and $x'_0(t)$ are non-trivial). Thus these functions are not linearly independent, and $\lambda = \lambda'$. But since $1, t_0, t_0^2, \ldots$ are linearly independent it follows thar $r = r'$ and

$$x_{\nu}(t) - x'_{\nu}(t) \equiv 0 \quad \text{for} \quad \nu = 0, \ldots, r.$$

Moreover (2.91) determines the minors in terms of the generators, and hence they are also unique.

3.3. Next we notice that the property of being a pseudo-solution of a given order is invariant under addition and under multiplication by a constant. This fact can easily be verified, so no proof will be given.

Lemma 3. Let $A(t)$ be a continuous square matrix function, λ a real number, and r a non-negative integer (or the symbol $+\infty$). Then the set of pseudo-solutions of (1.61) of order λ and degree $\leqq r$ form with the trivial solution a linear manifold.

Definition. If $A(t)$ is a continuous square matrix function, the manifold consisting of the trivial solution together with set of pseudo-solutions of (1.61) of order λ and degree $\leqq r$ will be called the pseudo-solution manifold of (1.61) of order λ and degree r. If r is not specified, it will be understood to be the symbol $+\infty$.

3.4. Another lemma that holds under fairly general conditions is the following.

Lemma 4. Let $A(t)$ be any continuous n-by-n matrix function and let $y_r(t)$ be a pseudo-solution of (1.61) of order λ and degree r having the generators $x_0(t), \ldots, x_r(t)$ and minors $y_0(t), \ldots, y_r(t)$. Then if $y_0(t)$ is bounded away from zero for sufficiently great positive values of t, we have for $\mu \leqq r$

$$x_\mu(t) = O^*[e^{\lambda t} t^\mu] \quad \text{at} \quad +\infty. \tag{3.41}$$

Moreover for each t the vectors $x_0(t), \ldots, x_r(t)$ are linearly independent, so that $r < n$ and $y_\mu(t)$ is different from zero for every value of μ and t.

For (3.41) is an obvious consequence of (2.92), and the linear independence of the vector functions $x_\mu(t)$ follows from (3.41). But if there is a linear relation between the $x_\mu(t)$ at a certain point $t = t_0$, the same relation holds for all t, since a solution which vanishes at one point vanishes identically. Thus the linear independence of the vector functions implies the linear independence of the vectors obtained by giving t a particular value.

3.5. Returning now to the case in which $A(t)$ is a. p. we shall show that under reasonable conditions the property of being a pseudo-solution is not altered by a limiting translation of t.

Lemma 5. Let $A(t)$ be an a. p. matrix function, and let $y_r(t)$ be a pseudo-solution of (1.61) of order λ and degree r whose minors and generators are $y_0(t), \ldots, y_r(t)$ and $x_0(t), \ldots, x_r(t)$ respectively. Also let $h_1, h_2, \ldots$ be a sequence such that $\tilde{A}(t) \equiv \lim\limits_{i\to\infty} A(t+h_i)$ exists uniformly in t and $\lim\limits_{i\to\infty} y_\mu(h_i)$ exists for each μ and does not vanish when $\mu = 0$. Then $\tilde{y}_r(t) \equiv \lim\limits_{i\to\infty} y_r(t+h_i)$ exists for all t and is a pseudo-solution of order λ and degree r of

$$D[\tilde{x}(t)] = \tilde{A}(t) \cdot \tilde{x}(t). \tag{3.51}$$

Moreover its minor of degree μ is

$$\tilde{y}_\mu(t) = \lim_{i \to \infty} y_\mu(t + h_i) \qquad (\mu = 0, \ldots, r) \tag{3.52}$$

and its generator of degree μ is

$$\tilde{x}_\mu(t) = \lim_{i \to \infty} e^{-\lambda h_i} \sum_{\nu=0}^{\mu} \frac{(-h_i)^\nu}{\nu!} x_{\mu-\nu}(t + h_i) \qquad (\mu = 0, \ldots, r). \tag{3.53}$$

To obtain this result, we first express the function $y_\mu(t+h_i)$ in terms of the $x_\mu(t+h_i)$ by means of (2.91), and then expand the binomials on the right, and interchange the order of summation.

We find that

$$y_\mu(t + h_i) = e^{-\lambda t} \sum_{\nu=0}^{\mu} \frac{(-t)^\nu}{\nu!} x^*_{\mu-\nu}(t; h_i), \tag{3.54}$$

where

$$x^*_\mu(t; h_i) = e^{-\lambda h_i} \sum_{\nu=0}^{\mu} \frac{(-h_i)^\nu}{\nu!} x_{\mu-\nu}(t + h_i).$$

Since $x^*_\mu(0; h_i) = y_\mu(h_i)$, $\lim_{i \to \infty} x^*_\mu(0, h_i)$ exists for each μ; and as $i \to \infty$ the sequence of systems $D[x(t)] = A(t+h_i) \cdot x(t)$ approach a limiting system uniformly and its sequence of solutions $x^*_\mu(t; h_i)$ approaches a limit at one point, $t = 0$. Thus the sequence of solutions approaches a limit for all t, and this limit is a solution of the limiting system; or in other words, for each μ, $\tilde{x}_\mu(t)$ exists for all t as defined by (3.53), and is a solution of (3.51). Hence we can take limits as $i \to \infty$ in (3.54), and we find that for each μ, $\tilde{y}_\mu(t)$ exists for all t as defined by (3.52), and $\tilde{y}_\mu(t) = e^{-\lambda t} \sum_{\nu=0}^{\mu} \frac{(-t)^\nu}{\nu!} \tilde{x}_{\mu-\nu}(t)$. The latter equation taken in conjunction with the fact that $\tilde{x}_0(0) = \lim_{i \to \infty} y_0(h_i) \neq 0$, shows that $\tilde{y}_r(t)$ is a pseudo-solution of order λ and degree r of (3.51), having the $\tilde{x}_\mu(t)$ as generators and the $\tilde{y}_\mu(t)$ as minors.

3.6. We now show in Lemmas 6 and 7 that if $A(t)$ is a. p. and all the solutions of (1.61) are of the a. p. type, then every pseudo-solution of (1.61) is bounded away from zero.

Lemma 6. Let λ be a real number and $A(t)$ an a. p. matrix function such that every pseudo-solution of (1.61) of order λ and degree zero is bounded away from zero for positive t. Then all of the pseudo-solutions of (1.61) of order λ are bounded away from zero for all positive t.

For suppose that there is a pseudo-solution $y_r(t)$ of order λ and some degree r which comes arbitrarily close to zero for positive t. Let its minors be $y_0(t), \ldots, y_r(t)$ and let $h_1, h_2, \ldots$ be a sequence of positive numbers such that $\lim\limits_{i\to\infty} y_r(h_i) = 0$. Since each of the $y_\mu(t)$ is bounded and because of Bochner's theorem on normal functions[1] we can choose a subsequence $h'_1, h'_2, \ldots$ of $h_1, h_2, \ldots$ such that $\lim\limits_{i\to\infty} y_\mu(h'_i)$ exists for each μ and such that $\tilde{A}(t) \equiv \lim\limits_{i\to\infty} A(t+h'_i)$ exists uniformly in t. By hypothesis, $\lim\limits_{i\to\infty} y_0(h'_i) \neq 0$, and hence by Lemma 5, $\tilde{y}_\mu(t) \equiv \lim\limits_{i\to\infty} y_\mu(t+h'_i)$ exists for all μ and t and $\tilde{y}_r(t)$ is a pseudo solution of $D[x(t)] = \tilde{A}(t)\cdot x(t)$ of order λ and degree r having the minors $\tilde{y}_0(t), \ldots, \tilde{y}_r(t)$. But $\tilde{y}_0(t)$ is obviously bounded away from zero, and hence Lemma 4 applies and shows that $\tilde{y}_r(0) \neq 0$, in spite of our assumption that $\lim\limits_{i\to\infty} y_r(h_i) = 0$. Thus our lemma holds.

It is of course clear that if the hypothesis had been that the pseudo-solutions of order λ and degree zero were bounded away from zero for negative t (or for all t) the corresponding conclusion would hold for negative t (or for all t).

3.7. **Lemma 7.** Let $A(t)$ be an a. p. matrix function. Then all of the a. p. pseudo-solutions of (1.61) of degree zero are bounded away from zero.

For suppose that there exists an a. p. pseudo-solution $y_0(t)$ of degree zero and some order λ which comes arbitrarily close to zero. Let $x_0(t) \equiv e^{\lambda t} y_0(t)$ be the generator of $y_0(t)$ and $h_1, h_2, \ldots$ a sequence such that $\lim\limits_{i\to\infty} y_0(h_i) = 0$. Let $h'_1, h'_2, \ldots$ be a subsequence of $h_1, h_2, \ldots$ such that $\tilde{A}(t) \equiv \lim\limits_{i\to\infty} A(t+h'_i)$ and $\tilde{y}_0(t) \equiv \lim\limits_{i\to\infty} y_0(t+h'_i)$ exist uniformly in t for all t. Then if we let $e^{\lambda t}\tilde{y}_0(t) \equiv \tilde{x}_0(t)$ we have

$$\lim_{i\to\infty} e^{-h'_i} x_0(t+h'_i) \equiv \tilde{x}_0(t),$$

[1] Fastperiodische Funktionen I, Mathematische Annalen, vol. 96, pp. 119—147, esp. p. 143.

so that $\tilde{x}_0(t)$ is a solution of $D[x(t)] = \tilde{A}(t) \cdot x(t)$. But $\tilde{x}(0) = \tilde{y}_0(0) = 0$; so $\tilde{x}_0(t) \equiv 0$ and $\tilde{y}_0(t) \equiv 0$; and because of the uniformity of the limits $y_0(t) = \lim_{i \to \infty} \tilde{y}_0(t - h'_i) \equiv 0$. Thus $x_0(t)$ is identically zero, which is contrary to the assumption that it is the generator of $y_0(t)$.

3.8. Having proved all the necessary preliminary lemmas, we state our main conclusion for this section.

Theorem II. Let $A(t)$ be an a. p. matrix function. Then a necessary condition that all the solutions of (1.61) be of the a. p. type is that the system (1.61) satisfy Condition I.

To prove this theorem, let $x^{(0)}(t), \ldots, x^{(p)}(t)$ be any set of solutions of (1.61) (not all identically zero), and let

$$z(t) \equiv \sum_{\nu=0}^{p} t^\nu x^{(\nu)}(t). \tag{3.81}$$

Clearly by the argument used in Lemma I, $z(t) \not\equiv 0$, for if it were we should have $\sum_{\nu=0}^{p} t_0^\nu x^{(\nu)}(t) \equiv 0$ identically in t and t_0, and all the $x^{(\nu)}(t)$ would be trivial. Moreover by Theorem I,

$$x^{(\nu)}(t) = \sum_{\sigma=1}^{g_\nu} e^{\lambda_{\nu,\sigma} t}\, t^{\varrho_{\nu,\sigma}} y_{\nu,\sigma}(t) \qquad (\nu = 1, \ldots, p), \tag{3.82}$$

where the $\lambda_{\nu,\sigma}$ are real, the $\varrho_{\nu,\sigma}$ are non-negative integers, and the $y_{\nu,\sigma}(t)$ are pseudo-solutions of (1.61) of order $\lambda_{\nu,\sigma}$. But from Lemmas 6 and 7, for all ν and σ, $y_{\nu,\sigma}(t) = O^*(1)$ at $+\infty$, and hence

$$x^{(\nu)}(t) = O^*(e^{\lambda_\nu t}\, t^{\varrho_\nu}) \text{ at } +\infty, \tag{3.83}$$

where λ_ν is the greatest of $\lambda_{\nu,1}, \lambda_{\nu,2}, \ldots, \lambda_{\nu,g_\nu}$, and ϱ_ν is the greatest of the $\varrho_{\nu,\sigma}$ for which $\lambda_{\nu,\sigma} = \lambda_\nu$. But if we substitue (3.82) in (3.81) we obtain an expression of the same form. For when terms having the same exponential and power of t are combined, pseudo-solutions having the same orders (though not the same degrees) will be combined, and the result will thus still be a pseudo-solution (or else zero, in which the term drops out). Thus the same reasoning that was used in establishing (3.83) holds for $z(t)$, and it follows that the system satisfies Condition I.

3.9. It is quite clear that if in the statement of Condition I we replace $+\infty$ by $-\infty$, Theorem II will still hold. Of course, we can not replace $+\infty$ by $\pm\infty$, because the λ and r occurring in our asymptotic equation will not usually be the same at $+\infty$ and $-\infty$. We could on the other hand require the asymptotic equation to hold both at $-\infty$ and at $+\infty$ for separate pairs λ, r; and Theorem II would be somewhat stronger since it would prove the necessity of a stronger condition. However in the future Condition I will usually appear in the hypothesis of our theorems rather than in the conclusions and therefore we will restrict it to be a one-sided condition as stated. Our theorems will be much stronger in consequence.

§ 4. Decomposable Solutions.

4.1. In this section we shall show the connection between Condition I and systems of the form (1.61) all of whose solutions are decomposable.

In the first place we note that the proof of Theorem II could be carried through just as well if the hypothesis that all the solutions are of the a. p. type were replaced by the hypothesis that all the solutions are decomposable, provided that we know that the hypothesis of Lemma 6 is satisfied for every λ. Thus we have

Theorem III. Let $A(t)$ be an a. p. matrix function. Then if all the solutions of (1.61) are decomposable and all the pseudo-solutions of (1.61) of order zero are bounded away from zero for positive t, it follows that Condition I is satisfied.

4.2. We shall show in this section (§ 4) that the converse of Theorem III is also true. The proof of this fact is rather long, and before carrying it through we shall give an outline of it using for convience the following terminology.

Definition. If $x(t) = O^*(e^{\lambda t} t^r)$ at $+\infty$, where λ is real and r is a non-negative integer, let $\varLambda^+(x)$ denote λ and $\varXi^+(x)$ denote r. Similarly if $x(t) = O^*(e^{\lambda t} t^r)$ at $-\infty$ or at $\pm\infty$, let $\varLambda^-(x)$ and $\varXi^-(x)$ or $\varLambda(x)$ and $\varXi(x)$ denote λ and r respectively.

The converse of Theorem III of course has as its hypothesis the assumption the $A(t)$ is a. p. and Condition I is satisfied. Since Condition I holds, $\varLambda^+(x)$ exists for every non-trivial solution x, and as x ranges over all solutions it takes on only a finite number of values, say $\lambda_1, \ldots, \lambda_p$, where p does not exceed the order of $A(t)$. These numbers λ_ν will play a prominent role in what follows, for our conclusion will be proved by mathematical induction on the λ_ν. To carry through the induction we shall show the following three things.

(a). There are no pseudo-solutions of order $< \lambda_1$ and degree zero, and every pseudo-solution of order λ_1 is bounded away from zero for positive t.

(b). If every pseudo-solution of order $\leqq \lambda_\nu$ and degree zero is bounded away from zero for positive t, and every solution $x(t)$ such that $\Lambda^+(x) < \lambda_\nu$ is decomposable of order $< \lambda_\nu$, then every solution $x^*(t)$ such that $\Lambda^+(x^*) = \lambda_\nu$ is decomposable of order λ_ν.

(c). If every solution $x(t)$ such that $\Lambda^+(x) \leqq \lambda_\nu$ is decomposable of order $\leqq \lambda_\nu$ and every pseudo-solution of order $\leqq \lambda_\nu$ and degree zero is bounded away from zero for positive t, then every pseudo-solution of order $\leqq \lambda_{\nu+1}$ and degree zero is bounded away from zero for positive t.

It is obvious that these three statements imply that all of the solutions are decomposable and all pseudo-solutions of zero degree and order $\leqq \lambda_p$ are bounded away from zero for positive t. Moreover the restriction on the order of the pseudo-solutions can easily be removed, and we have the desired conclusion. The proof of (a) is almost obvious, but for the sake of completeness is given in Lemma 8. The proofs of (b) and (c) are given in Lemmas 11 and 12, following the preliminary Lemmas 9 and 10.

4.3. **Lemma 8.** *Let $A(t)$ be an a. p. matrix function such that (1.61) satisfies Condition I. Then if λ is the least value which $\Lambda^+(x)$ assumes for any solution $x(t)$ of (1.61), there is no pseudo-solution of order less than λ; and every pseudo-solution of order λ is bounded away from zero for all t.*

For if there were a pseudo-solution of order less than λ, say μ, its minor $y_0(t)$ of degree zero would also be of order μ, and $e^{\mu t} y_0(t)$ would be a solution $x_0(t)$ of (1.61). But $x_0(t) = O(e^{\mu t})$ at $+\infty$, so that $\Lambda^+(x_0) \leqq \mu$ contrary to the assumption that λ is the least value of $\Lambda^+(x)$. Again, if $y_0(t)$ is a pseudo-solution of order λ and degree zero, $x_0(t) = e^{\lambda t} y_0(t)$ is a solution, and $x_0(t) = O(e^{\lambda t})$ at $+\infty$. Thus $\Lambda^+(x_0) \leqq \lambda$; and this implies that $\Lambda^+(x_0) = \lambda$ and $\Xi^+(x_0) = 0$, so that $x_0(t) = O^*(e^{\lambda t})$ at $+\infty$ and $y_0(t) = O^*(1)$ at $+\infty$.

4.4. As a preliminary to the proofs of (b) and (c) of (4.2) we give two lemmas concerning transformations on pseudo-solutions.

Lemma 9. *Let λ be a real number and $A(t)$ an a. p. n-by-n matrix function such that (1.61) has no pseudo-solution of order λ and degree zero which* comes arbitrarily close to zero for positive t. Let $h_1, h_2, \ldots$ be a sequence of positive numbers such that $\tilde{A}(t) \equiv \lim_{i\to\infty} A(t+h_i)$ exists uniformly for all t. Then there exists a subsequence $h'_1, h'_2, \ldots$ of $h_1, h_2, \ldots$ such that for each pseudo-solution $y(t)$ of (1, 61) of order λ, $T[y](t) \equiv \lim_{i\to\infty} y(t+h'_i)$ exists for all t. Moreover if such a subsequence is chosen and $y(t)$ and $y^*(t)$ are distinct pseudo-solutions of (1.61) of order λ, then $T[y](t) \not\equiv T[y^*](t)$.

For in the first place the set of pseudo-solutions of order λ forms with the trivial solution a linear manifold, and this manifold must be of finite dimensionality since by Lemma 4 no pseudo-solution of order λ is of degree $\geqq n$. Let $y^{(1)}(t), \ldots, y^{(p)}(t)$ be a basis for it, and let $h'_1, h'_2, \ldots$ be a subsequence of $h_1, h_2, \ldots$ such that $\lim_{i\to\infty} y^{(\nu)}(h'_i)$ exists, $(\nu = 1, \ldots, p)$. Then if $y(t)$ is any pseudo-solution of (1.61) of order λ, $\lim_{i\to\infty} y(h'_i)$ exists, and the same applies to the minors of $y(t)$ which are also pseudo solutions. Moreover the corresponding limit is not zero for the minor of zero order. Thus by Lemma 5, $T[y](t)$ exists for all t.

Now suppose $y(t)$ and $y^*(t)$ are two distinct pseudo-solutions of (1.61) of order λ. Then $y(t)-y^*(t)$ is also a pseudo-solution of order λ, and by Lemma 5, $T[y-y^*](t) = T[y](t) - T[y^*](t)$ is a pseudo-solution of $D[x(t)] = \tilde{A}(t)\cdot x(t)$ of order λ and hence is not identically zero.

Lemma 10. Let λ be a real number and $A(t)$ an a. p. n-by-n matrix function such that (1.61) has no pseudo-solution of order λ and degree zero which comes arbitrarily close to zero for positive t. Then every pseudo-solution of (1.61) of order λ is bounded away from zero for all t. Moreover if $h_1, h_2, \ldots$ is a sequence such that $T[y](t) \equiv \lim_{i\to\infty} y(t+h_i)$ exists for each pseudo-solution $y(t)$ of (1.61) of order λ and such that uniformly in t, $\lim_{i\to\infty} A(t+h_i) = A(t)$, then the transformation T has a single-valued inverse T^{-1} defined over the entire pseudo-solution manifold S of (1.61) of order λ.

The second statement follows from the fact that according to Lemma 5, T takes S into a sub-manifold S' of itself, which is of the same dimensionality as S since T takes a non-trivial element into a non-trivial element. Thus $S' = S$, and T^{-1} is defined and single valued over the whole of S.

The first statement follows from the second statement and Lemmas 4 and 9. For there exists a sequence $h'_1, h'_2, \ldots$ which approaches $+\infty$ and for which

$\lim_{i\to\infty} A(t+h'_i) = A(t)$ uniformly in t; and this sequence has a subsequence $h''_1, h''_2, \ldots$ such that $\lim_{i\to\infty} y(t+h''_i)$ exists for all t and each pseudo-solution $y(t)$ of order λ. Then by what we have just proved, to every pseudo-solution $\tilde{y}(t)$ of order λ there corresponds a pseudo-solution of order λ such that $\lim_{i\to\infty} y(t+h''_i) = \tilde{y}(t)$ for all t. But $y(t)$ is bounded away from zero for positive t, and $\tilde{y}(t)$ is bounded away from zero for all t.

4. 5. We now proceed to the proof of proposition (b) of (4. 2), whose conclusion is that each solution $x(t)$ for which $\Lambda^+(x) = \lambda$ is decomposable of order λ. To obtain this result we shall have to actually carry out the decomposition of $x(t)$; and this process will be similar to but more involved than the decomposition of a solution of the a. p. type given in (2. 2—2. 8). As in the simpler case, we decompose $x(t)$ step by step by mathematical induction. But now each step will involve defining and showing the existence of the new quantities involved, as well as showing that they are solutions or pseudo-solutions of (1. 61). Moreover each step will involve two new solutions and two new pseudo-solutions, the pseudo-solutions being related by a transformation of the type defined in Lemma 10. Finally, each step will depend on a formula obtained in the preceding step, and we shall therefore give these formulas in the statement of our lemma in order to carry on an induction proof based on them.

Lemma 11. Let $A(t)$ be an a. p. n-by-n matrix function such that (1. 61) satisfies Condition I. Let λ be a number such that every solution $x'(t)$ of (1. 61) for which $\Lambda^+(x') < \lambda$ is decomposable of order $< \lambda$ and such that every pseudo-solution of order λ and degree zero is bounded away from zero for positive t. Let $x(t)$ be a solution of (1. 61) for which $\Lambda^+(x) = \lambda$, and let $\Xi^+(x) = r$. Under these conditions, $x(t)$ is decomposable of order λ.

Moreover corresponding to each non-negative integer $s \leqq r$ there exist a sequence $h_1, h_2, \ldots \to +\infty$ and a pseudo-solution $y_s(t)$ of (1. 61) having the following properties:

(1) $\lim_{i\to\infty} A(t+h_i) = A(t)$ uniformly for all t.

(2) For every pseudo-solution $y(t)$ of order λ, $\lim_{i\to\infty} y(t+h_i)$ exists for all t.

(3) If $y_0(t), \ldots, y_s(t)$ are the minors of $y_s(t)$, then for all t and each non-negative $\mu \leqq s$,

$$(4.51) \qquad \lim_{i\to\infty} \frac{x(t+h_i)e^{-\lambda(t+h_i)} - \sum\limits_{\varrho=r-\mu}^{r} \frac{(t+h_i)^\varrho}{\varrho!} y_{r-\varrho}(t+h_i)}{(t+h_i)^{r-\mu}} = 0.$$

To begin the induction proof of the lemma, we show that the second statement holds if $s=0$. It is obvious that a sequence $h_1'', h_2'', \ldots$ can be found which approaches $+\infty$ and has property (1). Moreover it follows from Lemma 9 that this sequence has a subsequence $h_1', h_2', \ldots$ which satisfies (2). But since $x(t) = O^*(e^{\lambda t}t^r)$, $x(t)e^{-\lambda t}t^{-r}$ is bounded and bounded away from zero for positive t. Thus $h_1', h_2', \ldots$ has a subsequence $h_1, h_2, \ldots$ such that $\lim\limits_{i\to\infty} x(h_i)e^{-\lambda h_i}h_i^{-r}$ exists. Hence $\tilde{x}_0(t) \equiv r!\lim\limits_{i\to\infty} x(t+h_i)e^{-\lambda h_i}h_i^{-r}$ exists for each t and is a non-trivial solution of (1.61). Moreover if $\tilde{y}_0(t) \equiv e^{-\lambda t}\tilde{x}_0(t)$,

$$(4.52) \qquad \tilde{y}_0(t) = r!\lim_{i\to\infty} x(t+h_i)e^{-\lambda(t+h_i)}(t+h_i)^{-r}$$

for all t; and since $x(t)e^{-\lambda t}t^{-r} = O^*(1)$ at $+\infty$, $\tilde{y}_0(t)$ is bounded for all t. Thus $\tilde{y}_0(t)$ is a pseudo-solution of (1.61) of order λ and degree zero and has the generator $\tilde{x}_0(t)$. By Lemma 10, there is a unique pseudo-solution $y_0(t)$ of order λ and degree zero such that $\lim\limits_{i\to\infty} y_0(t+h_i) \equiv \tilde{y}_0(t)$. Moreover if we substitute $\lim\limits_{i\to\infty} y_0(t+h_i)$ for $\tilde{y}_0(t)$ in (4.52) we find that (4.51) is satisfied for $s=0$. Hence $h_1, h_2, \ldots$ and $y_0(t)$ satisfy (1), (2), (3); and our statement holds when $s=0$.

Continuing the induction, we assume that the second statement of the lemma is true when $s = p-1 < r$. Let us designate the corresponding sequence by $h_1', h_2', \ldots$, the corresponding pseudo-solution by $y_{p-1}(t)$; and the minors and generators of $y_{p-1}(t)$ by $y_0(t), \ldots, y_{p-1}(t)$ and $x_0(t), \ldots, x_{p-1}(t)$ respectively. Then (4.51) holds for the sequence $h_1', h_2', \ldots$ with $\mu = s = p-1$, and if

$$(4.53) \qquad z(t) \equiv x(t) - e^{\lambda t}\sum_{\varrho=r-p+1}^{r} \frac{t^\varrho}{\varrho!} y_{r-\varrho}(t),$$

it follows that

$$(4.54) \qquad \liminf_{t\to\infty} \| z(t)e^{-\lambda t}t^{-(r-p+1)} \| = 0.$$

But

$$z(t) \equiv x(t) - \sum_{\varrho=r-p+1}^{r} \frac{t^\varrho}{\varrho!} \sum_{\nu=0}^{r-\varrho} \frac{(-t)^\nu}{\nu!} x_{r-\varrho-\nu}(t);$$

and hence by Condition I either $z(t) \equiv 0$ or $\Lambda^+(z)$ and $\Xi^+(z)$ both exist. Moreover it follows from (4.54) that either $z(t) \equiv 0$, or $\Lambda^+(z) < \lambda$, or $\Lambda^+(z) = \lambda$ and $\Xi^+(t) < r - p + 1$; so in any case $z(t) = O(e^{\lambda t} t^{r-p})$ at $+\infty$. Hence there exists a subsequence $h_1, h_2, \ldots$ of $h'_1, h'_2, \ldots$ such that $\lim\limits_{i\to\infty} z(h_i) e^{-\lambda h_i} h_i^{-(r-p)}$ exists. And since $h'_1, h'_2, \ldots$ has property (2), $T[y](t) = \lim\limits_{i\to\infty} y(t+h_i)$ exists for all t and each pseudo-solution $y(t)$ of order λ. Thus if $T[y_{p-1}](t)$ has the minors $\tilde{y}_0(t), \ldots, \tilde{y}_{p-1}(t)$ and generators $\tilde{x}_0(t), \ldots, \tilde{x}_{p-1}(t)$, it follows from Lemma 5 that for $\mu \leqq p-1$, $T[y_\mu](t) \equiv \tilde{y}_\mu(t)$.

Let

$$X(t,h) \equiv x(t+h) - \sum_{\varrho=r-p+1}^{r} \frac{h^\varrho}{\varrho!} \sum_{\nu=0}^{r-\varrho} \frac{(-h)^\nu}{\nu!} x_{r-\varrho-\nu}(t+h);$$

so that for each h, $X(t,h)$ is a solution of $D[x(t)] = A(t+h)\cdot x(t)$ and $X(0,h) = z(h)$. Then $\lim\limits_{i\to\infty} X(0,h_i) e^{-\lambda h_i} h_i^{-(r-p)}$ exists, and

$$\tilde{x}_p(t) \equiv (r-p)! \lim_{i\to\infty} X(t,h_i) e^{-\lambda h_i} h_i^{-(r-p)}$$

exists for all t and is a solution of (1.61). Moreover

$$[z(t+h) - X(t,h)]\, e^{-\lambda(t+h)}$$

$$= - \sum_{\varrho=r-p+1}^{r} y_{r-\varrho}(t+h) \sum_{\sigma=0}^{\varrho} \frac{h^\sigma t^{\varrho-\sigma}}{\sigma!(\varrho-\sigma)!} +$$

$$+ \sum_{\varrho=r-p+1}^{r} \frac{h^\varrho}{\varrho!} \sum_{\nu=0}^{r-\varrho} \frac{(-h)^\nu}{\nu!} \sum_{\sigma=0}^{r-\varrho-\nu} \frac{(t+h)^\sigma}{\sigma!} y_{r-\varrho-\nu-\sigma}(t+h)$$

$$= - \sum_{\varrho=r-p+1}^{r} y_{r-\varrho}(t+h) \sum_{\sigma=0}^{r-p} \frac{h^\sigma t^{\varrho-\sigma}}{\sigma!(\varrho-\sigma)!},$$

so that for all t

$$(r-p)! \lim_{i\to\infty} [z(t+h_i) - X(t,h_i)]\, e^{-\lambda(t+h_i)} h_i^{-(r-p)}$$

$$= - \sum_{\varrho=r-p+1}^{r} \frac{t^{\varrho-r+p}}{(\varrho-r+p)!} T[y_{r-\varrho}](t) = - \sum_{\varrho=1}^{p} \frac{t^\varrho}{\varrho!} \tilde{y}_{p-\varrho}(t).$$

Thus

$$\tilde{y}_p(t) \equiv (r-p)! \lim_{i\to\infty} z(t+h_i)\, e^{-\lambda(t+h_i)} (t+h_i)^{-(r-p)} \tag{4.55}$$

exists for all t, and

$$\tilde{y}_p(t) = -\sum_{\varrho=1}^{p} \frac{t^\varrho}{\varrho!} e^{-\lambda t} \sum_{\sigma=0}^{p-\varrho} \frac{(-t)^\sigma}{\sigma!} \tilde{x}_{p-\varrho-\sigma}(t) + e^{-\lambda t} \tilde{x}_p(t) = e^{-\lambda t} \sum_{\nu=0}^{p} \frac{t^\nu}{\nu!} \tilde{x}_{p-\nu}(t).$$

Since $z(t) = O(e^{\lambda t} t^{r-p})$ at $+\infty$, it follows from (4. 55) that $\tilde{y}_p(t)$ is bounded for all t, and hence is a pseudo-solution of (1. 61) of order λ and degree p having $\tilde{y}_0(t), \ldots, \tilde{y}_p(t)$ as minors and $\tilde{x}_0(t), \ldots, \tilde{x}_p(t)$ as generators.

Let $y_p(t) \equiv T^{-1}[\tilde{y}_p](t)$, and let $y_0^*(t), \ldots, y_{p-1}^*(t), y_p(t)$ be its minors. By Lemma 5, the minors of $\tilde{y}_p(t) = T[y_p](t)$ are $T[y_0^*](t), \ldots, T[y_{p-1}^*](t), T[y_p](t)$, and hence by Lemma 2, $T[y_\mu^*](t) \equiv \tilde{y}_\mu(t) \equiv T[y_\mu](t)$ for $\mu \leqq p-1$. Thus by Lemma 10, $y_\mu^*(t) \equiv y_\mu(t)$ for $\mu \leqq p-1$, and $y_p(t)$ has the minors $y_0(t), \ldots, y_p(t)$. Moreover we have from (4. 55) and (4. 53)

$$\lim_{i\to\infty} y_p(t+h_i) = (r-p)! \lim_{i\to\infty} \left\{ x(t+h_i) e^{-\lambda(t+h_i)} - \right.$$
$$\left. - \sum_{\varrho=r-p+1}^{r} \frac{(t+h_i)^\varrho}{\varrho!} y_{r-\varrho}(t+h_i) \right\} (t+h_i)^{-(r-p)},$$

so that (4. 51) holds for the pseudo-solution $y_p(t)$ and the sequence $h_1, h_2, \ldots$ Thus the induction is complete, and the second statement of this lemma holds for all $s \leqq r$.

Finally, let $s = r$, and let $x_r(t)$ be the leader of the pseudo-solution $y_r(t)$. Then by (4. 51),

$$\lim_{i\to\infty} [x(t+h_i) - x_r(t+h_i)] e^{-\lambda(t+h_i)} \equiv 0,$$

so that if $x^*(t) \equiv x(t) - x_r(t)$, $\liminf_{t\to\infty} \|x^*(t)\| e^{-\lambda t} = 0$. Hence either $x^*(t) \equiv 0$ or $\Lambda^+[x^*] < \lambda$; and in either case $x(t)$ is decomposable of order λ.

4. 6. Proposition (c) of (4. 2) is established in the following lemma.

Lemma 12. Let $A(t)$ be an a. p. square matrix function such that (1. 61) satisfies Condition I. Let λ be a number such that every solution $x(t)$ of (1. 61) for which $\Lambda^+(x) \leqq \lambda$ is decomposible of order $\leqq \lambda$, and such that every pseudo-solution of order $\leqq \lambda$ and degree zero is bounded away from zero for positive t. Let λ' be the least number greater than λ such that there exists a solution $x'(t)$ for which $\Lambda^+(x') = \lambda'$. Then every pseudo-solution of order $\leqq \lambda'$ and degree zero is bounded away from zero for positive t.

For suppose that there exists a pseudo-solution $y_0(t)$ of order $\lambda'' \leqq \lambda'$ and degree zero which comes arbitrarily close to zero for positive t and let $x_0(t) = e^{\lambda'' t} y_0(t)$ be its generator. Then $\lambda'' > \lambda$; and $\Delta^+(x_0)$ is less than λ' and hence less than or equal to λ. It follows that $x_0(t)$ is decomposable of order $\leqq \lambda$, so that

$$x_0(t) \equiv \sum_{\sigma=1}^{p} e^{\lambda_\sigma t} \sum_{\nu=0}^{r_\sigma} \frac{t^\nu}{\nu!} y_{\sigma, r_\sigma - \nu}(t),$$

where $\lambda \geqq \lambda_1 > \lambda_2 > \cdots > \lambda_p$ and the $y_{\sigma,\mu}(t)$ are pseudo-solutions of order λ_σ. But by Lemma 10, every $y_{\sigma,\mu}(t)$ is bounded away from zero for all t, and hence $x_0(t) = O^*(e^{\lambda_p t} t^{r_p})$ at $-\infty$. Thus $y_0(t) = O^*(e^{(\lambda_p - \lambda'')t} t^{r_p})$ at $-\infty$, which is impossible since $y_0(t)$ is bounded. It follows that all pseudo-solutions of order $\leqq \lambda'$ and degree zero are bounded away from zero for positive t.

4.7. We now state the main theorem of this section, which includes the converse of Theorem III.

Theorem IV. Let $A(t)$ be an a. p. square matrix function such that (1.61) satisfies Condition I. Then every solution of (1.61) can be expressed in one and only one way as a sum of satisfactory solutions having distinct orders. Moreover every pseudo-solution of (1.61) is bounded away from zero for all t.

Almost all of the proof of this theorem has already been given, since the three proposition (a), (b), (c), have been proved in Lemmas 8, 11, 12. Thus the induction outlined in (4.2) is complete; and it follows that all solutions of (1.61) are decomposable and all pseudo-solutions of order not greater than the λ_p of (4.2) and degree zero are bounded away from zero for positive t. But there are no solutions of order greater than λ_p and degree zero, for if there were a pseudo-solution $y_0(t)$ of order $\lambda' > \lambda_p$ and degree zero, $x_0(t) \equiv e^{\lambda' t} y_0(t)$ would be decomposable, $\Delta^-(x_0)$ would exist and be less than or equal to λ_p, and $y_0(t)$ would be unbounded for negative t. Thus it follows from Lemma 10 that all pseudo-solutions are bounded away from zero for all t.

Moreover a solution can be decomposed in only one way, for otherwise the trivial solution would be decomposable and $\Delta^+(0)$ would exist.

Corollary 1. If $A(t)$ is an a. p. square matrix function such that (1.61) satisfies Condition I, it follows that (1.61) satisfies the condition obtained by replacing $+\infty$ by $-\infty$ in the statement of Condition I.

It is of course also clear from symmetry that this negative reflection of Condition I implies Condition I.

Corollary 2. If $A(t)$ is an a. p. square matrix function such that (1.61) satisfies Condition I, it follows that every primary solution of (1.61) of order λ and degree r is a satisfactory solution of order λ and degree $\leqq r$.

For every solution $x(t)$ is the sum of satisfactory solutions of distinct orders, and it is clear that if two or more are present and $x(t) = O(e^{\lambda_1 t} t^{r_1})$ at $+\infty$ and $x(t) = O(e^{\lambda_2 t} t^{r_2})$ at $-\infty$, then $\lambda_1 > \lambda_2$ and the solution in question is not primary.

§ 5. The Invariance of Condition I.

We will now show that Condition I is invariant under limiting translations.

Theorem V. Let $A(t)$ be an a. p. n-by-n matrix function such (1.61) satisfies Condition I. Then if $h_1, h_2, \ldots$ is a sequence of real numbers such that

$$\tilde{A}(t) \equiv \lim_{i \to \infty} A(t + h_i)$$

exists uniformly in t, it follows that the transformed system

$$D[x(t)] = \tilde{A}(t) \cdot x(t) \tag{5.1}$$

also satisfies Condition I.

To establish this theorem, let $\lambda_1 < \lambda_2 < \cdots < \lambda_p$ be the set of values $\Lambda^+(x)$ can take on as $x(t)$ ranges over all solutions of (1.61), and for each $\nu \leqq p$ let L_ν be the linear manifold consisting of all satisfactory solutions of order $= \lambda_\nu$ together with the trivial solution. Since every solution is decomposable into a sum of satisfactory solutions, the sum of the dimensionalities $n_1, \ldots, n_p$ of $L_1, \ldots, L_p$ is n. Moreover for each $\nu \leqq p$ the pseudo-solution manifold L'_ν of (1.61) of order λ_ν has n_ν dimensions, for each non-trivial element of L_ν is a leader of an element of L'_ν and the correspondence is 1—1 since by Lemma 4 and Theorem IV no pseudo-solution has the trivial solution as a leader.

Now let $h'_1, h'_2, \ldots$ be a subsequence of $h_1, h_2, \ldots$ such that for each $\nu \leqq p$ and each element $y(t)$ of L'_ν, $T[y](t) \equiv \lim_{i \to \infty} y(t + h'_i)$ exists for all t. That such a subsequence can be chosen follows from Lemma 9 and Theorem IV, as does also the fact that T sets up a 1—1 correspondence between L'_ν and the manifold L'^*_ν into which it takes L'_ν. Thus L'^*_ν has n_ν dimensions, and by Lemma 5 its non-trivial elements are pseudo-solutions of (5. 1) of order λ. Since the non-trivial elements of L'_ν are bounded away from zero, so are the non-trivial elements of L'^*_ν.

For each $\nu \leqq p$, let L^*_ν be the linear manifold consisting of the trivial solution together with the leaders of the non-trivial elements of L'^*_ν. Then it follows from Lemmas 2 and 4 that the correspondence between L'^*_ν and L^*_ν is $1-1$, so that L^*_ν has n_ν dimensions. Moreover it follows from Lemma 4 that if $\tilde{x}(t)$ is a non-trivial element of L^*_ν, $\Lambda^+(\tilde{x}) = \lambda_\nu$; and hence the manifolds L^*_μ and L^*_ν have only the trivial element in common if $\mu \neq \nu$. Thus the manifold $L^*_1 + \cdots + L^*_p$ has n dimensions and is therefore the entire manifold of solutions of (5. 1); and every solution of (5. 1) is decomposable.

Finally, let $\tilde{y}_0(t)$ be any pseudo-solution of (5. 1) of degree zero, and let λ be its order. Then $e^{\lambda t}\tilde{y}_0(t)$ is a non-trivial solution of (5. 1) and is decomposable into the sum $x^{(\nu_1)}(t) + \cdots + x^{(\nu_g)}(t)$; where each $x^{(\nu_\varrho)}(t)$ is the leader of a non-trivial element $y^{(\nu_\varrho)}(t)$ of the manifold $L'^*_{\nu_\varrho}$; and $\nu_1 < \cdots < \nu_g$. By Lemma 4, $\Lambda(x^{(\nu_\varrho)}) = \lambda_\nu$ for each $\varrho \leqq g$, and hence $\Lambda^+(\tilde{y}_0) = \lambda_{\nu_g} - \lambda$ and $\Lambda^-(\tilde{y}_0) = \lambda_{\nu_1} - \lambda$. But since $\tilde{y}_0(t)$ is bounded, $\lambda_{\nu_1} = \lambda_{\nu_g} = \lambda$, $g = 1$, and $\tilde{y}_0(t) \equiv y^{(\nu_1)}(t)$. Thus every pseudo-solution of (5. 1) of order zero is bounded away from zero. Thus it follows from Theorem III that (5. 1) satisfies Condition I.

§ 6. Stationary Pseudo-solutions.

In this section we shall obtain sufficient conditions that a pseudo-solution of (1. 61) be a. p. We shall assume at the outset that $A(t)$ is a. p. with a module contained in M and that (1. 61) satisfies Condition I; and we shall show that a pseudo-solution $y(t)$ is a. p. with a module contained in M if and only if $y(t)$ is what we shall call a positive stationary function with respect to M.

Definition. A vector function $f(t)$ will be called (positive) stationary with respect to a module M if for each real t

$$\lim_{i\to\infty} f(t+h_i) = f(t) \tag{6. 11}$$

whenever $h_1, h_2, \ldots$ is a sequence of (positive) real numbers such that for each element Φ of M,

$$\lim_{i\to\infty} \Phi h_i = 0 \qquad (\text{mod } 2\pi).$$

We call particular attention to the fact that no uniformity is postulated in connection with (6. 11). As a matter of fact, if (6. 11) were assumed to hold uniformly for all t, a function which is stationary with respect to a module M would be a. p. with the module M. However, as the definition stands, a stationary function need not be a. p. at all, nor even uniformly continuous. Thus the theorem quoted above as the subject of this section is by no means a mere triviality; and as a matter of fact it forms the basis for all our later theorems.

The theorem of this section will be proved by the method of Favard, which is based on Bochner's definition of a normal function. As this process involves certain iterated limits, we shall first prove in Lemma 13, that a function which is stationary with respect to a module M has a similar property involving iterated limits. As another primilary to the main theorem, we shall show in Lemma 14 that a part of the hypothesis of Lemma 5 may be replaced by the hypothesis that (1. 61) satisfies Condition I.

6. 2. **Lemma 13.** Let $f(t)$ be positive stationary with respect to a module M, and let $h_1, h_2, \ldots$ and $k_1, k_2, \ldots$ be sequences of positive numbers such that for each element Φ of M

$$\lim_{i,j\to\infty} \Phi(h_i + k_j) = 0 \qquad (\bmod 2\pi).$$

Then if t_0 is a real number such that $g = \lim\limits_{j\to\infty} \lim\limits_{i\to\infty} f(t_0 + h_i + k_j)$ exists, it follows that $g = f(t_0)$.

For corresponding to each positive integer n there exists an index $j_n > n$ such that $\left| g - \lim\limits_{i\to\infty} f(t_0 + h_i + k_{j_n}) \right| \leqq \frac{1}{n}$ and there exists an index $i_n > n$ such that

$$\left| f(t_0 + h_{i_n} + k_{j_n}) - \lim_{i\to\infty} f(t_0 + h_i + k_{j_n}) \right| \leqq \frac{1}{n}\cdot$$

Then $\lim\limits_{n\to\infty} f(t_0 + h_{i_n} + k_{j_n}) = g$, and for each element Φ of M $\lim\limits_{n\to\infty} \Phi(h_{i_n} + k_{j_n}) = 0$ (mod 2π). It follows from the definition of a positive stationary function that $\lim\limits_{n\to\infty} f(t_0 + h_{i_n} + k_{j_n}) = f(t_0)$, so that the lemma is proved.

6. 3. **Lemma 14.** Let $A(t)$ be an a. p. matrix function such that (1. 61) satisfies Condition I, let $y(t)$ be any pseudo-solution of (1. 61), and let r and λ be the degree and order of $y(t)$. Then if $h_1, h_2, \ldots$ is a sequence such that

$\lim_{i\to\infty} y(h_i)$ exists and $\tilde{A}(t) = \lim_{i\to\infty} A(t+h_i)$ exists uniformly in t, it follows that $\tilde{y}(t) = \lim_{i\to\infty} y(t+h_i)$ exists for all t and is a pseudo-solution of order λ and degree r of $D[\tilde{x}(t)] = \tilde{A}(t)\cdot\tilde{x}(t)$. Moreover if $y_\mu(t)$ is the minor of $y(t)$ of degree μ, $\tilde{y}_\mu(t) = \lim_{i\to\infty} y_\mu(t+h_i)$ is the minor of $\tilde{y}(t)$ of degree μ.

For suppose that there exists a minor $y_p(t)$ of $y(t)$ such that $\lim_{i\to\infty} y_p(h_i)$ does not exist. Then since $y_\mu(t)$ is bounded for $\mu \leqq r$, there exist two subsequences $h'_1, h'_2, \ldots$ and $h''_1, h''_2, \ldots$ of $h_1, h_2, \ldots$ such that $y^*_\mu = \lim_{i\to\infty} y_\mu(h'_i)$ and $y^{**}_\mu = \lim_{i\to\infty} y_\mu(h''_i)$ exist for each $\mu \leqq r$ and $y^*_p \neq y^{**}_p$. It follows from Theorem IV that $y^*_0 \neq 0$ and $y^{**}_0 \neq 0$, and from Lemma 5 that $y^*(t) = \lim_{i\to\infty} y(t+h'_i)$ and $y^{**}(t) = \lim_{i\to\infty} y(t+h''_i)$ are pseudo-solutions of $D[x(t)] = \tilde{A}(t)\cdot\tilde{x}(t)$ of order λ and degree r. But $y^*(0) = y^{**}(0)$; so by Lemma 4, $y^*(t) \equiv y^{**}(t)$, and by Lemma 2 the minors of degree p, $\lim_{i\to\infty} y_p(t+h'_i)$ and $\lim_{i\to\infty} y_p(t+h''_i)$ are identical. Thus $y^*_p = y^{**}_p$; and this contradiction shows that $\lim_{i\to\infty} y_\mu(h_i)$ must exist for each μ. Hence the lemma follows from Lemma 5.

6.4. We can now prove

Theorem VI. Let $A(t)$ be a. p. with a module contained in a certain module M, and let (1.61) satisfy Condition I. Then a pseudo-solution $y(t)$ of (1.61) is a. p. with a module contained in M if and only if $y(t)$ is positive stationary with respect to M.

It is obvious that if $y(t)$ is a. p. with a module contained in M, it must be positive stationary with respect to M. We therefore assume that it is positive stationary with respect to M and seek to show that it must be a. p. with a module contained in M.

Procceeding along the general lines of Favard's method of proof, we assume that there exists a sequence of positive numbers $h_1, h_2, \ldots$ such that for each Φ in M, $\lim_{i\to\infty} \Phi h_i = 0 \pmod{2\pi}$, and such that $\lim_{i\to\infty} y(t+h_i)$ does not converge to $y(t)$ uniformly in t for all positive t. Then there exist a number $\varepsilon > 0$, a subsequence $h'_1, h'_2, \ldots$ of $h_1, h_2, \ldots$ and a sequence of positive numbers $t_1, t_2, \ldots$ such that for each index i,

$$\|y(t_i+h'_i) - y(t_i)\| > \varepsilon. \tag{6.41}$$

Let $n_1, n_2, \ldots$ be a sequence of indices such that $\lim_{i\to\infty} y(t_{n_i} + h'_{n_i})$ and $\lim_{i\to\infty} y(t_{n_i})$ exist and for each element Φ of M, $\lim_{i\to\infty} \Phi t_{n_i}$ converges (mod. 2π). Then $A^*(t) \equiv \lim_{i\to\infty} A(t + t_{n_i})$ exists uniformly in t and $\lim_{i\to\infty} A(t + t_{n_i} + h'_{n_i}) = A^*(t)$ uniformly in t. Moreover by Lemma 14, $y^*(t) \equiv \lim_{i\to\infty} y(t + t_{n_i})$ and $y^{**}(t) \equiv \equiv \lim_{i\to\infty} y(t + t_{n_i} + h'_{n_i})$ exist for all t and are pseudo-solutions of $D[x^*(t)] = = A^*(t) \cdot x^*(t)$.

Let $k_1, k_2, \ldots$ be a sequence of positive numbers such that $\lim_{i\to\infty} y^*(k_i)$ and $\lim_{i\to\infty} y^{**}(k_i)$ exist and for each element Φ of M, $\lim_{i,j\to\infty} \Phi(t_{n_i} + k_j) = 0 \pmod{2\pi}$. (It can easily be seen that such a sequence exists.) Then for each element Φ of M, $\lim_{i,j\to\infty} \Phi(t_{n_i} + h'_{n_i} + k_j) = 0 \pmod{2\pi}$ and by Lemma 13, $\lim_{i\to\infty} y^*(k_i) = y(0)$ and $\lim_{i\to\infty} y^{**}(k_i) = y(0)$. Moreover $\lim_{i\to\infty} A^*(t + k_i) = A(t)$ uniformly in t.

Now it follows from Theorem V that $D[x^*(t)] = A^*(t) \cdot x^*(t)$ satisfies Condition I, and from Theorem IV that $y^*(t) - y^{**}(t)$ is either identically zero or bounded away from zero. Since $\lim_{i\to\infty} \{y^*(k_i) - y^{**}(k_i)\} = 0$, the latter alternative is impossible, and $y^*(t) \equiv y^{**}(t)$. Thus we deduce the statement $\lim_{i\to\infty} \{y(t_{n_i}) - - y(t_{n_i} + h'_{n_i})\} = 0$ which contradicts (6.41). Hence we may conclude that for every sequence of positive numbers $h_1, h_2, \ldots$ such that for each Φ in M, $\lim_{i\to\infty} \Phi h_i = 0$ (mod M), $\lim_{i\to\infty} y(t + h_i) = y(t)$ uniformly for all positive t. Since $y(t)$ is positive stationary, $\lim_{i\to\infty} y(t + h_i) = y(t)$ for each negative t also. We shall now show that this limit is uniform for all t, positive and negative.

In the first place, suppose that $h_1, h_2, \ldots \to +\infty$. Let $\varepsilon > 0$; let N be so great that when $i > N$, $\|y(t + h_i) - y(t)\| \leqq \varepsilon$ for all positive t; and let t_0 be any value of t. Then for $i > N$,

$$\|y(t_0 + h_i) - y(t_0)\| = \lim_{j\to\infty} \|y(t_0 + h_i + h_j) - y(t_0 + h_j)\| \leqq \varepsilon.$$

Thus $\lim_{i\to\infty} y(t + h_i) = y(t)$ uniformly in t for all t. If we remove the restriction that $h_1, h_2, \ldots \to +\infty$, we can still find a sequence $l_1, l_2, \ldots \to +\infty$ such that $h_1 + l_1, h_2 + l_2, \ldots \to +\infty$ and such that for each element Φ in M, $\lim_{i\to\infty} \Phi l_i = 0 \pmod{2\pi}$. Then uniformly for all t, $\lim_{i\to\infty} y(t+h_i+l_i) = \lim_{i\to\infty} y(t+l_i) = y(t)$;

so that uniformly for all t, $\lim\limits_{i\to\infty} y(t+h_i) = y(t)$. Finally, assume merely that $h_1, h_2, \ldots$ is a sequence such that for each Φ of M, $\lim\limits_{i\to\infty} \Phi h_i$ exists (mod 2π). Then $\lim\limits_{i,j\to\infty} \Phi(h_i - h_j) = 0 \pmod{2\pi}$ and uniformly in t, $\lim\limits_{i\to\infty} \| y(t+h_i-h_j) - y(t) \| = 0$. Thus $\lim\limits_{i\to\infty} y(t+h_i)$ exists uniformly in t, and $y(t)$ is a. p. with a module contained in M.

§ 7. Almost Periodic Pseudo-solutions.

7. 1. In this section we shall use Theorem VI of Section 6 to prove several theorems concerning a. p. pseudo-solutions. In order to prove such theorems we shall have to show that the pseudo-solutions in question are stationary. We therefore prove Lemma 15, which shows how pseudo-solutions behave under transformations which leave $A(t)$ invariant.

7. 2. The lemma of this section deals with what we shall call the range of a function, and which is defined as follows.

Definition. If $f(t)$ is a vector function, the closure of the set of values in n-dimensional space which $f(t)$ assumes as t takes on all real values is called the range of $f(t)$ and is denoted by $R(t)$. If t is restricted to take on only positive vales, the closure of the set of values $f(t)$ is called the positive range of $f(t)$ and is denoted by $R^+(t)$.

Lemma 15. Let $A(t)$ be an a. p. square matrix function whose module is contained in a certain module M, let (1.61) satisfy Condition I, and let the pseudo-solution manifold S of (1.61) of order λ and degree r contain at least one non-trivial element. Let $h_1, h_2, \ldots$ be a sequence such that for each element $y(t)$ of S,

$$(7.21) \qquad T[y](t) = \lim_{i\to\infty} y(t+h_i) \text{ exists for all } t,$$

and such that uniformly in t, $\lim\limits_{i\to\infty} A(t+h_i) = A(t)$. Then T is a $1-1$ linear transformation which does not alter the positive range or the range of any element of S. Moreover if S is understood to include all possible complex elements even though $A(t)$ is real, there exists a basis $y^{(1)}(t), \ldots, y^{(L)}(t)$ of S such that for all t

$$(7.22) \qquad T[y^{(\nu)}](t) = \theta_\nu y^{(\nu)}(t) \qquad (\nu = 1, \ldots, L);$$

where the θ_ν are constants having the absolute value unity.

The fact that T is linear is obvious from (7. 21), as is also the fact that

$$(7.23) \qquad R^{(+)}[y(t)] > R^{(+)}\{T[y](t)\}$$

for each element of S. It is also clear from Lemma 14 that $T[y](t)$ is an element of S whenever $y(t)$ is an element of S. Hence (if S is taken in the wider complex sense) the matrix which represents T can be reduced to the classic canonical form. Thus there exists a set of indices $1 = p_1 < p_2 < \cdots < p_g < p_{g+1} = L + 1$, a set of constant multipliers $\theta_1, \ldots, \theta_g$, and a set of linearly independent elements $y^{(1)}(t), \ldots, y^{(L)}(t)$ of S such that

$$(7.24\,a) \qquad T[y^{(p_\varrho)}](t) \equiv \theta_\varrho y^{(p_\varrho)}(t) \qquad (\varrho = 1, \ldots, g)$$

$$(7.24\,b) \qquad T[y^{(\sigma)}](t) \equiv \theta_\varrho y^{(\sigma)}(t) + y^{(\sigma-1)}(t)$$

$$(\varrho = 1, \ldots, g;\ \sigma = p_\varrho + 1, p_\varrho + 2, \ldots, p_{\varrho+1} - 1).$$

Now by Theorem IV, for each $\nu \leqq L$, $\| y^{(\nu)}(t) \|$ has a positive greatest lower bound as well as a finite least upper bound. Thus it follows from (7 23) and (7. 24 a) that $|\theta_\varrho| = 1$, $\varrho = 1, \ldots, g$. Moreover for each ϱ, $p_{\varrho+1} = p_\varrho + 1$. For if $p_{\varrho+1} > p_\varrho + 1$, we have by iteration of (7. 24) that

$$T^m[y^{(p_\varrho)}](t) = \theta_\varrho^m y^{(p_\varrho)}(t)$$

$$T^m[y^{(p_\varrho+1)}](t) = \theta_\varrho^m y^{(p_\varrho+1)}(t) + m\,\theta_\varrho^{m-1} y^{(p_\varrho)}(t).$$

But this is impossible since for all positive integers m the set $R\{T^m[y^{(p_\varrho+1)}](t)\}$ is contained in the bounded set $R[y^{(p_\varrho+1)}(t)]$. Hence the matrix representing T is in the diagonal form and (7. 22) holds.

Finally, let $m_1, m_2, \ldots$ be a sequence of positive integers such that for all $\nu \leqq L$, $\lim\limits_{j\to\infty} \theta_\nu^{m_j} = 1$. Then for each ν, $\lim\limits_{j\to\infty} T^{m_j}[y^{(\nu)}](t) \equiv y^{(\nu)}(t)$; and since T^{m_j} is linear, it follows that for each element $y(t)$ of S, $\lim\limits_{j\to\infty} T^{m_j}[y](t) \equiv y(t)$. But for each (positive) t and each j, $T^{m_j}[y](t)$ is an element of the closed set $R^{(+)}\{T[y](t)\}$. Hence the limit point $y(t) = \lim\limits_{j\to\infty} T^{m_j}[y](t)$ also belongs to it, and $R^{(+)}[y(t)] < R^{(+)}\{T[y](t)\}$. It follows that T does not alter the range or the positive range of any element of S, (S being taken either in the wide complex or the narrow real sense), and the lemma holds.

7.3. Lemma 15 can be used to to prove the following theorem.

Theorem VII. Let $A(t)$ be a. p. with a module contained in a certain module M, and let (1.61) satisfy Condition I. Then a pseudo-solution $y(t)$ of (1.61) must be a. p. with a module contained in M if every pseudo-solution of (1.61) which is distinct from $y(t)$ but has the same order, degree, and (positive) range as $y(t)$ is a. p. with a module contained in M.

In order to establish this theorem it is of course only necessary in view of Theorem VI to show that $y(t)$ is positive stationary with respect to M. With this end in view let $h_1, h_2, \ldots$ be a sequence of positive numbers such that for each element Φ of M, $\lim_{i\to\infty} \Phi h_i = 0 \pmod{2\pi}$, and assume that it is not true that $\lim_{i\to\infty} y(t+h_i) = y(t)$ for all t. Then there exists a subsequence $h'_1, h'_2, \ldots$ of $h_1, h_2, \ldots$ such that $\lim_{i\to\infty} y(h'_i)$ exists and is not equal to $y(0)$; for otherwise $y(t) - \lim_{i\to\infty} y(t+h_i)$ would be a pseudo-solution of (1.61) which would vanish for $t = 0$. By Lemma 14, $\tilde{y}(t) \equiv \lim_{i\to\infty} y(t+h'_i)$ exists for all t and is a pseudo-solution of the same degree r and order λ as $y(t)$. Moreover it follows from Lemma 9 that we can choose a subsequence $h''_1, h''_2, \ldots$ of $h'_1, h'_2, \ldots$ such that $T[z](t) \equiv$ $\equiv \lim_{i\to\infty} z(t+h''_i)$ exists for each pseudo-solution $z(t)$ of order λ and for all t. Then by Lemma 15, $R^{(+)}[y(t)] = R^{(+)}\{T[y](t)\} = R^{(+)}[\tilde{y}(t)]$, and by hypothesis $\tilde{y}(t)$ is a. p. with a module contained in M. Since an a. p. function is stationary with respect to its module, $T[\tilde{y}](t) \equiv \tilde{y}(t) \equiv T[y](t)$; and it follows from Lemma 9 that $\tilde{y}(t) \equiv y(t)$. But this is impossible in view of our choice of $h'_1, h'_2, \ldots$; and therefore $\lim_{i\to\infty} y(t+h_i) = y(t)$ for all t, and $y(t)$ is positive stationary.

7.4. In the preceding theorem we used a part of the conclusion of Lemma 15; namely, the statement concerning ranges. In the following theorem we shall use the other part of the conclusion of Lemma 15 which deals with a canonical form of the transformation T. We first make the following definition.

Definition. A vector function $f(t)$ will be called (positive) twistable with respect to a module M if there exists a sequence of (positive) numbers $h_1, h_2, \ldots$ such that for each element Φ of M, $\lim_{i\to\infty} \Phi h_i = 0 \pmod{2\pi}$ and such that for all t $\lim\limits_{i\to\infty} f(t+h_i) = \theta f(t)$, where θ is a constant whose absolute value is unity but which is not equal to unity.

Theorem VIII. Let $A(t)$ be a. p. with a module contained in a certain module M, and let (1.61) satisfy Condition I. Then if for a certain λ there are no pseudo-solutions of (1.61) (neither real nor complex ones) of order λ and degree zero which are positive twistable with respect to M, it follows that all of the pseudo-solutions of (1.61) of order λ and degree zero are a. p. with a module contained in M.

As in Theorem VII it is only necessary to show that the pseudo-solutions are positive stationary with respect to M. Thus we let $h_1, h_2, \ldots$ be a sequence of positive numbers such that for each element Φ of M, $\lim\limits_{i\to\infty} \Phi h_i = 0 \pmod{2\pi}$, and assume that there exists a pseudo-solution $y(t)$ of order λ and degree zero for which it is not true that $\lim\limits_{i\to\infty} y(t+h_i) = y(t)$ for all t. As before, there exists a subsequence $h'_1, h'_2, \ldots$ of $h_1, h_2, \ldots$ such that $\lim\limits_{i\to\infty} y(h'_i)$ exists and does not equal $y(0)$. Moreover there exists a subsequence $h''_1, h''_2, \ldots$ of $h'_1, h'_2, \ldots$ such that $T[z](t) \equiv \lim\limits_{i\to\infty} z(t+h''_i)$ exists for each pseudo-solution $z(t)$ of order λ and for all t.

By Lemma 15, the set of pseudo-solutions of (1.61) of degree zero has a basis $y^{(1)}(t), \ldots, y^{(L)}(t)$ such that $T[y^{(\nu)}](t) = \theta_\nu y^{(\nu)}(t)$ for all t and all $\nu \leqq L$, where $\theta_1, \ldots, \theta_L$ are constants whose absolute value is unity. By hypothesis, $\theta_1 = \theta_2 = \cdots = \theta_L = 1$; and hence T is the identity transformation so far as pseudo-solution of order λ and degree zero are concerned. It follows that $\lim\limits_{i\to\infty} y(h''_i) = y(0)$ contrary to the definition of $h'_1, h'_2, \ldots$ Hence all of the pseudo-solutions of order λ and degree zero are stationary with respect to M.

7.5. The following theorem shows that pseudo-solution and its minors have similar a. p. properties.

Theorem IX. Let $A(t)$ be a. p. with a module contained in a certain module M, and let (1.61) satisfy Condition I. Then if a pseudo-solution of (1.61) is a. p. with a module contained in M, so are all of its minors. On the other hand, if all of the pseudo-solutions of (1.61) of a certain order λ and degree zero are a. p. with a module contained in M, the same is true of all pseudo-solutions of (1.61) of order λ irrespective of their degrees.

The truth of the first statement follows immediately from the fact that an a. p. pseudo-solution is stationary and from Lemma 14 which shows that the minors of a pseudo-solution have similar convergence properties to the pseudo-solution itself.

In order to prove the second statement, assume that there is a pseudo-solution of order λ which is not a. p. with a module contained in M, let p be the least degree which any such pseudo-solution has, and let $y_p(t)$ be such a pseudo-solution of order λ and degree p. By hypothesis, $p > 0$; and because of the choice of p all of the minors $y_0(t), \ldots, y_{p-1}(t)$ of $y_p(t)$ except $y_p(t)$ itself are a. p. with modules contained in M. As in the two preceding theorems, let $h_1, h_2, \ldots$ be a sequence of positive numbers such that for each element Φ of M, $\lim_{i\to\infty} \Phi h_i = 0 \pmod{2\pi}$, and as before let $h'_1, h'_2, \ldots$ be subsequence of $h_1, h_2, \ldots$ such that $\lim_{i\to\infty} y_p(h'_i)$ exists and is not equal to $y(0)$. It follows from Lemma 14 that $T[y_p](t)$ exists for all t, where $T[z](t) \equiv \lim_{i\to\infty} z(t+h_i)$.

Since $y_0(t), \ldots, y_{p-1}(t)$ are a. p. with modules contained in M, $T[y_\mu](t) \equiv \equiv y_\mu(t)$ for $\mu < p$. Thus $y_p(t)$ and $T[y_p](t)$ have the same minors of degree less than p, so that $y^*(t) \equiv y_p(t) - T[y_p](t)$ is a pseudo-solution of order λ and degree zero. It follows that $y^*(t)$ is a. p. with a module contained in M and that for all non-negative integers m,

$$(7.51) \qquad y^*(t) \equiv T^m[y^*](t) \equiv T^m[y_p](t) - T^{m+1}[y_p](t).$$

Putting $m = 0, 1, \ldots, r-1$ and adding, we obtain $r y^*(t) \equiv y_p(t) - T^r[y_p](t)$. But this is impossible since the right member is bounded for all t and $y^*(t) \not\equiv 0$. Thus our assumption was incorrect and all of the pseudo-solutions of order λ are a. p. with modules contained in M.

7.6. In concluding this section, we shall prove a theorem which although stated for the homogeneous rather than the non-homogeneous equation is based directly on a theorem of Favard.[1] Its proof depends upon a specific device of Favard, and not merely upon his general method as Theorem VI does. This device will be given in Lemma 16.

Definition. The norm of a point set S in n-dimensional space is the least upper bound of the norms of the elements of S. It will be denoted by $\|S\|$.

[1] Sur les équations differentielles linéaires à coefficients presque-periodiques, Acta mathematica, vol. 51, pp. 31—81, esp. p. 59.

Lemma 16. Let $A(t)$ be a continuous n-by-n matrix function such that (1.61) satisfies Condition I, and let $\mathfrak{A}$ be the pseudo-solution manifold of (1.61) of order λ and degree r. Let S be a closed convex point set in n-dimensional space which contains the range of at least one element of $\mathfrak{A}$, and let s be the greatest lower bound of $\| R[y(t)] \|$ for all elements $y(t)$ of $\mathfrak{A}$ whose ranges lie in S. Then there exists one and only one element $y^*(t)$ of $\mathfrak{A}$ such that $\| R[y^*(t)] \| = s$.

For if $y^{(1)}(t)$, $y^{(2)}(t)$, ... is a sequence of elements of $\mathfrak{A}$ whose ranges lie in S such that $\lim\limits_{\varrho \to \infty} \| R[y^{(\varrho)}(t)] \| = s$, there must exist a sequence of indices $\varrho_1, \varrho_2, \cdots$ such that $\lim\limits_{i \to \infty} y^{(\varrho_i)}(0)$ exists; and hence $y^*(t) = \lim\limits_{i \to \infty} y^{(\varrho_i)}(t)$ must exist for all t and be an element of $\mathfrak{A}$ whose range is contained in S. (The convergence property of the pseudo-solutions of order λ follows from the fact that they form a finite linear manifold and that $y(0) \neq 0$ for any pseudo-solution.) It is also clear that $\| R[y^*(t)] \| = s$, so that $y^*(t)$ has the property described in the lemma.

To show that $y^*(t)$ is unique, assume that there exists another element $y^{**}(t)$ of $\mathfrak{A}$ having its range in S and satisfying $\| R[y^{**}(t)] \| = s$. Then $\tilde{y}(t) = \frac{1}{2} y^*(t) + \frac{1}{2} y^{**}(t)$ is also an element of $\mathfrak{A}$ whose range is contained in S; and since s is a minimum and $\left\| \frac{1}{2}(a+b) \right\| \leqq \delta$ if $\|a\| \leqq \delta$ and $\|b\| \leqq \delta$, it follows that $\| R[\tilde{y}(t)] \| = s$. Let $h_1, h_2, \ldots$ be a sequence such that $\lim\limits_{i \to \infty} \| \tilde{y}(h_i) \| = s$. Then since

$$\left\| \frac{1}{2}(a-b) \right\|^2 = \frac{1}{2} \left\| a \right\|^2 + \frac{1}{2} \left\| b \right\|^2 - \left\| \frac{1}{2}(a+b) \right\|^2,$$

$$\lim_{i \to \infty} \left\| \frac{1}{2} [y^*(t) - y^{**}(t)] \right\| \leqq \frac{1}{2} s + \frac{1}{2} s - s.$$

But this is impossible since no pseudo-solution comes arbitrarily close to zero; and it follows that $y^*(t)$ is unique.

7.7. **Theorem X.** Let $A(t)$ be an a. p. square matrix function having the module M such that (1.61) satisfies Condition I, and let S be a closed convex point set in n-dimensional space which contains the range of a certain pseudo-solution of (1.61) of order λ and degree r. Then if S does not contain the origin, there exists an a. p. pseudo-solution $y^*(t)$ of (1.61) of order λ and degree $\leqq r$ having its range contained in S and its module contained in M.

For by Lemma 16, $\| R[y(t)] \|$ has a proper minimum for elements of the pseudo-solution manifold of (1.61) of order λ and degree r which have their ranges in S. Since S does not contain the origin, the element $y^*(t)$ which has this minimum range norm is not identically zero but is actually a pseudo-solution. But by Lemma 15, ranges and hence range norms are unaltered by the transformations there considered, and hence the transform of $y^*(t)$ has a minimum range norm and must be $y^*(t)$ itself. It readily follows that if $h_1, h_2, \ldots$ is such that $\lim\limits_{i\to\infty} \Phi h_i = 0 \pmod{2\pi}$ for each element Φ of M, then $\lim\limits_{i\to\infty} y^*(t+h_i)$ exists for all t, and of course equals $y^*(t)$. Thus $y^*(t)$ is stationary and hence a. p. with a module contained in M.

§ 8. Solutions of the Almost Periodic Type. — The Homogeneous Case.

8.1. The theorems of the last section enable us to prove theorems concerning solutions of the a. p. type. Thus Theorem VII, the first part of Theorem IX and Lemma 1 yield immediately

Theorem XI. Let $A(t)$ be an a. p. square matrix function which has its module contained in a certain module M, and let (1.61) satisfy Condition I. Let $x(t)$ be a primary solution of (1.61) having $y(t)$ as its associated pseudo-solution. Then if every other pseudo-solution which has the same order, degree, and range as $y(t)$ is a. p. with a module contained in M, it follows that $x(t)$ is of the a. p. type with a module contained in M.

In the following corollaries, $A(t)$ is understood to be a. p. with its module contained in M, and (1.61) is understood to satisfy Condition I.

Corollary 1. Let $x(t)$ be a bounded solution of (1.61). Then if every other solution which has the same range as $x(t)$ is a. p. with a module contained in M, so is $x(t)$.

Corollary 2. Let $x(t)$ be a bounded solution of (1.61). Then if no other solution has the same range as $x(t)$, $x(t)$ is a. p. with a module contained in M.

Corollary 3. Let $x(t)$ be a primary solution of (1.61) having $y(t)$ as its associated pseudo-solution. Then if no other pseudo-solution has the same order, degree and range as $y(t)$, $x(t)$ is of the a. p. type with a module contained in M.

8.2. The above theorem leads immediately to a theorem giving necessary and sufficient conditions that all the solutions be of the a. p. type. The fact that Condition I is a necessary condition was shown in Theorem I.

Theorem XII. Let $A(t)$ be an a. p. n-by-n matrix function which has its module contained in a certain module M. Then all of the solutions of (1.61) will be of the a. p. type with a module contained in M if and only if (1.61) satisfies Condition I and there exist n pseudo-solutions $y^{(1)}(t), \ldots, y^{(n)}(t)$ of (1.61) whose leaders are linearly independent and for each of which the following statement holds. With the exception of a. p. ones whose modules are contained in M, there exists no pseudo-solution of (1.61) which is distinct from but has the same order, degree, and range as $y^{(\nu)}(t)$.

8.3. Another set of necessary and sufficient conditions can be obtained from Theorem VIII and the second part of Theorem IX.

Theorem XIII. Let $A(t)$ be an a. p. square matrix function whose module is contained in a certain module M. Then all of the solutions of (1.61) will be of the a. p. type with modules contained in M if and only if (1.61) satisfies Condition I and there exists no solution $x(t)$ of (1.61) (neither a real nor a complex one) having an exponential multiplier $e^{-\lambda t}$ such that $e^{-\lambda t}x(t)$ is bounded and positive twistable with respect to M.

8.4. A some what neater but less incisive theorem can be obtained from Theorem XIII by use of the following definition and lemma.

Definition. A vector function $f(t)$ will be called symmetric if $R[f(t)] = R[cf(t)]$ for some constant scalar multiplier c which is not equal to unity but has unity as its absolute value.

Lemma 17. If there exists a module with respect to which a scalar function $f(t)$ is twistable, it follows that $f(t)$ is symmetric.

For there exists a sequence $h_1, h_2, \ldots$ such that for all t $\lim\limits_{i\to\infty} f(t+h_i) = cf(t)$ for some constant c which is not equal to but has the absolute value unity. Thus $R[cf(t)] < R[f(t)]$; and by repetition it follows that for all positive integers m and all t, $c^m f(t)$ is an element of $R[cf(t)]$. But the set $c, c^2, c^3, \ldots$ has unity either as an element or as a limit point, and hence for each t, $f(t)$ is an element of $R[cf(t)]$. Hence $R[cf(t)] = R[f(t)]$ and $f(t)$ is symmetric.

Theorem XIV. Let $A(t)$ be an a. p. square matrix function whose module is contained in a certain module M. Then all of the solutions of (1.61) will be of the a. p. type with modules contained in M if and only if (1.61) satisfies Condition I and every bounded and symmetric product $e^{-\lambda t} x(t)$ of an exponential and a (real or complex) solution $x(t)$ of (1.61) is a. p. with a module contained in M.

For $e^{-\lambda t} x(t)$ cannot be bounded and twistable with respect to M, since if it were it would be symmetric and hence a. p. with a module contained in M.

8.5. Finally, another theorem concerning a particular solution can be obtained from Theorem X and the first part of Theorem IX.

Theorem XV. Let $A(t)$ be an a. p. square matrix function having the module M, let (1.61) satisfy Condition I, and let S be closed convex point set in the n-dimensional space which contains the range of a certain pseudo-solution of order λ and degree r. Then if S does not contain the origin, there exists a satisfactory solution $x(t)$ of (1.61) of order λ and degree $\leqq r$ which is of the a. p. type with its module contained in M and the range of its associated pseudo-solution contained in S.

§ 9. Solutions of the Almost Periodic Type. — The Non-Homogeneous Case.

9.1. The non-homogeneous case will not be considered at length as the homogeneous case was, since it has been pointed out in (1.6) how theorems dealing with the homogeneous equation may be restated to fit the non-homogeneous equation. We shall however apply Theorem XV in order to show that in the non-homogeneous case the condition analogous to Condition I alone implies the existence of a solution of the a. p. type. The condition mentioned is

Condition II. The system (1.21) will be said to satisfy Condition II if to every non-trivial vector function $z(t)$ of the form $\sum_{\nu=0}^{p} [c_\nu x(t) + x^{(\nu)}(t)] t^\nu$ where $x(t)$ is a solution of (1.21), the $x^{(\nu)}(t)$ are solutions of (1.61), and the c_ν are constants, there correspond a real number λ and a non-negative integer r such that $z(t) = O^*(e^{\lambda t} t^r)$ at $+\infty$.

9.2. We can now state

Theorem XVI. Let $A(t)$ be an a. p. n-by-n matrix function and $b(t)$ an a. p. n-dimensional vector function, let M be the least common module of $A(t)$ and $b(t)$, and let (1.21) satisfy Condition II. Then (1.21) has at least one solution of the form

$$x^*(t) = y_0(t) + t y_1(t) + \cdots + t^n y_n(t),$$

where $y_0(t), \ldots, y_n(t)$ are a. p. vectors having their modules conditioned in M. In accordance with the method given in (1.6), we consider the system

$$(1.62) \qquad \begin{cases} D[x(t)] = A(t) \cdot x(t) + b(t)\xi(t) \\ \dfrac{d}{dt}\xi(t) = 0; \end{cases}$$

where $\xi(t)$ is a scalar function and the $(n+1)$-dimensional vector $[x(t), \xi(t)]$ is the unknown. Since (1.21) satisfies Condition II, it is clear that (1.62) satisfies Condition I; and every solution of (1.62) is decomposable. But if $x(t)$ is a solution of (1.21), $[x(t), 1]$ is a solution of (1.62); and at least one of the satisfactory solutions into which it is decomposable is of the form $[c x_r(t); c]$, where $x_r(t)$ is a solution of (1.21) and c is a non-zero constant. Let $[y(t); \eta(t)]$ be the associated pseudo-solution of $[x_r(t); 1]$; let λ and r be its order and degree, and let $[x_0(t); c_0], \ldots, [x_r(t); c_r]$ be the minors. Then the bounded function $\eta(t) \equiv e^{-\lambda t} \sum_{\nu=0}^{r} \frac{t^\nu}{\nu!} c_{r-\nu}$, where $c_r = 1$; so $\lambda = c_0 = \cdots = c_{r-1} = 0$. Thus if S is the set of points in $(n+1)$-dimensional space whose last coordinates are equal to unity, it follows that (1.62) has a pseudo-solution of order zero whose range is contained in S. Since S does not contain the origin we can apply Theorem XV, and it follows that (1.62) has a satisfactory solution of order zero of the a. p. type with a module contained in M whose associated pseudo-solution $[y^*(t), 1]$ has its range contained in S. But a pseudo-solution and its leader are equal when $t = 0$, and hence the solution of (1.62) in question is of the form $[x^*(t), 1]$; and $x^*(t)$ is a solution of (1.21) having the property demanded by the theorem.

9.3. In conclusion, we note that Theorem XVI can be combined with Theorem XII, Theorem XIII, or Theorem XIV to give necessary and sufficient conditions that all of the solutions of (1.21) be of the a. p. type.

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