ON THE UNSYMMETRIC TOP.*

BY

JOHN J. CORLISS

of ANN ARBOR, Mich., U. S. A.

The problem of the motion of a heavy rigid body about a fixed point is an old problem, — one of which much has been written but of which little is known. Euler[1] first stated the equations of motion in the final definitive and elegant form in use today. They are

$$(1) \qquad I_1 \dot{\omega}_1 + (I_3 - I_2)\,\omega_2\,\omega_3 = H_1$$

$$(2) \qquad I_2 \dot{\omega}_2 + (I_1 - I_3)\,\omega_1\,\omega_3 = H_2$$

$$(3) \qquad I_3 \dot{\omega}_3 + (I_2 - I_1)\,\omega_1\,\omega_2 = H_3.$$

The angular velocities ω_1, ω_2 and ω_3 are connected with Euler's angles Θ, Φ and Ψ by the equations:

$$(4) \qquad \dot{\Theta} = \omega_1 \cos \Phi - \omega_2 \sin \Phi$$

$$(5) \qquad \dot{\Phi} = -\omega_1 \sin \Phi \cot \Theta - \omega_2 \cos \Phi \cot \Theta + \omega_3$$

$$(6) \qquad \dot{\Psi} = \omega_1 \sin \Phi \csc \Theta + \omega_2 \cos \Phi \csc \Theta$$

or by the equations:

$$(7) \qquad \omega_1 = \dot{\Theta} \cos \Phi + \dot{\Psi} \sin \Theta \sin \Phi$$

$$(8) \qquad \omega_2 = -\dot{\Theta} \sin \Phi + \dot{\Psi} \sin \Theta \cos \Phi$$

$$(9) \qquad \omega_3 = \dot{\Phi} + \dot{\Psi} \cos \Theta.$$

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[1] Euler: Mémoires de L'Académie de Berlin, 1758.

The origin is the fixed point, and the following notation is used: (x, y, z) denotes the fixed system of axes; (x_1, y_1, z_1) the moving system, which is taken coincident with the principal axes of the momental ellipsoid at O; (f, g, h) are the coordinates of the center of gravity; ω_1, ω_2, and ω_3 are the components of the instantaneous angular velocity vector along the moving axes; H_1, H_2, and H_3 are the components of the instantaneous moment vector, H; I_1, I_2 and I_3 are the principal moments of inertia at O; M is the angular momentum vector; and w is the weight of the body. The figure illustrates the geometric meaning of Euler's angles, and the table gives a convenient means of finding components.

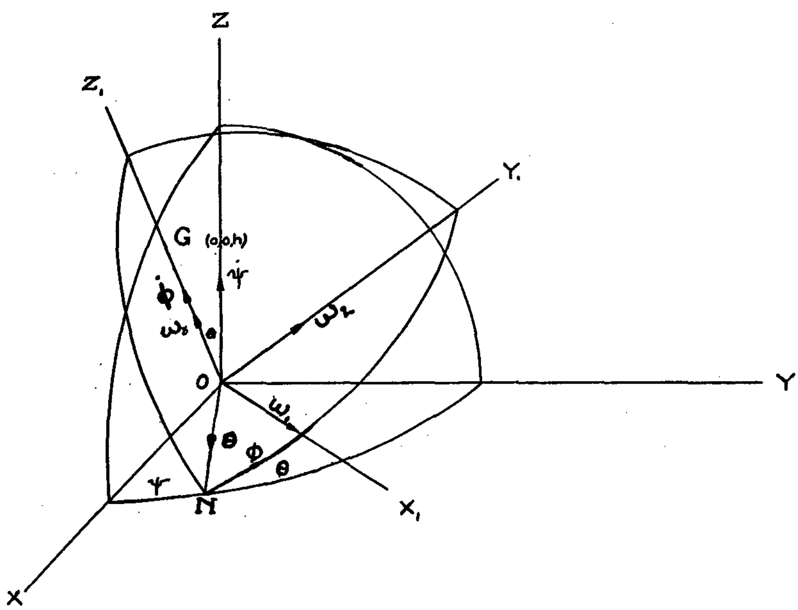

Fig. 1.

		M_1	M_2	M_3
		ω_1	ω_2	ω_3
		X_1	Y_1	Z_1
$\dot{\Theta}$	N	$\cos \Phi$	$\sin \Phi$	0
$\dot{\Phi}$	Z_1	0	0	1
$\dot{\Psi}$	$Z(-w)$	$\sin \Theta \sin \Phi$	$\sin \Theta \cos \Phi$	$\cos \Theta$

The values of H_1, H_2, and H_3 are:

(10) $$H_1 = g(-w\cos\Theta) \qquad - h(-w\sin\Theta\cos\Phi)$$

(11) $$H_2 = h(-w\sin\Theta\sin\Phi) - f(-w\cos\Theta)$$

(12) $$H_3 = f(-w\sin\Theta\cos\Phi) - g(-w\sin\Theta\sin\Phi).$$

The classical integrals are:

(13) $$I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 = E - 2w\,[f\sin\Theta\sin\Phi + g\sin\Theta\cos\Phi + h\cos\Theta]$$

(14) $$I_1\omega_1\sin\Theta\sin\Phi + I_2\omega_2\sin\Theta\cos\Phi + I_3\omega_3\cos\Theta = k$$

(15) $$(\sin\Theta\sin\Phi)^2 + (\sin\Theta\cos\Phi)^2 + \cos^2\Theta = 1\,.$$

The first integral expresses the fact that the total energy remains constant, the second that the projection of the angular momentum upon the vertical is constant, the third is a trigonometric identity.

The literature upon the Top Problem is extensive but it is a literature of special cases. Euler[1] and Poinsot[2] studied the case in which the exterior forces are zero, or are such that their resultant passes through the fixed point. Lagrange[3] and Poisson[4] solved the case for which (1) $I_1 = I_2$ and (2) $f = g = 0$. Mme. S. Kowalevski[5] solved the case for which (1) $I_1 = I_2 = 2I_3$ and (2) $h = 0$. It may be remarked that nothing is gained in generality by not assuming $g = h = 0$, since any pair of perpendicular axes in the equatorial plane may be taken as principal axes. The difference in this case and that of Lagrange lies in the fact that in the case of Lagrange the center of gravity lies on the unsymmetric axis while in the case of S. Kowalevski it lies on one of the symmetric axes.

R. Liouville[6] has shown that for (1) $I_1 = I_2$, (2) $h = 0$, and (3) the ratio $\frac{2I_3}{I_1} = n$ (where n can be any integer, which because of the relation $I_3 \leq I_1 + I_2$ cannot exceed 4) a fourth integral independent of the classical integrals (13), (14), (15) exists, and hence the problem is solvable.

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[1] loc. cit.

[2] Journal de Liouville 1re série 16. Théorie nouvelle de la rotation des corps, Paris, 1834.

[3] Mécanique Analytique, p. 251.

[4] Journal de l'École Polytechnique, 16th book, 1815. Traité de Mécanique, 1811.

[5] Mme. S. Kowalevski: Acta mathematica 12, p. 177, 1889.

[6] Mémoire présenté au concours du prix Bordin, en 1894. Acta math. 20, p. 239, 1897.

For the four cases just given, the initial conditions are perfectly arbitrary, hence there enter in the solution six constants of integration. Three of the six quantities giving the mass distribution are arbitrary so that for these cases there enter in all nine arbitrary constants. These four cases are unique in that they are the only cases for which a fourth algebraic integral (not a combination of the classical integrals) can exist. They exhaust all the possibilities of solving the problem for perfectly general initial conditions. R. Liouville[6], Poincaré[7] and Ed. Husson[8] have all obtained the theorem: In order that in the case of the movement of a heavy rigid body around a fixed point, an algebraic integral may exist, that is not a combination of the classical integrals, and which does not implicitly involve the time, it is necessary that the momental ellipsoid (of the body) belonging to the fixed point, be an ellipsoid of revolution, the initial conditions being assumed to be perfectly arbitrary.

Since there are no more algebraic cases that can be solved for perfectly general initial conditions, recent writers have sought to solve the problem for special initial conditions and for special mass distributions. This has been done in a very few cases.

W. Hess[9] assumes that for the mass distribution two conditions hold:

$$(1)\ g = 0 \quad \text{and} \quad (2)\ \frac{I_1 f^2 + I_3 h^2}{I_1 I_3} = \frac{f^2 + h^2}{I_2}.$$

N. Joukowsky[10] has shown that these two restrictions imply that the center of gravity, G, lies on the perpendicular through the fixed point, o, to a circular cross section of the reciprocal momental ellipsoid

$$\frac{x_1^2}{I_1} + \frac{y_1^2}{I_2} + \frac{z_1^2}{I_3} = 1.$$

In addition Hess assumes that the extremity of the impulse vector, M, lies initially in said circular cross section. The motion is such that throughout the whole movement the impulse vector, M, lies in this plane.

[6] loc. cit.

[7] Les méthodes nouvelles de la Mécanique céleste, Paris, 1892.

[8] Acta math. 31, p. 71, 1907; Toulouse Ann. 8, p. 73, 1906.

[9] Math. Annalen 37, p. 153, 1890.

[10] Jahresbericht der deutschen Mathematikervereinigung. V (3), p. 62, 1894.

O. Staude[11] studied the case for which the motion consists of a uniform rotation about a vertical axis fixed in the body. He shows that there are ∞^2 such axes and that they form a cone of the second order whose equation is

$$(I_2 - I_3) f y_1 z_1 + (I_3 - I_1) g z_1 x_1 + (I_1 - I_2) h x_1 y_1 = 0.$$

S. Tschaplygin[12] and D. N. Gorjatscheff[13] independently solved the problem under the restrictions:

$$(1)\ I_1 = I_2 = 4 I_3 \quad (2)\ g = h = 0 \quad (3)\ k = 0.$$

W. Stekloff[14] and D. Bobyleff[15] studied a special case of the Kowalevski case. They assumed

$$(1)\ I_2 = I_3 = 2 I_1$$

$$(2)\ f = h = 0$$

and found a particular solution subject to the initial conditions

$$(3)\ \omega_3 = 0 \text{ and}$$

$$(4)\ \omega_1 \omega_2 = - \frac{w g \sin \Theta \sin \Phi}{I_1}.$$

With these initial conditions the instantaneous axis of rotation remains throughout the course of the movement in the $x_1 y_1$ plane. The two solutions were published in the same volume, the treatment of Bobyleff following that of Stekloff.

W. Stekloff[16] has another case in which he assumes

$$(1)\ h = g = 0 \qquad (2)\ \sin \Theta \cos \Phi = A \omega_1 \omega_2$$
$$\cos \Theta = B \omega_1 \omega_3$$

where A and B are undetermined coefficients. The solution contains one arbitrary constant and if the movement is to be real the I's must satisfy the inequalities

$$I_1 > 2 I_3, \quad I_2 > I_1 > I_3.$$

[11] Crelle, Journal für Math. 113, p. 318, 1894.
[12] Moskau Phys. Sect. 10, Bd. 2, 1901.
[13] Moskau Math. Samml. 21, p. 431.
[14] Moskau Phys. Sect. 8 heft. 2, p. 19, 1896.
[15] Moskau Phys. Sect. 8 heft. 2, p. 21, 1896.
[16] Moskau Phys. Sect. 10, Lief. 1, p. 1, 1899.

D. Gorjatscheff[17] following in the footsteps of Stekloff found a particular solution by assuming:

$$(1)\quad g = h = 0$$
$$(2)\quad wf \sin\Theta \cos\Phi = A\,\omega_1\omega_2$$
$$wf \cos\Theta = (B + C\omega_1^2)\,\omega_1\omega_3$$

where A, B, and C are undetermined coefficients.

The I's must satisfy the equation

$$I_1 I_3 = \sin\Theta \sin\Phi\,(I_2 - I_3)(I_1 - 2I_2).$$

The solution involves one arbitrary constant.

D. Gorjatscheff[18] has two other cases which are practically the same. The restrictions for both cases are:

$$(1)\quad I_1 = I_2 = 4I_3$$
$$(2)\quad f = g = 0$$
$$(3)\quad k = 0.$$

S. Tschaplygin[19] generalized the work of Gorjatscheff. He assumes

$$(1)\quad I_1 = I_2 = 4I_3$$
$$(2)\quad h = 0$$
$$(3)\quad k = 0.$$

S. Tschaplygin[20] generalized the assumptions of Stekloff and Gorjatscheff. He assumes

$$(1)\quad g = h = 0$$
$$(2)\quad a \sin\Theta \cos\Phi = A\,\omega_1\omega_2 + D\,\omega_2\omega_1^n$$
$$(3)\quad a \cos\Theta = B\,\omega_1\omega_3 + C\,\omega_3\omega_1^n$$

where A, B, C, and D are undetermined coefficients.

[17] Moskau Phys. Sect. 10, Lief. 1, p. 22, 1899.
[18] Moskau Math. Ges. Am. 16, 1899. Moskau Samml. 21, p. 431, 1899.
[19] Moskau Phys. Sect. 10, Lief. 2, p. 31, 1899.
[20] Moskau Phys. Sect. 12, Lief. 1, p. 1, 1904.

The I's must satisfy the relation

$$\omega_2(2I_2-I_3)(2I_3-I_1)=4I_2I_3.$$

We come now to a development in the recent history of the top problem.

P. A. Schiff[21] suggested that instead of using as variables the vectors ω_1, ω_2 and ω_3, that the three scalars T, U, and S be used, where

$$T=\frac{I_1\omega_1^2+I_2\omega_2^2+I_3\omega_3^2}{2}$$

$$U=\frac{I_1^2\omega_1^2+I_2^2\omega_2^2+I_3^2\omega_3^2}{2}$$

$$S=fI_1\omega_1+gI_2\omega_2+hI_3\omega_3.$$

T is the kinetic energy. U is the square of the angular momentum divided by two and S is the dot product of the impulse vector and the coordinates of the center of gravity. Schiff very simply and elegantly transformed the Euler-Poisson equations into equations involving T, U, and S. These equations have come to be known as the Hess-Schiff reduced differential equations. Hess first suggested the idea of using T, U, and S as variables.

The Hess-Schiff equations created widespread interest.

P. Stäckel[22] and O. Lazzarino[23] considered the Hess-Schiff equations at length especially with reference to their equivalence to the Euler-Poisson equations. If S, T, and U are all variable then the two systems are equivalent. Assuming each constant separately merely leads to known cases.

N. Kowalevski[24] restricted himself to the case for which the center of gravity lies on a principal axis, i. e., $f=g=0$. N. Kowalevski observed that although the integrals of Stekloff, Gorjatscheff, and Tschaplygin had been obtained under widely different conditions, that they all have a similar analytical character. He set out to find all possible cases for which ω_1^2 and ω_2^2 can be expressed as polynomials of the third degree in ω_3. He shows that the conditions he obtains give rise to the cases of Stekloff, Gorjatscheff, and Tschaplygin, and also to a new case, the details of which he does not carry out. N. Kowa-

[21] Moskau Math. Samml. 24, p. 169, 1903.

[22] Math. Ann. 65, p. 538, 1908. Math. Ann. 67, p. 399, 1909.

[23] Rend. d. Soc. reale di Napoli, (3 a) 17, p. 68, 1911. R. Accademia dei Lincei atti 28_1, p. 266; p. 325; p. 341, 1919. R. Accademia dei Lincei atti 28_2, p. 9; p. 259; p. 329, 1919.

[24] Math. Annalen 65, p. 528, 1908.

levski's conditions are long and complicated and hence better suited to verifying old cases, than to picking out restrictions which lead to new cases. It is, therefore, no wonder that he missed the case of P. Field[25] and those given in this paper.

New Cases.

First Case.

For the first case treated the assumptions are: (1) that the center of gravity is on a principal axis, i. e., $f = g = 0$, and (2) that the projection of the angular momentum upon the vertical is zero, i. e., $k = 0$. The results obtained are extremely simple and interesting for that reason if for no other. With these assumptions equations (1), (2), and (3) may be written in the form

$$(16) \qquad I_1 \dot{\omega}_1 + (I_3 - I_2)\,\omega_2\omega_3 = wh \sin\Theta\cos\Phi$$

$$(17) \qquad I_2 \dot{\omega}_2 + (I_1 - I_3)\,\omega_1\omega_3 = -wh \sin\Theta\sin\Phi$$

$$(18) \qquad I_3 \dot{\omega}_3 + (I_2 - I_1)\,\omega_1\omega_2 = 0$$

and (13) becomes

$$(19) \qquad I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 = E - 2wh\cos\Theta.$$

In order to simplify the study of the problem let us eliminate $\sin\Theta\sin\Phi$, $\sin\Theta\cos\Phi$, $\cos\Theta$, and dt from equation (14) by means of equations (16), (17), (18), and (19). This gives

$$(20) \qquad I_1 I_2 (I_1 - I_2)\left[-\omega_1^2(2\omega_2 d\omega_2) + \omega_2^2(2\omega_1 d\omega_1)\right] + \\ + \left[(I_1I_3 - 2I_1^2)\omega_1^2 + (I_2I_3 - 2I_2^2)\omega_2^2 - I_3^2\omega_3^2 + I_3E\right] I_3\omega_3 d\omega_3 = \\ 2whkI_3 d\omega_3.$$

Eliminating the same quantities from the identity

$$(\sin\Theta\sin\Phi)^2 + (\sin\Theta\cos\Phi)^2 + (\cos\Theta)^2 = 1$$

gives

$$(21) \qquad \omega_1^2\left[I_2(I_1 - I_2)\,2\omega_2 d\omega_2 + I_3(I_1 - I_3)\,2\omega_3 d\omega_3\right]^2 + \\ \omega_2^2\left[I_1(I_1 - I_2)\,2\omega_1 d\omega_1 + I_3(I_3 - I_2)\,2\omega_3 d\omega_3\right]^2 = \\ \{4w^2h^2 - [E - (I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)]^2\}\, I_3^2 d\omega_3^2.$$

[25] Acta mathematica, Vol. 56.

Equations (20) and (21) may be simplified by a change of variables. Let

(22) $$du = 2I_2(I_1 - I_2)\,\omega_2 d\omega_2 + 2I_3(I_1 - I_3)\,\omega_3 d\omega_3$$
$$\text{or}\quad u = I_2(I_1 - I_2)\,\omega_2^2 + I_3(I_1 - I_3)\,\omega_3^2$$

(23) $$dv = 2I_1(I_1 - I_2)\,\omega_1 d\omega_1 + 2I_3(I_3 - I_2)\,\omega_3 d\omega_3$$
$$\text{or}\quad v = I_1(I_1 - I_2)\,\omega_1^2 + I_3(I_3 - I_2)\,\omega_3^2$$

we see that

(24) $$I_1 v + I_2 u = (I_1 - I_2)\,[I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2]$$

(25) $$\text{and}\quad u + v = (I_1 - I_2)\,[I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2].$$

In terms of the new variables, u and v, equations (20) and (21) become

(26) $$-[v - I_3(I_3 - I_2)\,\omega_3^2]\,du + [u - I_3(I_1 - I_3\,\omega_3^2]\,dv +$$
$$[E(I_1 - I_2) - (u+v)]\,I_3^2\,\omega_3 d\omega_3 = 2whkI_3(I_1 - I_2)\,d\omega_3$$

(27) $$I_2[v - I_3(I_3 - I_2)\,\omega_3^2]\,du^2 + I_1[u - I_3(I_1 - I_3)\,\omega_3^2]\,dv^2$$
$$+ \frac{I_1 I_2 I_3^2}{(I_1 - I_2)}[E(I_1 - I_2) - (u+v)]^2\,d\omega_3^2 = 4I_1 I_2 I_3^2(I_1 - I_2)w^2h^2 d\omega_3^2.$$

Finally we may still further simplify the problem by another change of variables. (Of course both these changes might be combined into one, but it seemed best to preserve the logical sequence.)

Let us assume

(28) $$\xi = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$$

(29) $$\eta = \frac{E(I_1 - I_2) - (u+v)}{(I_1 - I_2)}.$$

From equation (25) we see that

(30) $$\eta = E - (I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$$

and consequently from equation (19) that

(31) $$\cos\Theta = \frac{\eta}{2wh}.$$

Solving equations (28) and (29) for u and v we have:

$$u = -[I_1\eta + \xi - I_1 E] \tag{32}$$

$$v = [I_2\eta + \xi - I_2 E] \tag{33}$$

In terms of the new variables equations (26) and (27) become after dividing out the common factor $(I_1 - I_2)$

$$[E - \eta - I_3\omega_3^2]\,d\xi + [\xi - I_3^2\omega_3^2]\,d\eta + I_3^2\eta\omega_3 d\omega_3 = 2whkI_3 d\omega_3 \tag{34}$$

$$\begin{aligned}&[-\xi - \eta(I_1 + I_2) + E(I_1 + I_2) + I_3(I_3 - I_1 - I_2)\omega_3^2]\,d\xi^2 + \\ &I_1 I_2[\xi - I_3^2\omega_3^2]\,d\eta^2 + 2I_1I_2[-\eta + E - I_3\omega_3^2]\,d\xi d\eta + \\ &I_1I_2I_3^2\eta^2 d\omega_3^2 = 4w^2h^2I_1I_2I_3^2 d\omega_3^2.\end{aligned} \tag{35}$$

Squaring equation (34) and multiplying through by $I_1 I_2$ gives

$$\begin{aligned}&I_1I_2[E - \eta - I_3\omega_3^2]^2 d\xi^2 + I_1I_2[\xi - I_3^2\omega_3^2]^2 d\eta^2 + \\ &2I_1I_2[E - \eta - I_3\omega_3^2]\,[\xi - I_3^2\omega_3^2]\,d\xi d\eta = \\ &I_1I_2[2whkI_3 - I_3^2\eta\omega_3]^2 d\omega_3^2.\end{aligned} \tag{36}$$

Multiplying equation (35) by $[\xi - I_3^2\omega_3^2]$ and from the result subtracting equation (36) gives

$$\begin{aligned}&\{[-\xi - \eta(I_1 + I_2) + E(I_1 + I_2) + I_3(I_3 - I_1 - I_2)\omega_3^2]\,[\xi - I_3^2\omega_3^2] - \\ &I_1I_2[E - \eta - I_3\omega_3^2]^2\}\,d\xi^2 = \\ &\{[I_1I_2I_3^2(4w^2h^2 - \eta^2)]\,[\xi - I_3^2\omega_3^2] - \\ &I_1I_2[2whkI_3 - I_3^2\eta\omega_3]^2\}\,d\omega_3^2.\end{aligned} \tag{37}$$

Equation (37) may be written in the form

$$\begin{aligned}&\left\{I_1I_2\frac{\left(\frac{d\xi}{d\omega_3}\right)^2}{I_1I_2I_3^2} - \xi\right\}\eta^2 + \left\{I_1[\xi - I_2E + I_3(I_2 - I_3)\omega_3^2]\frac{\left(\frac{d\xi}{d\omega_3}\right)^2}{I_1I_2I_3^2} + \right. \\ &\left. I_2[\xi - I_1E + I_3(I_1 - I_3)\omega_3^2]\frac{\left(\frac{d\xi}{d\omega_3}\right)^2}{I_1I_2I_3^2} + 4whkI_3\omega_3\right\}\eta +\end{aligned} \tag{38}$$

$$+\left\{4w^2h^2(\xi-I_3^2\omega_3^2)+[\xi-I_1E+I_3(I_1-I_3)\omega_3^2]\cdot\right.$$
$$\left.\cdot[\xi-I_2E+I_3(I_2-I_3)\omega_3^2]\frac{\left(\frac{d\xi}{d\omega_3}\right)^2}{I_1I_2I_3^2}-4w^2h^2k^2\right\}=0.$$

Equation (38) is worthy of study. First we note that it is a quadratic in η. Consequently we see that if, in our attempts to find solutions, we assume ξ to be an algebraic function of ω_3, that the solution for η will be algebraic. Hence in attempting to solve equation (34) we shall choose only such functions of ω_3 for ξ as will lead to a solution of equation (34) which will be algebraic, since the solution for η must satisfy simultaneously equations (34) and (38), or the equations (34) and (35).

If we rewrite equation (34) in the form (39) we see that it is linear in type:

$$(39)\qquad \frac{d\eta}{d\omega_3}-\frac{\left(\frac{d\xi}{d\omega_3}-I_3^2\omega_3\right)}{(\xi-I_3^2\omega_3^2)}\eta=-\frac{(I_3\omega_3^2-E)\frac{d\xi}{d\omega_3}+2whkI_3}{(\xi-I_3^2\omega_3^2)}$$

$$(35)\qquad \begin{aligned}&[-\xi-\eta(I_1+I_2)+E(I_1+I_2)+I_3(I_3-I_1-I_2)\omega_3^2]\,d\xi^2+\\&I_1I_2[\xi-I_3^2\omega_3^2]\,d\eta^2+2I_1I_2[-\eta+E-I_3\omega_3^2]\,d\xi\,d\eta+\\&I_1I_2I_3^2\eta^2d\omega_3^2=4w^2h^2I_1I_2I_3^2\,d\omega_3^2.\end{aligned}$$

So far we have shown that: (1) If ξ be an algebraic function of ω_3 then η will also be an algebraic function of ω_3; and (2) we have reduced our problem to the study of the two equations (39) and (35).

Stäckel* and Lazzarino* have shown that for S, T, and U all variable the Hess-Schiff reduced differential equations are equivalent to the Euler-Poisson equations. The variables used here differ only by constants from the Hess-Schiff variables, consequently we may regard these two equations as necessary and sufficient conditions that the values of ξ and η (ξ, η and ω_3 being all variable) satisfying equations (39) and (35) will lead to values of ω_1, ω_2, ω_3 satisfying equations (16), (17), and (18). As a check upon the numerical work the values obtained in the cases given below have been substituted directly in equations (16), (17), (18), (7), (8), and (9).

Equations (39) and (35) are entirely general, the only restriction so far being that (1) $f=g=0$. They are especially adapted to the search for new

* See references already given.

particular solutions. As a first effort toward finding a new particular solution let us assume

$$\xi = I_3^2 (a\omega_3^2 + b) \tag{40}$$

$$\eta = A\omega_3^2 + B \tag{41}$$

where a, b, A and B are arbitrary.

Let us now see if we can choose a, b, A, and B so that ξ and η shall satisfy simultaneous equations (39) and (35). It is to be expected that in order to satisfy these conditions that we shall have to impose certain restrictions upon the constants of integration, and perhaps on the I's.

Substituting for ξ and η in equations (39) and (35) and equating coefficients of like powers of ω_3 gives us after a great deal of numerical work the following values of a, b, A, and B, together with certain restrictions on E and k:

$$k = 0 \tag{42}$$

which means that the angular momentum vector remains in the xy plane throughout the motion

$$a = \frac{(I_3 - I_1)(I_3 - I_2)}{(I_3 - 2I_1)(I_3 - 2I_2)} \tag{43}$$

$$b = \frac{[2I_1 I_2 - (I_1 + I_2)I_3]wh}{I_3(I_3 - I_1)(I_3 - I_2)} \tag{44}$$

$$A = -2aI_3 \tag{45}$$

$$B = 2wh \tag{46}$$

$$E = \frac{[2I_1 I_2 - 2I_1 I_3 - 2I_2 I_3 + I_3^2]\,wh}{(I_3 - I_1)(I_3 - I_2)}. \tag{47}$$

Equation (47) imposes a restriction upon the total energy. This is essentially a positive quantity so that the I's must be chosen so as to make E positive.

Retracing our steps we find for ω_1^2 and ω_2^2 the values

$$\omega_1^2 = \frac{(I_3 - I_2)I_3}{(I_3 - 2I_1)(I_1 - I_2)}\left[\omega_3^2 - \frac{wh(I_3 - 2I_2)(I_3 - 2I_1)}{(I_3 - I_1)(I_3 - I_2)^2}\right] \tag{48}$$

$$\omega_2^2 = -\frac{(I_3 - I_1)I_3}{(I_3 - 2I_2)(I_1 - I_2)}\left[\omega_3^2 - \frac{wh(I_3 - 2I_2)(I_3 - 2I_1)}{(I_3 - I_1)^2(I_3 - I_2)}\right]. \tag{49}$$

To find the values of Φ. From (48) and (49) the derivatives of ω_1 and ω_2 with respect to ω_3 are

$$(50)\qquad \frac{d\omega_1}{d\omega_3} = \frac{(I_3 - I_2) I_3 \omega_3}{(I_3 - 2I_1)(I_1 - I_2)\omega_1}$$

$$(51)\qquad \frac{d\omega_2}{d\omega_3} = -\frac{(I_3 - I_1) I_3 \omega_3}{(I_3 - 2I_2)(I_1 - I_2)\omega_2}$$

and using equation (18) we find

$$(52)\qquad \frac{d\omega_1}{dt} = \frac{(I_3 - I_2)\omega_2\omega_3}{(I_3 - 2I_1)}$$

$$(53)\qquad \frac{d\omega_2}{dt} = -\frac{(I_3 - I_1)\omega_1\omega_3}{(I_3 - 2I_2)}.$$

Substituting these values in equations (16) and (17) and dividing one by the other gives us tan Φ

$$(54)\qquad \tan\Phi = \frac{(I_3 - 2I_1)\omega_1}{(I_3 - 2I_2)\omega_2}.$$

To find the value of Ψ. From equations (7) and (8) we get

$$(55)\qquad \frac{\omega_2 - \dot{\Psi}\sin\Theta\cos\Phi}{\omega_1 - \dot{\Psi}\sin\Theta\sin\Phi} = -\tan\Phi$$

and making use of equations (52), (53), (16), and (17) we find that $\dot{\Psi}$ has the value

$$(56)\qquad [(I_3 - 2I_2)^2\omega_2^2 + (I_3 - 2I_1)^2\omega_1^2]\,\omega_3\dot{\Psi} = \frac{wh(I_3 - 2I_1)(I_3 - 2I_2)}{(I_3 - I_1)(I_3 - I_2)}[(I_3 - 2I_1)\omega_1^2 + (I_3 - 2I_2)\omega_2^2].$$

From equations (48) and (49) we find that

$$(57)\qquad (I_3 - 2I_1)\omega_1^2 + (I_3 - 2I_2)\omega_2^2 = I_3\omega_3^2$$

consequently equation (56) may be written

$$(58)\qquad [(I_3 - 2I_1)^2\omega_1^2 + (I_3 - 2I_2)^2\omega_2^2]\,\omega_3\dot{\Psi} = \frac{wh(I_3 - 2I_1)(I_3 - 2I_2)}{(I_3 - I_1)(I_3 - I_2)}[I_3\omega_3^2].$$

To integrate $\dot{\Psi}$ we shall now show that it is equal to $\dot{\Phi}$.

Differentiating equation (54) gives

$$(59) \qquad \sec^2 \Phi \dot{\Phi} = \frac{I_3 - 2I_1}{I_3 - 2I_2} \cdot \frac{\omega_2 \dot{\omega}_1 - \omega_1 \dot{\omega}_2}{\omega_2^2}.$$

Replacing $\sec^2 \Phi$ by its value from equation (54) and ω_1 and ω_2 by their values from equations (52) and (53) and simplifying gives

$$(60) \qquad [(I_3 - 2I_1)^2 \omega_1^2 + (I_3 - 2I_2)^2 \omega_2^2] \dot{\Phi} = \\ \omega_3 [(I_3 - I_2)(I_3 - 2I_2)\omega_2^2 + (I_3 - I_1)(I_3 - 2I_1)\omega_1^2].$$

Again from equations (48) and (49) we have

$$(61) \qquad (I_3 - I_1)(I_3 - 2I_1)\omega_1^2 + (I_3 - I_2)(I_3 - 2I_2)\omega_2^2 = \\ \frac{w h I_3 (I_3 - 2I_2)(I_3 - 2I_1)}{(I_3 - I_2)(I_3 - I_1)}.$$

Hence the equation for $\dot{\Phi}$ becomes

$$(62) \qquad [(I_3 - 2I_1)^2 \omega_1^2 + (I_3 - 2I_2)\omega_2^2] \dot{\Phi} = \frac{w h I_3 (I_3 - 2I_1)(I_3 - 2I_2)\omega_3}{(I_3 - I_2)(I_3 - I_1)}.$$

Comparing equations (58) and (62) we see that

$$(63) \qquad \dot{\Psi} = \dot{\Phi}$$

and hence

$$(64) \qquad \Psi = \Phi + C_0.$$

For convenience we shall take $C_0 = 0$. To obtain the equation of the body, or the polhodal cone we must eliminate the ω's from its general equation

$$(65) \qquad \frac{x_1^2}{\omega_1^2} = \frac{y_1^2}{\omega_2^2} = \frac{z_1^2}{\omega_3^2}.$$

Carrying out this work gives as the equation of the body cone

$$(66) \qquad (2I_1 - I_3)x_1^2 + (2I_2 - I_3)y_1^2 + I_3 z_1^2 = 0.$$

Let us assume $I_1 > I_2$.

Then if

(66[1]) $$2I_1 > I_3 > 2I_2$$

equation (66) is that of a cone whose axis is OY_1.

If $I_3 > 2I_1$ is that of a cone whose axis is OZ_1.

To obtain the equation of the space or herpolhodal cone. Let ω_x, ω_y, and ω_z denote the components of the angular velocity, ω, along the fixed axes OX, OY, and OZ respectively. Then referring to the figure we see that

(67) $$\omega_x = \dot{\Theta}\cos\Psi + \dot{\Phi}\sin\Theta\sin\Psi$$

(68) $$\omega_y = \dot{\Theta}\sin\Psi - \dot{\Phi}\sin\Theta\cos\Psi$$

(69) $$\omega_z = \dot{\Psi} + \dot{\Phi}\cos\Theta.$$

Remembering that (for $c_0 = 0$) $\Phi = \Psi$ we may rewrite these as

(70) $$\omega_x = \dot{\Theta}\cos\Phi + \dot{\Psi}\sin\Theta\sin\Phi = \omega_1$$

(71) $$\omega_y = \dot{\Theta}\sin\Phi - \dot{\Psi}\sin\Theta\cos\Phi = -\omega_2$$

(72) $$\omega_z = \dot{\Phi} + \dot{\Psi}\cos\Theta = \omega_3.$$

(See equations (7), (8), and (9).)

The general equation of the space cone is

(73) $$\frac{x^2}{\omega_x^2} = \frac{y^2}{\omega_y^2} = \frac{z^2}{\omega_z^2}.$$

It follows that the equations of the body and space cones are the same only that of the body cone is referred to the moving system of axes while that of the space cone is referred to the fixed system of axes. The equation of the space cone is

(74) $$(2I_1 - I_3)x^2 + (2I_2 - I_3)y^2 + I_3 z^2 = 0.$$

To find the angular velocity, we must form the sum

(75) $$\omega^2 = \omega_1^2 + \omega_2^2 + \omega_3^2.$$

Using equations (48) and (49) we find

(76) $$\omega^2 = \frac{4(I_3 - I_1)(I_3 - I_2)}{(I_3 - 2I_1)(I_3 - 2I_2)}\omega_3^2 - \frac{2whI_3}{(I_3 - I_1)(I_3 - I_2)}$$

To find the time we use equation (18) which may be written

$$(77) \qquad t = \int \frac{I_3 \, d\omega_3}{(I_1 - I_2)\,\omega_1 \omega_2}.$$

We see that it is necessary to know for what values of ω_3 the product $\omega_1 \omega_2$ is real, since the time is real. Equating to zero the right hand members of equations (48) and (49) and solving for ω_3 we get

$$(78) \qquad \omega_3 = \pm \sqrt{\frac{(I_3 - 2I_1)(I_3 - 2I_2) w h}{(I_3 - I_1)(I_3 - I_2)^2}} = \pm A_3^{(1)}$$

$$(79) \qquad \omega_3 = \pm \sqrt{\frac{(I_3 - 2I_1)(I_3 - 2I_2) w h}{(I_3 - I_1)^2 (I_3 - I_2)}} = \pm A_3^{(2)},$$

where the superscript (1), for example, indicates that this value of ω_3 results from setting $\omega_1^2 = 0$.

We must now study the restrictions that must be placed upon the I's in order that the time shall be real.

If we assume that $I_1 > I_2$ then we have:

First:

$$(80) \qquad \text{For } I_3 > I_1; \quad 2I_2 < I_3 < 2I_1$$

we see that $A_3^{(1)}$ and $A_3^{(2)}$ are both imaginary hence for this choice of I's we cannot have real motion.

Second:

$$(81) \qquad \text{For } I_3 < I_1; \quad 2I_2 < I_3 < 2I_1$$

we see that $A_3^{(1)}$ is real, while $A_3^{(2)}$ is imaginary.

Plotting ω_3 against $\omega_1^2 \omega_2^2$ we obtain the graph

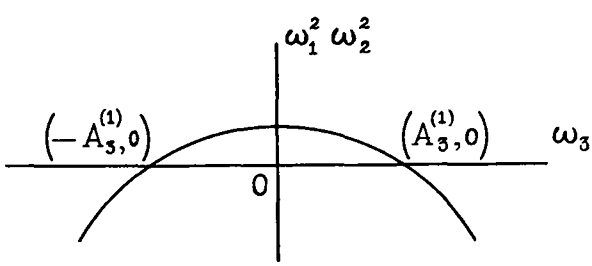

Fig. 2.

Third:

(82) If $I_3 > 2I_1 > 2I_2$.

This case is impossible since the moments must be such as to form the sides of a triangle. We have

(83) $$I_3 < I_1 + I_2$$

but from equation (82) it follows

$$I_3 > I_1 + I_2.$$

Thus we can have real motion only in the second case.

The inequalities of equation (81) may both be expressed by the single inequality

(84) $$2I_2 < I_3 < I_1.$$

Making use of equations (48), (49), (78), and (79) the integral for the time may be written

(85) $$t = \pm \sqrt{\frac{(I_3 - 2I_1)(I_3 - 2I_2)}{(I_3 - I_1)(I_3 - I_2)}} \int_{-A_3^{(1)}}^{\omega_3} \frac{d\omega_3}{\sqrt{-[(\omega_3 - A_3^{(1)})(\omega_3 + A_3^{(1)})(\omega_3 - A_3^{(2)})(\omega_3 + A_3^{(2)})]}},$$

where the sign is to be chosen so as to make the time, t, positive. Note in figure that ω_3 may become zero.

It is easy to verify that equation (84) is the only restriction on the I's necessary to satisfy the conditions imposed on the I's by equations (47), (66^1), and the fact that $\cos\Theta$ must be less or equal to one.

To obtain a geometric picture of the motion. As $\cos\Theta = 1$ when $\omega_3 = 0$ this is a convenient starting point.

From equations (48) and (49) we find that

(86) $$\omega_1^2 = -\frac{whI_3(I_3 - 2I_2)}{(I_1 - I_2)(I_3 - I_1)(I_3 - I_2)}$$

(87) $$\omega_2^2 = \frac{whI_3(I_3 - 2I_1)}{(I_1 - I_2)(I_3 - I_1)(I_3 - I_2)}$$

when $\omega_3 = 0$.

Since $\Psi = \Phi$ we may use equation (54) to find their value when $\omega_3 = 0$

(88) $$\tan^2 \Phi = -\frac{I_3 - 2I_1}{I_3 - 2I_2}$$

hence

(89) $$\Psi = \Phi = \arctan \sqrt{-\frac{(I_3 - 2I_1)}{(I_3 - 2I_2)}}$$

and

(90) $$\Psi + \Phi = 2 \arctan \sqrt{-\frac{(I_3 - 2I_1)}{(I_3 - 2I_2)}}.$$

The angle between OX and OX_1 is $\Phi + \Psi$ so that at the instant at which $\omega_3 = 0$ the diagram giving the motion is as shown in Fig. 3.

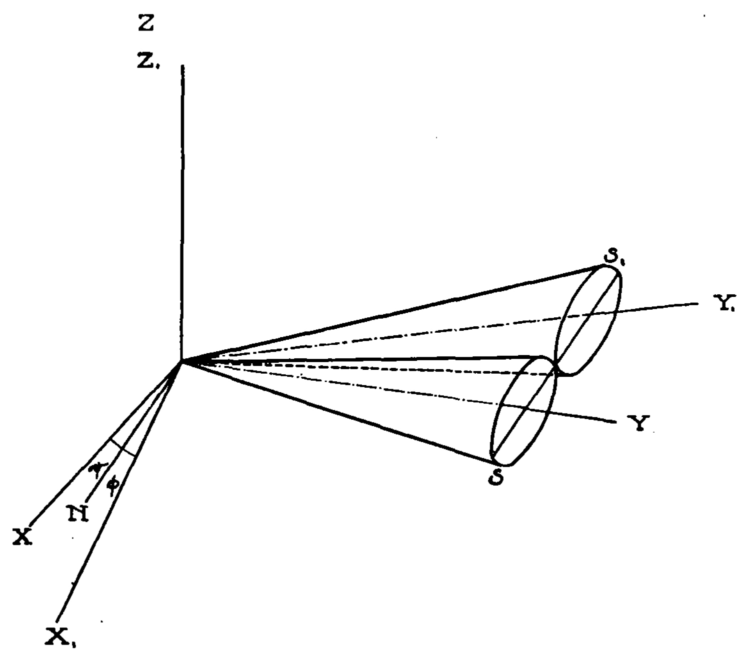

Fig. 3.

The motion consists of the rolling of the body cone, s_1, over the space cone s.

Second Case.

This case is a special case of the S. Kowalevski case. The method of procedure is identical with that just given for the First Case, so I have merely given the results.

For this case the restrictions are:

$$(1)\quad f = g = 0$$

$$(2)\quad I_1 = I_3 = 2 I_2$$

$$(3)\quad k = I_3 b \sqrt{\frac{n+1}{2}}$$

$$(4)\quad E = n I_3 b^2$$

where n must be greater than or equal to one, and b^2 is given by the equation

$$(5)\quad b^2 = \frac{wh}{I_3}\sqrt{\frac{2}{n+1}}.$$

The expressions for ω_1^2 and ω_2^2 as functions of ω_3 are

$$(6)\quad \omega_1^2 = -[\omega_3^2 - 2b\omega_3 - b^2]$$

$$(7)\quad \omega_2^2 = -2[\omega_3^2 + 2b\omega_3 - nb^2].$$

The time and the angle Ψ are given as functions of ω_3 by elliptic integrals.

Further study of equations (39) and (35) should lead to a few more particular solutions.

We remark once again that though the literature upon the Top Problem is extensive, it is entirely a literature of special cases. Klein and Sommerfeld in their huge work on the »Theorie des Kreisels», page 391, have suggested the possibility of interpolating between known cases, and upon the basis of continuity between the two movements. Hence if we can find enough special cases, we may yet hope to know something of the top problem.