ON FOURIER TRANSFORMS OF MEASURES WITH COMPACT SUPPORT

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Introduction

This paper will deal with the set $\mathfrak{M}$ of measures with compact support on the real line. To each positive number a we associate the set $\mathfrak{M}_a$ consisting of measures with support contained in $[-a, a]$. $\hat{\mathfrak{M}}$ and $\hat{\mathfrak{M}}_a$ will denote the sets of Fourier transforms $\hat{\mu}$ for μ belonging to $\mathfrak{M}$ and $\mathfrak{M}_a$ respectively. By reason of convenience the identically vanishing measure shall not be included in $\mathfrak{M}$ or $\mathfrak{M}_a$.

Our main objective is to decide if for each $a>0$ there exists $\mu \in \mathfrak{M}_a$ which tend to 0 in a prescribed sense as $x \to \pm\infty$. Since each $\hat{\mu}(x) \in \hat{\mathfrak{M}}$ is the restriction to the real axis of an entire function of exponential type $\leqslant a$, bounded for real x, we know by a classical theorem that

$$J(\log^-|\hat{\mu}|) = \int_{-\infty}^{\infty} \frac{\log^-|\hat{\mu}(x)|}{1+x^2}\,dx > -\infty. \tag{0.0}$$

This property is therefore a necessary condition.

Let $w(x) \geqslant 1$ be a measurable function on the real line and let L_w^p $(1 \leqslant p \leqslant \infty)$ be the space of measurable functions $f(x)$ with norm

$$\|f\| = \left\{\int_{-\infty}^{\infty} |f(x)|^p w(x)^p dx\right\}^{1/p}.$$

The following problem will be considered. Determine for a given p the set W_p of all weight functions $w(x) \geqslant 1$ subject to these two conditions:

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(a) The translation operators $f(x) \to f(x+t)$ are bounded in L^p_w.

(b) For each $a > 0$, L^p_w contains elements of $\hat{\mathfrak{M}}_a$.

On defining $\omega(x) = \log w(x)$ we find that each of our postulates leads trivially to a necessary condition on $\omega(x)$. Thus (a) implies that

$$\operatorname*{true\,max}_{-\infty<x<\infty} |\omega(x+t) - \omega(x)| < \infty, \tag{0.1}$$

and (b) implies

$$\int_{-\infty}^{\infty} \frac{\omega(x)}{1+x^2} dx < \infty. \tag{0.2}$$

We shall prove

THEOREM I. *The sets W_p are independent of p and W consists of all weight functions $w(x) = e^{\omega(x)} \geqslant 1$ satisfying* (0.1) *and* (0.2).

The main step in the proof of this result is not elementary and requires the development of new techniques, basically depending on a variational problem in a certain Hilbert space.

The same method will also yield:

THEOREM II. *Let $g \not\equiv 0$ be an entire function of exponential type such that $J(\log^+ |g|) < \infty$. Then each $\hat{\mathfrak{M}}_a$ contains element $\hat{\mu}$ with the property $\hat{\mu}(x) g(x) \in \hat{\mathfrak{M}}$.*

The preceding result can also be expressed in terms of the convolution algebra $\mathfrak{M}$: Let ν, $\mu \in \mathfrak{M}$ and assume that μ divides ν in the sense that the function $\hat{\nu}/\hat{\mu}$ is entire. Then for each $\varepsilon > 0$, there exists an $\alpha \in \mathfrak{M}_\varepsilon$ such that $\alpha \ast \nu$ is contained in the ideal generated by μ.

Another formulation of Theorem II deserves to be recognized, viz.: The sets

$$\{f(x) \,|\, f \text{ entire}, \ f = \frac{\hat{\nu}}{\hat{\mu}}, \ \hat{\nu}, \ \hat{\mu} \in \hat{\mathfrak{M}}\}$$

and

$$\{f(x) \,|\, f \text{ entire of exponential type}, \ J(|\log |f||) < \infty\}$$

are identical.

The property described above can be considered as a formal analogue of a theorem of Nevanlinna stating that a meromorphic function with bounded characteristic in the unit disc can be expressed as the quotient of two bounded analytic functions.

We should also like to point out that Theorem I combined with a result by Beurling ([1], Theorem IV, lecture 3) give rise to this striking conclusion: If translations are bounded operators in a space $L^p_w\,(w(x)\geqslant 1,\ 1\leqslant p\leqslant\infty)$ then one of the following two alternatives holds true. The space either contains elements $f\neq\phi$ with Fourier transforms $\hat{f}$ vanishing outside any given interval $[a, b]$, or the space does not contain any $f\neq\phi$ with a transform $\hat{f}$ vanishing on any interval.

1. Preliminaries on Harmonic Functions

In the following sections we shall frequently be concerned with functions $u(x+iy)$ harmonic in the upper half plane and with boundary values $u(x)$ on the real axis. It will always be assumed, although not always explicitly stated, that the relation between $u(z)$ and its boundary values $u(x)$ is such that

$$\lim_{y\downarrow 0}\int_{x_1}^{x_2}|u(x+iy)-u(x)|\,dx=0 \tag{1.1}$$

for finite intervals (x_1, x_2). If in addition

$$\int_{-\infty}^{\infty}\frac{|u(x)|}{1+x^2}\,dx<\infty, \tag{1.2}$$

then $u(x)$ has a well defined Poisson integral which we shall denote

$$P_z u=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{yu(\xi)\,d\xi}{y^2+(x-\xi)^2}.$$

If therefore $u(z)$ satisfies (2.1) and (2.2), then $u(z)-P_z u$ is harmonic in the upper half plane with boundary values vanishing almost everywhere on the real line. By an application of the symmetry principle it follows that

$$u(z)-P_z u=\mathfrak{Im}\left\{\sum_0^{\infty}c_n z^n\right\}\quad (y>0),$$

where c_n are real constants such that the series represent an entire function. The sets $\mathcal{D}_0$ and $\mathcal{D}_1$ are defined as follows: $u\in\mathcal{D}_0$ if $c_n=0,\ n>0$, and thus $u(z)=P_z u$; $u\in\mathcal{D}_1$ if $c_n=0\ (n>1)$, and consequently $u(z)=P_z u+c_1 y$.

Let ϱ be a positive measure on $[0,\infty)$ such that the integral

$$U^{\varrho}(z)\equiv\int_0^{\infty}\log\left|1-\frac{z^2}{t^2}\right|d\varrho(t)$$

converges for $y>0$. If $U^{\varrho}(z)$ is bounded from above for real z and if

$$\int_0^t d\varrho = O(t),$$

then the boundary values

$$U^\varrho(x) = \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\varrho(t)$$

are finite almost everywhere and satisfy (1.1) and (1.2). By a Tauberian theorem of Paley–Wiener it follows that the limit

$$a = \pi \lim_{T\to\infty} \frac{1}{T}\int_0^T d\varrho$$

exists and is finite. Moreover $U^\varrho(z) \in \mathcal{D}_1$ and the constant c_1 equals a.

2. Atomizing of Positive Measures

This section will contain an elementary but important step in establishing the existence of functions $\hat{\mu} \in \mathfrak{M}_a$ with prescribed properties.

We shall denote by Ω the collection of all measurable functions $\omega(x) \geq 0$ satisfying (0.2) and in addition meeting this condition: For each $a > 0$ there exists on $[0, \infty($ a continuous positive measure ϱ such that

$$U^\varrho(x) \leq -\omega(x) + \text{const.} \quad \text{for a.a. real } x, \tag{2.1}$$

$$\overline{\lim_{T\to\infty}} \frac{\pi}{T}\int_0^T d\varrho \leq a. \tag{2.2}$$

It should be observed that (0.1) is not included as a condition for Ω. We recognize that Ω is a convex cone: If $\omega_1, \omega_2 \in \Omega$ then the same is true of $\lambda_1\omega_1 + \lambda_2\omega_2$ for $\lambda_1, \lambda_2 \geq 0$. Moreover, if $\omega(x)$ belongs to Ω so does $\omega(-x)$ as well as $\omega(x) + \omega(-x)$. Each non-negative measurable minorant of an $\omega \in \Omega$ will also belong to Ω. The set Ω is therefore uniquely determined by the even functions it contains.

LEMMA I. *Assume $\omega \in \Omega$ and let γ be a given positive number < 1. Then for each $a > 0$ there exists a $\hat{\mu} \in \hat{\mathfrak{M}}_a$ such that*

$$\int_{-\infty}^\infty |\hat{\mu}(x)| \exp(\omega(x) + 2|x|^\gamma)\, dx < \infty. \tag{2.3}$$

Proof. We recall the formula

$$\int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| dt^\gamma = |x|^\gamma \pi \operatorname{cotg}\frac{\pi\gamma}{2} \quad (0 < \gamma < 2).$$

Thus, if $s(t)=at-2t^{\gamma}\pi^{-1}\operatorname{tg}\frac{1}{2}\pi\gamma$ $(a>0,\ 0<\gamma<1)$, then

$$\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right| ds(t) = -2|x|^{\gamma}.$$

The function $s(t)$ is obviously increasing for $t \geqslant t_0$, where t_0 depends on a and γ. If therefore τ is the measure obtained by restricting s to (t_0, ∞) we shall have

$$U^{\tau}(x) \leqslant -|x|^{\gamma} + \text{const.} \tag{2.4}$$

Hence, $|x|^{\gamma} \in \Omega$ for $0<\gamma<1$. Let $a>0$ be given and let ϱ be a measure satisfying the stipulated conditions with respect to a and to $\omega_1(x)=2\omega(x)+5|x|^{\gamma}$. We construct an atomized measure ϱ^* by the procedure:

$$\varrho^*(t)=\int_0^t d\varrho^* = \left[\varrho(t)+\frac{1}{2}\right], \quad \varrho(t)=\int_0^t d\varrho, \tag{2.5}$$

where $[x]$ denotes the integral part of x.

Since ϱ is positive and continuous, ϱ^* is uniquely determined. Define for $z=x+iy$ $(y>0)$,

$$h(z)=\exp\left\{\int_0^\infty \log\left(1-\frac{z^2}{t^2}\right) d\varrho(t)\right\}, \tag{2.6}$$

$$f(z)=\exp\left\{\int_0^\infty \log\left(1-\frac{z^2}{t^2}\right) d\varrho^*(t)\right\}, \tag{2.7}$$

where the logarithm is real for $z=iy$ $(y>0)$. We observe that $f(z)$ is an entire function,

$$f(z)=\prod_1^\infty\left(1-\frac{z^2}{\lambda_n^2}\right), \quad \varrho(\lambda_n)+\frac{1}{2}=n.$$

Our conditions on ϱ and on ω imply that

$$\log|h(z)| \leqslant -P_z\omega_1 + by + \text{const.} \quad (y>0), \tag{2.8}$$

where b is a constant $\leqslant a$. The function

$$\log\frac{f(z)}{h(z)} = u(z)+iv(z)$$

is holomorphic in the upper half plane and its imaginary part v is bounded there and vanishes for $z=iy$ $(y>0)$. For $x>0$ the boundary value of v is

$$v(x) = \pi(\varrho^*(x) - \varrho(x)) = \pi\left(\left[\varrho(x) + \frac{1}{2}\right] - \varrho(x)\right).$$

Since $v(-x+iy) = -v(x+iy)$ we shall have $-\frac{1}{2}\pi \leqslant v(x) \leqslant \frac{1}{2}\pi$ on the real axis and those inequalities will hold throughout the upper half plane by virtue of the maximum–minimum principle. Assume $0 < k < 1$ and set

$$\left(\frac{f}{h}\right)^k = e^{ku}\cos kv + ie^{ku}\sin kv \equiv U_k + iV_k.$$

Then U_k is a positive harmonic function and

$$\cos k\frac{\pi}{2}\left|\frac{f}{h}\right|^k \leqslant U_k. \tag{2.9}$$

By an inequality of Harnack

$$U_k(z) \leqslant U_k(i)\,\frac{|z+i|+|z-i|}{|z+i|-|z-i|} \quad (y>0). \tag{2.10}$$

In the half plane $y \geqslant 1$, the factor in (2.10) is majorized by $(1+|x|)^2$. On combining (2.8), (2.9) and (2.10) taking $k=\frac{1}{2}$, we obtain for $y \geqslant 1$,

$$\log|f(z)| \leqslant -P_z\omega_1 + by + 4\log(1+|x|) + \text{const.} \tag{2.11}$$

Since the same inequality holds for $z = x - iy$ it follows that $f(z)$ is of exponential type $\leqslant a$. By virtue of the definition of ω_1 we conclude that

$$|f(x+iy)| \leqslant Me^{-4|x|^\gamma} \quad (-1 \leqslant y \leqslant 1), \tag{2.12}$$

where M is a finite constant. This proves that $f \in \hat{\mathfrak{M}}_a$.

Since $U_k(z)$ is positive for $y>0$, and $U_k \in \mathfrak{D}_0$,

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{U_k(x)}{1+x^2}\,dx = U_k(i). \tag{2.13}$$

Hence, by (2.9), taking $k=\frac{1}{2}$,

$$\int_{-\infty}^{\infty}\left|\frac{f(x)}{h(x)}\right|^{\frac{1}{2}}\frac{dx}{1+x^2} < \infty. \tag{2.14}$$

By the definition of ϱ,

$$\log|h(x)| = U^\varrho(x) \leqslant -2\,\omega(x) - 5|x|^\gamma + \text{const.}$$

Therefore (2.14) implies that

$$\int_{-\infty}^{\infty} |f(x)|^{\frac{1}{2}} \exp(\omega(x) + 2|x|^{\gamma})\, dx < \infty,$$

and (2.3) follows since $\hat{\mu}(x) = f(x)$ is bounded.

We shall now derive a stronger result under the assumption that $\omega(x)$ has a certain weak continuity property.

Lemma II. *Suppose $\omega(x)$ is continuous and let there exist positive numbers α and $\beta < 1$ such that for all x outside some compact set and for $|h| \leqslant \exp(-|x|^{\beta})$,*

$$|\omega(x+h) - \omega(x)| \leqslant |x|^{\alpha}. \tag{2.15}$$

Then the summability (2.3) *for $\gamma > \max(\alpha, \beta)$ implies that*

$$|f(x)| \exp(\omega(x) + |x|^{\gamma}) \leqslant \text{const.} \tag{2.16}$$

Proof. The lemma is a simple consequence of the following minimum modulus theorem. There exists an absolute constant $\vartheta > 0$ such that if $g(z)$ is holomorphic for $|z| < R$ and $|g(z)| \leqslant M$, then

$$\min_{|z|=r} |g(z)| \geqslant \frac{|g(0)|}{M}$$

for a set of values r of measure $\geqslant \vartheta R$. If therefore (2.16) were false there would exist arbitrary large x_0 such that

$$|f(x_0)| \geqslant \exp(-\omega(x_0) - |x_0|^{\gamma}).$$

Since f is bounded by a constant M in the strip $-1 \leqslant y \leqslant 1$ we would have

$$|f(x)| \geqslant M^{-1} \exp(-\omega(x_0) - |x_0|^{\gamma})$$

on a set E contained in the interval $|\xi - x_0| \leqslant \exp(-|x_0|^{\beta})$ and of measure $\geqslant 2\vartheta \exp(-|x_0|^{\beta})$. This inequality together with (2.15) contradicts the summability expressed in (2.3) and the lemma is therefore true.

3. A Variational Problem in a Hilbert Space

The main objective of this section is to connect the set of functions Ω with a certain variational problem in a suitably chosen real Hilbert space. By definition $\mathcal{H}$ shall consist of all odd real valued measurable function on $(-\infty, \infty)$ satisfying the condition

$$\int_{-\infty}^{\infty} \frac{|u(x)|}{1+x^2}\,dx < \infty \tag{3.1}$$

and such that the harmonic function $u(z) = P_z u$ has a finite Dirichlet integral

$$\|u\|^2 = \int_0^{\infty}\int_0^{\infty} |\operatorname{grad} u|^2\,dx\,dy. \tag{3.2}$$

The norm in $\mathcal{H}$ shall be defined by (3.2). Because $u(iy) = 0$ $(y > 0)$, it follows by well established properties of the Dirichlet norm that $\mathcal{H}$ is complete.

Frequent use will be made of the inequality

$$\int_0^{\infty} u^2(x)\frac{dx}{x} \leqslant \frac{\pi}{2}\|u\|^2. \tag{3.3}$$

In order to prove (3.3) define $m(r) = \sup_{0<\theta<\frac{1}{2}\pi} |u(re^{i\theta})|$. Then

$$m^2(r) \leqslant \left(\int_0^{\frac{1}{2}\pi}\left|\frac{\partial u}{\partial \theta}\right| d\theta\right)^2 \leqslant \frac{\pi}{2}\int_0^{\frac{1}{2}\pi}\left(\frac{\partial u}{\partial\theta}\right)^2 d\theta.$$

Consequently $$\int_0^{\infty} m^2(r)\frac{dr}{r} \leqslant \frac{\pi}{2}\int_0^{\infty} r\,dr \int_0^{\frac{1}{2}\pi}\frac{1}{r^2}\left(\frac{\partial u}{\partial\theta}\right)^2 d\theta \leqslant \frac{\pi}{2}\|u\|^2,$$

and (3.3) follows. The norm in $\mathcal{H}$ can of course be expressed directly in terms of $u(x)$. One such expression is furnished by the Douglas functional

$$\|u\|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{u(x)-u(y)}{x-y}\right)^2 dx\,dy. \tag{3.4}$$

We shall later on define an equivalent norm in $\mathcal{H}$ more convenient than (3.4) for our specific purposes. It should be pointed out that $\mathcal{H}$ is a Dirichlet space in the sense of Beurling and Deny [2, 3]. We shall use the technique of these spaces without referring to the general theory.

For each $u(x) \in \mathcal{H}$ the harmonic function $u(z) = P_z u$ has a conjugate harmonic function $\tilde{u}(z)$ uniquely determined except for an additive constant. Since $u(z)$ and

$\tilde{u}(z)$ have the same Dirichlet integral we conclude that $\tilde{u}(z)$ has boundary values $\tilde{u}(x)$ which are at least locally L^2-summable. If $u, v \in \mathcal{H}$ the scalar product is formally expressed by the integrals

$$(u, v) = \int_0^\infty u(x)\, d\tilde{v}(x) = \int_0^\infty v(x)\, d\tilde{u}(x).$$

If, however, v belongs to the set $C \subset \mathcal{H}$ consisting of all odd real-valued differentiable function with compact support then we shall have

$$(u, v) = -\int_0^\infty \tilde{u}(x)\, dv(x), \tag{3.5}$$

where the integral is well defined. The proof of (3.5) is elementary.

The main result of this paper is contained in

Lemma III. *Let $\omega(x)$ be a non-negative measurable function such that for almost all $x > 0$*

$$\omega(x) \leqslant x\sigma(x) + \text{const.}, \tag{3.6}$$

where $\sigma \in \mathcal{H}$ and

$$\int_0^\infty \frac{\sigma(x)}{x}\, dx < \infty. \tag{3.7}$$

Then $\omega \in \Omega$.

Proof. In order to exhibit the existence of measures ϱ with the prescribed properties we assume $a > 0$ given and we choose b $(0 < b < a)$. Define

$$K_\sigma = \{u \mid u \in \mathcal{H},\ u(x) \geqslant \sigma(x), \text{ a.e. for } x > 0\}.$$

This set is convex and it is closed by virtue of (3.3). Define further

$$\Phi(u) = \|u\|^2 + 2b \int_0^\infty \frac{u(x)}{x}\, dx, \quad m = \inf_{u \in K_\sigma} \Phi(u). \tag{3.8}$$

Since $\sigma \in K_\sigma$, m is finite. Assume $u_1, u_2 \in K_\sigma$, $\Phi(u_1), \Phi(u_2) < m + \varepsilon$. Then $\Phi(\frac{1}{2}(u_1 + u_2)) \geqslant m$ and consequently

$$\tfrac{1}{2}\Phi(u_1) + \tfrac{1}{2}\Phi(u_2) - \Phi(\tfrac{1}{2}(u_1 + u_2)) < \varepsilon.$$

This inequality can also be written in the form

$$\|\tfrac{1}{2}(u_1 - u_2)\|^2 < \varepsilon.$$

If therefore $u_n \in K_\sigma$, $\Phi(u_n) \to m$, then $\{u_n\}_1^\infty$ is a Cauchy sequence and converges to an element $u \in K_\sigma$. By (3.3) we shall have for $0 < x_1 < x_2 < \infty$,

$$\lim_{n=\infty} \int_{x_1}^{x_2} |u(x) - u_n)x)| \frac{dx}{x} = 0.$$

Hence
$$\|u\|^2 + 2b \int_{x_1}^{x_2} u(x) \frac{dx}{x} \leqslant m,$$

and it follows that $\Phi(u) = m$ since $u(x) \geqslant 0$ a.e. for $x > 0$. Let now $v \in C$ and assume $v(x) \geqslant 0$ for $x \geqslant 0$. Then $u + \lambda v \in K_\sigma$ for $\lambda > 0$ and $\Phi(u + \lambda v) - \Phi(u) \geqslant 0$. This implies that

$$(u, v) + b \int_0^\infty \frac{v(x)}{x} dx \geqslant 0. \tag{3.9}$$

The left-hand side of this relation is therefore a linear form $F(v)$ defined for $v \in C$ and $F(v) \geqslant 0$ if $v > 0$ for $x > 0$. By a familiar argument we conclude that

$$F(v) = \int_0^\infty v(x)\, d\alpha(x), \tag{3.10}$$

where α is a non-negative measure on $(0, \infty)$.

We now introduce a normalized conjugate function $\tilde{u}(z)$ by the formula

$$u(z) + i\tilde{u}(z) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{u(t)}{t - z} dt = \frac{2}{\pi i} \int_0^\infty \frac{u(t)\, t}{t^2 - z^2} dt.$$

The integral is well defined because

$$\int_{-\infty}^{\infty} \frac{|u(t)|}{|t|} dt < \infty.$$

On combining (3.5) and (3.9) we obtain for $v \in C$

$$-\int_0^\infty \tilde{u}(x)\, dv(x) = -b \int_0^\infty \frac{v(x)}{x} dx + \int_0^\infty v(x)\, d\alpha(x). \tag{3.11}$$

This relation implies that $\tilde{u}(x)$ a.e. coincide with a function locally of bounded variation on $(0, \infty)$.

The precise pointwise limit

$$\tilde{u}(x) = \lim_{y \downarrow 0} \tilde{u}(x + iy)$$

is therefore of bounded variation on finite intervals $[x_1, x_2]$, $x_1 > 0$. This implies that the limits $\tilde{u}(x-0)$ and $\tilde{u}(x+0)$ exist. By another version of (3.3),

$$\int_0^\infty (\tilde{u}(x+t)-\tilde{u}(x-t))^2 \frac{dt}{t} \leqslant \pi \| u \|^2.$$

It follows that $\tilde{u}(x+0)=\tilde{u}(x-0)$, and $\tilde{u}(x)$ is thus continuous on $(0, \infty)$. In addition it follows by (3.11) that

$$\tilde{u}(x_2)-\tilde{u}(x_1) \geqslant -b \log \frac{x_2}{x_1} \quad (x_2 > x_1 > 0). \tag{3.12}$$

We shall next prove

$$\left.\begin{aligned} \lim_{x\uparrow\infty} (\tilde{u}(x)-\tilde{u}(ix)) &= 0,\\ \lim_{x\downarrow 0} (\tilde{u}(x)-\tilde{u}(ix)) &= 0.\end{aligned}\right\} \tag{3.13}$$

To this purpose we consider

$$J(r, \lambda)=\int_r^{\lambda r} |\tilde{u}(x)-\tilde{u}(ix)| \frac{dx}{x} \quad (r>0, \lambda>1),$$

and we observe that

$$\lim_{x\uparrow\infty} \tilde{u}(ix)=0, \quad \lim_{x\downarrow 0} \tilde{u}(ix)=-\frac{2}{\pi}\int_0^\infty \frac{u(t)}{t}\, dt.$$

By an application of Schwarz inequality and by the proof of (3.3),

$$J(r, \lambda) \leqslant \left(\frac{\pi}{2} \log \lambda \cdot D(r, \lambda)\right)^{\frac{1}{2}},$$

where $D(r, \lambda)$ denotes the Dirichlet integral of $\tilde{u}$ extended over the region

$$\{z \,|\, r < |z| < \lambda r, \ 0 < \arg z < \tfrac{1}{2}\pi\}.$$

Hence, for bounded λ, $J(r, \lambda)$ tends to 0 as $r\uparrow\infty$ or $r\downarrow 0$. If (3.13) were not true there would exist a positive η and arbitrary large (or small) $x>0$ such that

$$|\tilde{u}(x)-\tilde{u}(ix)| > 2\eta.$$

By virtue of (3.12) we conclude that for some fixed $\lambda > 1$ only depending on b and η we would have

$$|\tilde{u}(x)-\tilde{u}(ix)| > \eta \quad (x \in (r, \lambda r))$$

for some values of r arbitrary large (or small). This contradicts our result on $J(r, \lambda)$

and (3.13) is therefore established. Hence, $\tilde{u}(x)$ is a bounded continuous function tending to 0 at ∞, and to a finite limit at $x=0$.

We now turn to the construction of the measures ϱ. Since $u(z)$, $\tilde{u}(z) \in \mathcal{D}_0$ we shall have

$$u(z) + i\tilde{u}(z) = \lim_{T\to\infty} \frac{1}{\pi} \int_{-T}^{T} \frac{\tilde{u}(t)}{t-z}\, dt = \frac{2z}{\pi} \int_0^\infty \frac{\tilde{u}(t)}{t^2 - z^2}\, dt.$$

This function $u(z)$ coincides with the original $u(z) = P_z u$ because both vanish on the positive imaginary axis and both have the same conjugate function. By adding the constant a to $\tilde{u}(t)$, we obtain

$$u(z) + i\tilde{u}(z) + ia = \frac{2z}{\pi} \int_0^\infty \frac{\tilde{u}(t) + a}{t^2 - z^2}\, dt.$$

Consequently
$$z(u(z) + i\tilde{u}(z) + ia) = \frac{1}{\pi} \int_0^\infty t(\tilde{u}(t) + a) \frac{2z^2}{(t^2 - z^2)\, t}\, dt, \tag{3.14}$$

where the last factor in the integral is the derivative of $\log(1 - z^2/t^2)$ with respect to t. Since $a > b$ there exists a finite t_0 such that for $t \geq t_0$, $\tilde{u}(t) > b - a$. We also recall that the lower derivative of $\tilde{u}(t)$ is $\geq -b/t$ at each point $t > 0$. These properties imply that $s(t) = t(\tilde{u}(t) + a)$ is increasing for $t \geq t_0$ and of bounded variation on $[0, t_0)$. We obtain by first making a partial integration in (3.14) and then by letting $y \downarrow 0$,

$$-xu(x) = \frac{1}{\pi} \int_0^\infty \log \left| 1 - \frac{x^2}{t^2} \right| ds(t).$$

A continuous positive measure ϱ_1 is now readily obtained by defining

$$\pi\, d\varrho_1 = ds \qquad \text{for} \quad t \geq t_0,$$

$$\pi\, d\varrho_1 = \mathrm{a}\, dt + t\, d\tilde{u} \quad \text{for} \quad 0 < t < t_0.$$

By construction of $u(x)$,

$$\omega(x) \leq x\sigma(x) + \text{const} \leq xu(x) + \text{const.}, \quad \text{a.e. for } x > 0.$$

Therefore

$$U^{\varrho_1}(x) \leq -\omega(x) - \frac{1}{\pi} \int_0^{t_0} \tilde{u}(t) \log \left| 1 - \frac{x^2}{t^2} \right| dt + \text{const.}$$

Since $\tilde{u}(t)$ is bounded we conclude that for a.a. $x > 0$,

$$U^{\varrho_1}(x) \leq -\omega(x) + c_T \log(1 + x^2) + \text{const.}$$

In order to obtain a ϱ strictly satisfying all the conditions, we have only to form $\varrho = \varrho_1 + \tau$, where τ is one of the previously constructed measures satisfying (2.4) for $\gamma = \frac{1}{2}$ and (2.2) with the constant $a - b$.

This concludes the proof of Lemma III.

4. An Equivalent Norm in $\mathcal{H}$

In order to obtain simple and explicit conditions implying that functions $u(x)$ belong to $\mathcal{H}$ we shall introduce an equivalent norm in $\mathcal{H}$.

LEMMA IV. *For odd measurable functions $u(x)$ on $(-\infty, \infty)$ let*

$$\|u\|_0^2 = \int_{-\infty}^{\infty} u^2(e^{\xi})\,d\xi + \int_0^{\infty} \frac{d\eta}{\eta^2} \int_{-\infty}^{\infty} (u(e^{\xi+\eta}) - u(e^{\xi}))^2\,d\xi. \tag{4.1}$$

Then $\|u\|$ and $\|u\|_0$ are equivalent norms in $\mathcal{H}$, i.e. $\|u\|/\|u\|_0$ remains included between positive finite constants.

Proof. Any of the assumptions $\|u\| < \infty$ or $\|u\|_0 < \infty$ imply that

$$u(e^{\xi}) \in L^2(-\infty, \infty).$$

We may therefore assume that $\psi(\xi) = u(e^{\xi})$ has a Fourier transform $\hat{\psi}(t) \in L^2(-\infty, \infty)$. By an application of Parseval relation

$$\int_{-\infty}^{\infty} (\psi(\xi+\eta) - \psi(\xi))^2\,d\xi = 4\int_{-\infty}^{\infty} \sin^2\frac{t\eta}{2}\,|\hat{\psi}(t)|^2\,dt.$$

Consequently
$$\|u\|_0^2 = \int_{-\infty}^{\infty} |\hat{\psi}(t)|^2 \lambda_0(t)\,dt$$

where
$$\lambda_0(t) = 1 + 4|t| \int_0^{\infty} \sin^2\frac{s}{2}\,\frac{ds}{s^2} = 1 + \pi|t|.$$

On the other hand the function $\psi(\xi + i\eta) = u(e^{\xi + i\eta})$ is harmonic in the strip $0 < \eta < \frac{1}{2}\pi$ and vanishes for $\eta = \frac{1}{2}\pi$. Since the Dirichlet integral is invariant under conformal mapping

$$\|u\|^2 \equiv \|\psi\|^2 = \int_0^{\frac{1}{2}\pi} d\eta \int_{-\infty}^{\infty} |\operatorname{grad}\psi|^2\,d\xi.$$

The kernel
$$K(t, \xi, \eta) = e^{it\xi} \frac{\operatorname{sh}\left(\frac{\pi}{2} - \eta\right) t}{\operatorname{sh}\frac{\pi}{2} t}$$

is harmonic in (ξ, η) and $K(t, \xi, 0) = e^{it\xi}$, $K(t, \xi, \frac{1}{2}\pi) \equiv 0$. By this we conclude that

$$\psi(\xi + i\eta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K(t, \xi, \eta)\, \hat{\psi}(t)\, dt \quad (0 < \eta \leqslant \tfrac{1}{2}\pi).$$

By a straightforward computation using the Parseval relation,

$$\|u\|^2 = \int_{-\infty}^{\infty} |\psi(t)|^2 \lambda(t)\, dt$$

with
$$\lambda(t) = \frac{1}{2} t \frac{\operatorname{sh} \pi t}{\operatorname{sh}^2 \frac{1}{2}\pi t} = t \cdot \frac{e^{\frac{1}{2}\pi t} + e^{-\frac{1}{2}\pi t}}{e^{\frac{1}{2}\pi t} - e^{-\frac{1}{2}\pi t}}.$$

The ratio λ_0/λ is obviously bounded from below and from above by positive constants, and the lemma follows.

Lemma V. *Let $\omega(x)$ be an even non-negative function uniformly* Lip 1 *on the real axis and such that*

$$A = \int_0^{\infty} \frac{\omega(x)}{x}\, dx < \infty.$$

Then $\sigma(x) = \omega(x)/x \in \mathcal{H}$, and by Lemma III, $\omega \in \Omega$.

Proof. Without loss of generality we may assume that ω is differentiable for $x \neq 0$ and that its derivative ω' is bounded by a constant M. We define on $(-\infty, \infty)$,

$$\psi(\xi) = \sigma(e^{\xi}) = \omega(e^{\xi})\, e^{-\xi},$$

and observe that
$$A = \int_{-\infty}^{\infty} \psi(\xi)\, d\xi, \tag{4.2}$$

$$\psi'(\xi) + \psi(\xi) = \omega'(e^{\xi}). \tag{4.3}$$

If (4.3) is multiplied by ψ and then integrated over $(-\infty, \xi)$,

$$\frac{1}{2}\psi^2(\xi) + \int_{-\infty}^{\xi} \psi^2(\xi)\, d\xi \leqslant MA. \tag{4.4}$$

Thus
$$\int_{-\infty}^{\infty} \psi^2(\xi)\,d\xi \leqslant MA, \tag{4.5}$$

$$|\psi(\xi)| \leqslant \sqrt{2\,MA}, \tag{4.6}$$

$$|\psi'(\xi)| \leqslant M + \sqrt{2\,MA} = M_1. \tag{4.7}$$

By virtue of the definition of the equivalent norm the lemma is proved if we can show that (4.7) implies

$$\int_0^{\infty} \delta^2(\eta, \psi)\frac{d\eta}{\eta^2} \leqslant 4\,M_1 \int_{-\infty}^{\infty} \psi(\xi)\,d\xi, \tag{4.8}$$

where
$$\delta^2(\eta, \psi) = \int_{-\infty}^{\infty} (\psi(\xi+\eta) - \psi(\xi))^2\,d\xi = \int_{A_\eta} + \int_{A'_\eta}.$$

By A_η we denote the set where at least one of the functions $\psi(\xi)$, $\psi(\xi+\eta)$ is $>\eta$, and we define $E_\eta = \{\xi \,|\, \psi(\xi) > \eta\}$. Let $m(\eta)$ be the measure of E_η and observe that

$$\int_{-\infty}^{\infty} \psi^2(\xi)\,d\xi = -\int_0^{\infty} \eta^2\,dm(\eta), \qquad \int_{-\infty}^{\infty} \psi(\xi)\,d\xi = \int_0^{\infty} m(\eta)\,d\eta.$$

By reason of homogeneity it is sufficient to establish (4.8) in the particular case that $M_1 = 1$. Since the measure of A_η is less than $2\,m(\eta)$ we shall have

$$\int_{A_\eta} \leqslant 2\,\eta^2\,m(\eta),$$

$$\int_{A'_\eta} \leqslant 2\int_{E'_\eta} \psi^2(\xi)\,d\xi = -2\int_0^{\eta} t^2\,dm(t).$$

Consequently

$$\int_0^{\infty} \delta^2(\eta, \psi)\frac{d\eta}{\eta^2} \leqslant 2\int_0^{\infty} m(\eta)\,d\eta - 2\int_0^{\infty}\frac{d\eta}{\eta^2}\int_0^{\eta} t^2\,dm(t) = 4\int_{-\infty}^{\infty}\psi(\xi)\,d\xi.$$

This proves (4.8) and the lemma follows.

5. Proofs of Theorems I and II

The necessary condition (0.1) states that

$$\alpha(t) = \operatorname*{true\ max}_{-\infty < x < \infty} |\omega(x+t) - \omega(x)|$$

is finite for all t. If therefore M is sufficiently large the set $E = \{t \mid \alpha(t) \mid \leqslant M)$ has positive measure. By a well known argument the set

$$E_1 = \{t \mid t = t_1 - t_2,\ t_1,\ t_2 \in E\}$$

contains an interval. Since $\alpha(t)$ is subadditive and even we shall have $\alpha(t) \leqslant 2M$ on some interval $[a, b]$. Consequently $\alpha(t) \leqslant 4M$ for $|t| \leqslant b-a$. Again by subadditivity it follows that $\alpha(t) \leqslant M_0$ for $|t| \leqslant 1$, M_0 being a finite constant. Define

$$\omega_1(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \omega(x+t)\,dt.$$

Then $|\omega_1'(x)| \leqslant M_0$ and we shall have

$$|\omega_1(x_1) - \omega_1(x_2)| \leqslant M_0 |x_1 - x_2|, \tag{5.1}$$

$$|\omega(x) - \omega_1(x)| = \left| \int_{-\frac{1}{2}}^{\frac{1}{2}} (\omega(x) - \omega(x+t))\,dt \right| \leqslant M_0. \tag{5.2}$$

The last inequality implies that the weight functions $w(x) = e^{\omega(x)}$ and $w_1(x) = e^{\omega_1(x)}$ are equivalent. Without loss of generality we may also assume that ω_1 vanishes on $(-1, 1)$. The summability (0.2) and the Lipschitz condition (5.1) imply that Lemma V applies to

$$\sigma(x) = \frac{\omega_1(x) + \omega_1(-x)}{x}.$$

Thus, $\sigma \in \mathcal{H}$. By Lemma III, $\omega_1 \in \Omega$. Lemmas I and II ascertain the existence of functions $\hat{\mu}$ with the stipulated properties, and Theorem I follows.

The proof of Theorem II is also based on Lemma III, while Lemmas II and V are dispensable. If g is entire of exponential type, then the elementary theory of Fourier integrals implies that $\hat{\mu} g \in \hat{\mathfrak{M}}$, if $\hat{\mu} g$ is summable on the real line.

We also observe that it suffices to prove Theorem II for functions of the form

$$g(z) = \prod_1^\infty \left(1 - \frac{z^2}{\lambda_n^2}\right). \tag{5.3}$$

because each g has a majorant of this kind on the real axis, viz.

$$1 + z^2 (g(z)\overline{g(\bar{z})} + g(-z)\overline{g(-\bar{z})}).$$

As a substitute for Lemma V we shall use

LEMMA VI. *Let* (5.3) *be entire of exponential type and such that for real* x, $|g(x)| \geqslant 1$. *If* $J(\log|g|) < \infty$, *then*

$$u(x) = \frac{\log|g(x)|}{x} \in \mathcal{H}. \tag{5.4}$$

Proof. It is well known that our conditions imply

$$\sum_1^\infty \left| \Im\mathrm{m}\left(\frac{1}{\lambda_n}\right) \right| < \infty, \tag{5.5}$$

$$\pi \lim_{r\to\infty} \frac{N(r)}{r} = \lim_{|z|\to\infty} \sup \frac{\log|g(z)|}{|z|} = A, \tag{5.6}$$

where $N(r) = \sum_{|\lambda_n|<r} 1$ and where $\Im\mathrm{m}$ is the imaginary part. Assume, as we may, $\lambda_n = |\lambda_n| e^{i\theta_n}$ $(0 \leqslant \theta_n < \pi)$, and define

$$f(z) = \prod_1^\infty \left(1 + \frac{z}{\lambda_n}\right)\left(1 - \frac{z}{\bar{\lambda}_n}\right).$$

By (5.5) this product converges and represents an entire function $f(z)$ of the same exponential type A as $g(z)$. For real x, $|f(x)| = |g(x)|$. Since $f(z)$ is free from zeros in the upper half plane we shall have there

$$\log f(z) = \log|f(z)| + i\vartheta(z),$$

where $\vartheta(iy) = 0$ $(y > 0)$. At each real point x, $|f(x+iy)|$ increases with y and $\vartheta(x)$ is therefore a monotonic decreasing function. In particular, $\vartheta(x)$ has a jump $-\pi$ at each real zero of f. An elementary consequence of (5.5) and (5.6) is that

$$\lim_{x=\infty} \frac{\vartheta(x)}{x} = -\pi \lim_{r\to\infty} \frac{N(r)}{r} = -A.$$

There exists therefore a finite constant M such that

$$\frac{\vartheta(x)}{x} \geqslant -M \quad (x > 0). \tag{5.7}$$

We now define u and $\tilde{u}$ in the upper half plane by the relation

$$u(z) + i\tilde{u}(z) = \frac{\log f(z) + iAz}{z},$$

and observe that on the real axis,

$$u(x) = \frac{\log|f(x)|}{x}, \quad \tilde{u}(x) = \frac{\vartheta(x)}{x} + A.$$

Because of (5.7) and the fact that $\vartheta(x)$ is decreasing we shall have for $x > 0$, $\lambda > 1$,

$$\tilde{u}(\lambda x) - \tilde{u}(x) \leqslant M \log \lambda. \tag{5.8}$$

We recall that both $u(z)$ and $\tilde{u}(z)$ belong to $\mathcal{D}_0$, and that u is an odd and $\tilde{u}$ an even function of x. Our objective is to show that the Dirichlet integral of $u(z)$ is finite. By assumption on g, $u(z)$ is positive in the first quadrant, and

$$x^{-1} u(x) \in L^1(0, \infty).$$

Therefore
$$\int_0^\infty u(re^{i\theta}) \frac{dr}{r} = \left(1 - \frac{2\theta}{\pi}\right) \int_0^\infty u(r) \frac{dr}{r} \quad (0 \leqslant \theta \leqslant \tfrac{1}{2}\pi). \tag{5.9}$$

In particular
$$\int_{r_0}^{2r_0} u(re^{i\delta}) \frac{dr}{r} \to 0 \quad (r_0 \to \infty),$$

and we conclude by Harnack's inequality that $u(re^{i\delta}) \to 0$ as $r \to \infty$, δ being fixed. This implies that we have uniformly

$$u(z) = o(1) \quad (\delta < \theta \leqslant \tfrac{1}{2}\pi). \tag{5.10}$$

As a consequence of (5.10),

$$|\operatorname{grad} u| = |\operatorname{grad} \tilde{u}| = o\left(\frac{1}{r}\right) \quad (\delta < \theta \leqslant \tfrac{1}{2}\pi). \tag{5.11}$$

We now turn our attention to $\tilde{u}$. By virtue of (5.8) the function $\tilde{u}(\lambda z) - \tilde{u}(z)$ is bounded by $M \log \lambda$ on the real axis. The same bound therefore holds throughout the upper half plane. Consequently

$$\frac{\partial \tilde{u}(re^{i\theta})}{\partial r} \leqslant \frac{M}{r} \quad (r > 0, \ 0 < \theta < \pi). \tag{5.12}$$

The classical formula
$$\int_S |\operatorname{grad} u|^2 \, dx\, dy = \int_{\partial S} u \frac{d\tilde{u}}{ds} ds$$

is now valid for each sector $S = \{z = re^{i\theta},\ 0 < r < r_0,\ \delta < \theta < \tfrac{1}{2}\pi\}$. According to (5.10), (5.11), the integral extended over the circular arc tends to 0 as $r_0 \to \infty$. The Dirichlet integral for the angle $\delta < \theta < \tfrac{1}{2}\pi$ is therefore properly expressed by the integral

$$\int_0^\infty u(r e^{i\delta}) \frac{\partial}{\partial r} \tilde{u}(r e^{i\delta})\, dr$$

and consequently majorized by

$$M \int_0^\infty u(r) \frac{dr}{r}.$$

This proves the lemma.

6. Concluding Remarks

It should be observed that the lemmas admit a strengthening of Theorem I independently of whether (0.1) is satisfied or not. Assume for example that $\omega(x) \geqslant 0$ is even and that the necessary summability condition (0.2) is satisfied. If $\omega(x)/x \in \mathcal{H}$, then $f = \hat{\mu} \in \hat{\mathfrak{M}}_a$ can be constructed as in section 2 with ω replaced by $p\omega$ $(1 \leqslant p < \infty)$, so that

$$\int_{-\infty}^{\infty} |\hat{\mu}(x)|^p e^{p\omega(x)}\, dx < \infty. \tag{6.1}$$

The corresponding result for $p = \infty$,

$$|\hat{\mu}(x)| e^{\omega(x)} \leqslant \text{const.} \quad \text{for a.a. real } x, \tag{6.2}$$

is of course not true since our present condition does not imply that $\omega(x)$ is essentially bounded on any interval. If however, $\omega(x)$ has the continuity stipulated in Lemma II then again each $\hat{\mathfrak{M}}_a$ $(a > 0)$, contains elements $\hat{\mu}$ such that (6.2) holds for all real x.

In another paper we shall use the results of this study to resolve a closure problem for given systems of characters. This application together with some aspects of the present problem have been outlined in recent lectures by one of the authors [4].

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