THEORY OF x-IDEALS

BY

KARL EGIL AUBERT

Oslo

Table of Contents

Page

Introduction 2

CHAPTER 1: *General x-systems in commutative semi-groups*

1. The definition of an x-system 4
2. Comparison with the axiom system of Lorenzen 5
3. Operations on x-ideals 6
4. An alternative definition of the x-systems 9
5. (y, z)-homomorphisms and congruence modulo an x-ideal 11
6. Construction of x-systems from systems which do not satisfy the continuity axiom 14
7. The lattice of x-systems in S 15

CHAPTER 2: *The Krull theory for x-systems of finite character*

8. The Krull–Stone theorem. 16
9. A converse of the Krull–Stone theorem and other converses. 19
10. The non-associative case 21
11. Isolated primary components 22
12. Maximal prime x-ideals belonging to an x-ideal 23

CHAPTER 3: *x-Noetherian and x-Dedekindian semi-groups*

13. x-Noetherian semi-groups 25
14. Fractionary x-ideals 28
15. x-Dedekindian semi-groups 30

CHAPTER 4: *Structure spaces of x-ideals*

16. The Stone topology for maximal x-ideals 34
17. Characteristic function semi-groups 35

CHAPTER 5: *Applications to particular x-systems*

18. Lattices 38
19. Multiplicative lattices and semi-lattices 41
20. Radical differential ideals and perfect difference ideals 44
21. Rings with operators and monadic ideals 46
22. Convex subgroups of lattice-ordered groups and rings 48
23. Another characteristic function semi-group 50

Introduction

Since Kummer and Dedekind introduced ideals in connection with the problem of unique factorization of algebraic integers, numerous other notions of ideal have made their appearance in various branches of mathematics. In ring theory alone van der Waerden, Artin and especially Krull have introduced a whole series of new notions of ideal devised for different arithmetical purposes. These notions in ring theory can all be subsumed under a basic idea of Prüfer which has later been successfully applied in greater generality by Lorenzen and others to the arithmetics of semi-groups and ordered groups.

However, outside ring theory one finds that a considerable role is played by objects having a strong formal resemblance to ideals in rings. Many of these objects have therefore also appropriately been termed ideals. We have ideals (and filters) in Boolean algebras and more general lattices. We have radical (perfect) differential ideals in differential rings and various notions of ideal in semi-groups, m-lattices, ordered groups and ordered rings. Also, normal subgroups, the monadic ideals of Halmos and differential ideals in differential rings are pertinent to an axiomatic slightly more general than the one adopted in this paper. (See the lemma of section 19.)

These notions of ideal have been used for many different purposes. If we were to mention one group of questions outside the domain of general arithmetics for which various notions of ideal have played a decisive part, it would above all be the questions related to functional representation of various types of ordered and topological algebraic systems such as Boolean algebras, ordered groups, ordered rings and Banach algebras. It is sufficient to refer to the fundamental work of Stone, Gelfand and Kadison on the maximal ideal method and its numerous applications in connection with functional representation, compactification, etc.

The formal analogies between the existing notions of ideal suggest at once that a great number of results in the special ideal theories may be derived from a common source. The purpose of the present paper is to exhibit such a common axiomatic source and to lay the foundation of a general ideal theory based on it. The basic idea of the present approach which is to axiomatize the passage from a set to the ideal generated by that set, goes back to Prüfer [27]. This idea was generalized and used systematically by Krull and Lorenzen. But their investigations had a purpose entirely different from ours and were in fact directed exclusively towards the arithmetic of integral domains and ordered groups. The axioms of Lorenzen are, as they stand, so restrictive that they exclude application to a great number of the special concepts of ideal we have mentioned above. However, by an appropriate generalization these axioms become relevant for the general purposes we have in mind. One of the pleasent features of our axioms is that, in a precise sense, they represent the most general form of a reasonable ideal theory. In fact any weakening of the crucial axiom 1.3″ (at least beyond the condition given in the lemma of section 19) will imply that most of the basic rules of calculation, valid in the particular cases, will be lost.

The contents of the present paper are entirely elementary. In fact, only results which cover the most classical parts of the ordinary ideal theory of rings is given here. We shall in particular show that most of the results of the ideal theory of Krull [16] of rings without chain condition and the theory of Noetherian and Dedekindian rings carry over to this general setting. However, certain crucial results will require additional hypotheses. It is for instance not true for the general type of ideals considered here—called x-ideals—that an irreducible x-ideal is always primary in the presence of the ascending chain condition for x-ideals. Since the additive operation in ring theory is no longer present in our axioms it also seems difficult to carry over certain results from the ideal theory of rings which make strong use of additive properties. Still, certain arguments which appear to have an additive character can easily be reformulated so as to fit in the present theory. Examples of this are the results on relatively prime x-ideals, generalizing the exposition of van der Waerden [33; pp. 80–83]. One essential feature of ordinary ideals in rings is that they give rise to quotient rings or equivalently that they form kernels of ring homomorphisms. To give an entirely satisfactory imitation of this for general x-ideals seems difficult, but we can attach a notion of congruence to each x-ideal which, when specialized to rings, comes close to the usual congruence modulo an ideal.

In the last two chapters of the paper we have gathered some applications of the general theory. In the chapter on structure spaces we obtain a general characterization of a compact space X in terms of x-systems defined on semi-groups of continuous functions on X. This theorem contains well-known $C(X)$-theorems of Gelfand-Kolmogoroff, Stone and others. In the last chapter we have preferred to emphasize the variety of the possible applications rather than going into any detail. We prove in particular a representation theorem which shows that the most developed part of the theory of m-lattices is subsumed under the present theory. On the other hand, we prove that the crucial axiom 1.3″ cannot be formulated within the theory of m-lattices. This together with other facts seems to indicate clearly that the theory of x-ideals has considerable advantages over the ideal theory based on m-lattices.

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Chapter 1

General x-systems in commutative semi-groups

1. *The definition of an x-system.* By a semi-group we understand a set S in which there is defined a binary associative operation. We shall denote the operation multiplicatively and say that S is commutative if $ab = ba$ for all a, $b \in S$. For the sake of simplicity we shall here suppose S to be commutative. There is no difficulty in extending the following basic definitions and results to the non-commutative case. Indications concerning this extension will be given at the end of Chapter 4.

We shall say that there is defined a *system of x-ideals* or shortly an *x-system* in S if to every subset A of S there corresponds a subset A_x of S such that

1.1 $A \subseteq A_x$,

1.2 $A \subseteq B_x \Rightarrow A_x \subseteq B_x$,

1.3 $AB_x \subseteq B_x \cap (AB)_x$.

$A \cdot B$ here denotes the set of all products $a \cdot b$ with $a \in A$ and $b \in B$. The condition 1.3 is equivalent to the conjunction of the following two conditions

1.3′ $AB_x \subseteq B_x$,

1.3″ $AB_x \subseteq (AB)_x$.

In 1.3 we get an equivalent formulation if we replace A by a single element. We shall also refer to the passage from A to A_x as an *x-operation*. We remark that the conditions 1.1 and 1.2 just express that an x-operation is a closure operation. Condition 1.3′ is the multiplicative ideal property and the crucial axiom 1.3″ says that the multiplication in S is continuous with respect to the x-operation. We shall therefore also refer to this axiom as the *continuity axiom*. If $A = A_x$ we shall say that A is an *x-ideal*. In general A_x is the x-ideal *generated* by A. An x-system is said to be of *finite character* if for N finite

$$A_x = \bigcup_{N \subseteq A} N_x \tag{1}$$

i.e. the x-ideal generated by A equals the set-theoretic union of all the x-ideals generated by finite subsets of A. If we have defined only a *finite* x-system, i.e. supposing only that 1.1, 1.2, and 1.3 are satisfied for finite sets A and B we can use (1) to extend it to an x-system.

Examples. The above definition of an x-system includes as special cases nearly all the ideal concepts we have been able to find in the literature—for example all the ideal concepts in rings (see especially [18]), semi-groups, distributive lattices and m-lattices, perfect differential ideals in differential rings, closed ideals in topological rings, convex lattice-closed subgroups in lattice ordered groups, normal subgroups, monadic and polyadic ideals in Boolean algebras and numerous other more or less familiar instances. In most of these examples it is clear which semi-group is going to play the role of S. Let us just mention that in the case of convex subgroups the multiplication is $|a| \cap |b|$ and in case of normal subgroups it is the (non-commutative and non-associative) commutator multiplication $aba^{-1}b^{-1}$. For a closer examination of the above examples and their relationship to the general theory we can refer the reader to Chapter 5.

As to the term "x-ideal" this seems to be an appropriate name since various special cases bear names such as v-ideal, r-ideal, l-ideal, etc. The specialization is thus obtained by putting special letters in place of the indeterminate letter x.

2. *Comparison with the axiom system of Lorenzen.* The x-systems defined above should more precisely be termed *integral* x-systems in contradistinction to the fractional x-systems to be defined in Chapter 3. When comparing with the earlier "fractional" definitions of Prüfer, Krull and Lorenzen we should therefore rather have this latter definition in mind. If we formulate Lorenzen's definition in the case of integral r-ideals his axioms are as follows

2.1 $A \subseteq A_r$,

2.2 $A \subseteq B_r \Rightarrow A_r \subseteq B_r$,

2.3 $\{a\}_r = aS$,

2.4 $a \cdot A_r = (aA)_r$,

where S is now supposed to be a commutative semi-group with cancellation law ($ab = ac \Rightarrow b = c$) and an identity element $e(ea = a$ for all $a \in S)$. The condition 2.3 expresses that the r-ideal generated by a single element a consists of all the multiples of a. We shall also denote the set aS by (a). We note that the axioms 1.1 and 1.2 are the same as Lorenzen's axioms 2.1 and 2.2. Apart from the fact that we remove the condition that S shall satisfy the cancellation law and have an identity, the range of applications of the theory is also essentially broadened by our weakening of the conditions 2.3 and 2.4. 1.3′ is a consequence of 2.3 and 1.3″ is a consequence of 2.4. What we have retained of 2.3 is only the fact that any x-ideal is closed with respect to multiplication with an arbitrary element of S. We remark, however, that in the axiom system of Lorenzen we can replace 2.3 by the weaker form 1.3′ because of 2.4 and the presence of an identity element. Indeed, by 2.4 $a(e)_r = (a)_r$ and by 1.3′ $(e)_r = S$ which together give 2.3.

Among important x-systems which generally do not satisfy either 2.3 or 2.4, and where S both satisfies the cancellation law and has an identity, are the x-systems defined by the perfect differential ideals of a differential ring, the convex lattice-closed subgroups of a lattice ordered group and the closed ideals of a topological ring. Take for instance the differential polynomial ring $Z[x]$ in one variable over the rational integers and let derivation have its ordinary meaning. Denoting the passage from A to the perfect[1] differential ideal generated by A as the δ-operation, 2.3 is not satisfied since $\{x\}_\delta \neq xS = x\cdot Z[x]$. From the above remark on the implication 2.4 and $1.3' \Rightarrow 2.3$ in the presence of an identity we conclude that 2.4 is not satisfied. We have, for instance, $x\{1\}_\delta \neq \{x\}_\delta$. In a topological ring, with an identity, 2.3 and 2.4 fail to hold if there exist principal ideals which are not closed.

3. *Operations on x-ideals. Equivalent forms of the continuity axiom.* The basic operations in usual ideal theory are the operations of intersection, union, multiplication and residuation. We shall, in this section, state some of the most fundamental properties of these operations in the case of general x-ideals. It will turn out that these properties depend entirely on the validity of the continuity axiom $1.3''$.

It follows trivially from 1.1 and 1.2 that the (set-theoretic) intersection of any family of x-ideals is again an x-ideal and since S is an x-ideal we obtain

PROPOSITION 1. *The family of all x-ideals of S forms a complete lattice $L_x^{(S)}$ with respect to set-inclusion.*

In contradistinction to intersection, the set-theoretic union of two x-ideals is in general not an x-ideal. Thus the set-theoretic union is generally different from the union within the lattice $L_x^{(S)}$. We shall, therefore, term this latter operation *x-union* and denote it by $\cup_x$. Thus

$$\bigcup_{i\in I}{}_x A^{(i)} = \Big(\bigcup_{i\in I} A^{(i)}\Big)_x$$

In ring theory the product of two ideals $\mathfrak{a}$ and $\mathfrak{b}$ is defined as the ideal generated by $\mathfrak{a}\cdot\mathfrak{b}$. Similarly the *x-product* of two subsets A and B of S is defined as the set $(AB)_x$. We denote this product by $A\circ_x B$ or more briefly by $A\circ B$ and call it *x-multiplication*.

THEOREM 1. *The following statements are equivalent under the hypothesis that the passage $A\to A_x$ is a closure operation:*

A. *The continuity axiom $AB_x \subseteq (AB)_x$.*

B. $A\circ B = A\circ B_x$ (or $A\circ B = A_x \circ B_x$).

C. *The one-sided distributive law $A\cdot(B\cup_x C)\subseteq AB\cup_x AC$.*

D. *The x-multiplication is distributive with respect to x-union,* i.e. $A\circ(B\cup_x C) = A\circ B\cup_x A\circ C$.

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(1) A set B is perfect if $a^n \in B$ implies $a\in B$.

Proof. It is sufficient to establish, for instance, the following sequence of implications $\mathbf{C}\Rightarrow\mathbf{D}\Rightarrow\mathbf{B}\Rightarrow\mathbf{A}\Rightarrow\mathbf{C}$. This is a routine check and can be left to the reader.

Under the additional hypothesis that S has an identity we have the following slightly more astonishing equivalence.

THEOREM 2. *If S contains an identity e and the passage $A\to A_x$ is a closure operation, then the continuity axiom is equivalent to the associativity of the x-multiplication.*

Proof. Using the continuity axiom we obtain $A\circ(B\circ C)=(A(BC)_x)_x\subseteq(A(BC))_x=((AB)C)_x\subseteq((AB)_xC)_x=(A\circ B)\circ C$. In the same way $(A\circ B)\circ C\subseteq A\circ(B\circ C)$. Conversely, putting $C=\{e\}$ in $A\circ(B\circ C)=(A\circ B)\circ C$ we get $A\circ B_x=A\circ B$ which according to Theorem 1 is equivalent to the continuity axiom.

We now pass to the operation of residuation. If A and B are subsets of S we denote by $A:B$ the set of all $c\in S$ such that $cB\subseteq A$ and call $A:B$ the quotient of A by B. If $B=\{b\}$ consists of a single element, we write $A:b$ instead of $A:\{b\}$. From the definition it follows that $(A:B)B\subseteq A$ and, therefore, also that $(A_x:B)\circ B\subseteq A_x$. Because of 3′ we always have $A_x\subseteq A_x:B$. The identities

$$(\bigcap_{i\in I}A^{(i)}):B=\bigcap_{i\in I}(A^{(i)}:B), \tag{2}$$

and

$$A:\bigcup_{i\in I}B^{(i)}=\bigcap_{i\in I}(A:B^{(i)}). \tag{3}$$

are essentially set-theoretical and are readily seen to be valid. As shown by the following theorem, other essential properties of the operation of residuation are only valid under the assumption of the continuity axiom.

THEOREM 3. *The following statements are equivalent under the hypothesis that $A\to A_x$ is a closure operation:*

A. *The continuity axiom $AB_x\subseteq(AB)_x$.*

B. $(A_x:B)_x=A_x:B$.

C. $(A_x:b)_x=A_x:b$.

D. $A_x:B_x=A_x:B$.

E. $(A:B)_x\subseteq A_x:B$.

F. *The dual distributive law $A_x:\bigcup_{i\in I}{}_x B^{(i)}=\bigcap_{i\in I}(A_x:B^{(i)})$. If S contains an identity we may also add the equality*

G. $(A_x:B):C=A_x:(B\circ C)$.

Proof. To show the equivalence of the first six properties we may, for instance, establish the following sequence of implications

$$\mathbf{C} \Rightarrow \mathbf{B} \Rightarrow \mathbf{A} \Rightarrow \mathbf{F} \Rightarrow \mathbf{D} \Rightarrow \mathbf{E} \Rightarrow \mathbf{C}.$$

$\mathbf{C} \Rightarrow \mathbf{B}$: Using (3) and the fact that an intersection of x-ideals is again an x-ideal we find that $A_x : B = \bigcap_{b \in B} A_x : b$ is an x-ideal if **C** is satisfied.

$\mathbf{B} \Rightarrow \mathbf{A}$: Since $B \subseteq (AB)_x : A$ we conclude from **B**. that $B_x \subseteq (AB)_x : A$, which is **A**.

$\mathbf{A} \Rightarrow \mathbf{F}$: Obviously

$$A_x : \bigcup_{i \in I} {}_x B^{(i)} \subseteq A_x : B^{(i)},$$

for all i and therefore

$$A_x : \bigcup_{i \in I} {}_x B^{(i)} \subseteq \bigcap_{i \in I} (A_x : B^{(i)}).$$

Conversely, if $c \in \bigcap_{i \in I} (A_x : B^{(i)})$, we have $cB^{(i)} \subseteq A_x$ for all i. Hence,

$$c \bigcup_{i \in I} {}_x B^{(i)} = c\,(\bigcup_{i \in I} B^{(i)})_x \subseteq (c \bigcup_{i \in I} B^{(i)})_x = (\bigcup_{i \in I} cB^{(i)})_x \subseteq A_x.$$

$\mathbf{F} \Rightarrow \mathbf{D}$: **D** follows by putting $B^{(i)} = B$ for all i in **F**.

$\mathbf{D} \Rightarrow \mathbf{E}$: Condition **D** is equivalent to the implication $CB \subseteq A_x \Rightarrow CB_x \subseteq A_x$. Interchanging B and C we obtain the condition **E**.

$\mathbf{E} \Rightarrow \mathbf{C}$: We obtain **C** by putting A_x instead of A and $\{b\}$ instead of B in **E**. Using the continuity axiom we can easily prove **G**: If $d \in (A_x : B) : C$ this means that $d(BC) \subseteq A_x$ and, therefore, $d(BC)_x \subseteq (d(BC))_x \subseteq A_x$ showing that $d \in A_x : (B \circ C)$. The inclusion $A_x : (B \circ C) \subseteq (A_x : B) : C$ is equally obvious. Conversely, putting $C = \{e\}$ in **G**. we obtain **D**.

We shall refer to a set $A : a$ as a *residual* of A. For a fixed element $a \in S$ the mapping f_a which maps $b \in S$ into ab will be called a *translation* and the set aA is a *translate* of A. The following two propositions give a further clarification of the continuity axiom and its stronger counterpart in the Lorenzen theory.

PROPOSITION 2. *The condition $(aB)_x \subseteq aB_x$ is equivalent to the fact that the translates aB_x of an x-ideal B_x are all x-ideals.*

Proof. Assuming that B_x is an x-ideal and applying the inclusion in the proposition we get $(aB_x)_x \subseteq a(B_x)_x = aB_x$, showing that the translate aB_x is also an x-ideal. Conversely, if aB_x is an x-ideal we have $(aB)_x \subseteq (aB_x)_x = aB_x$.

PROPOSITION 3. *In a group the equality* $aB_x=(aB)_x$ *is equivalent to either of the two inclusions* $aB_x\subseteq(aB)_x$ *and* $aB_x\supseteq(aB)_x$ *and in this case the x-system will also be an (integral) r-system in the sense of Lorenzen.*

Proof. Assuming, for instance, $aB_x\subseteq(aB)_x$ and replacing B by aB and a by a^{-1} we get $a^{-1}(aB)_x\subseteq B_x$. Multiplication on both sides by a gives $(aB)_x\subseteq aB_x$ as desired. The latter half of the proposition follows from the implication 2.4 and $1.3'\Rightarrow 2.3$ mentioned in 2.

4. *An alternative definition of the x-systems.* Just as the notion of a topological space may be defined in various ways—for instance, by a closure operation or by a family of closed sets—the x-systems also permit similar alternative definitions. To the definition of a topological space by closed sets corresponds here the definition of an x-system by a family of x-ideals. The precise connection between the two definitions is given by the following:

THEOREM 4. *Let* S *be a commutative semi-group and let* $\mathfrak{X}$ *be a non-void family of subsets of* S, *called x-ideals, which satisfy the following two conditions:*

4.1 *The intersection of any non-void family of x-ideals is again an x-ideal.*

4.2 *Any residual of an x-ideal* A_x *is an x-ideal containing* A_x.

Let A *be a subset of* S *and put*

$$A_x=\bigcap_{\substack{B_x\in\mathfrak{X}\\ A\subseteq B_x}} B_x$$

then the correspondence $A\to A_x$ *defines an x-operation with respect to which the family of x-ideals coincides with* $\mathfrak{X}$. *This establishes a one-to-one correspondence between the x-systems in* S *and the families* $\mathfrak{X}$ *satisfying* 4.1 *and* 4.2.

Proof. It is well known that there is a one-to-one correspondence between the general closure operations on S satisfying 1.1 and 1.2 in § 1 and the families of closed sets satisfying 4.1 and the condition that the entire set S is closed. This latter condition is satisfied in our case. For by the second half of 4.2 $SA_x\subseteq A_x$ which implies that $A_x:a=S$ whenever $a\in A_x$. The first half of 4.2, therefore, assures that $S\in\mathfrak{X}$. The theorem now follows from Theorem 3 which shows that 1.3 and 4.2 are equivalent conditions.

The analogue of Theorem 4 for r-systems in the sense of Lorenzen can be formulated as follows:

THEOREM 5. *Let* S *be a commutative semi-group with an identity element. Then there is a one-to-one correspondence (defined in the same way as in Theorem* 4) *between the (integral) r-systems of Lorenzen satisfying* 2.1, 2.2, 2.3 *and* 2.4 *and the families* $\mathfrak{R}$ *consisting of subsets of* S *such that the following conditions are satisfied.*

4.1* *$\mathfrak{R}$ is closed under arbitrary (non-void) intersections.*

4.2* *$\mathfrak{R}$ is closed under the operations of taking residuals and translates. The translate of a set $A \in \mathfrak{R}$ is contained in A.*

Proof. From Theorem 3 and Proposition 2 it follows immediately that 4.1* and 4.2* are satisfied by the family of r-ideals defined by 2.1–2.4. Conversely, 4.2* implies $S \in \mathfrak{R}$ and the mapping

$$A \to \bigcap_{\substack{A \subseteq B_r \\ B_r \in}} B_r$$

is because of 4.1* a closure operation. Using 4.2*, Theorem 3 and Proposition 2 it is clear that this closure operation satisfies 2.4. By the second half of 4.2* $Sa \subseteq \{a\}_r$. On the other hand, $S \in \mathfrak{R}$ implies by 4.2* that $Sa \in \mathfrak{R}$. Since S is assumed to have an identity $a \in Sa$ and 2.3 follows.

Remark. The presence of an identity element is essential in the above argument. However, in the original paper of Lorenzen, where only semi-groups with cancellation law are considered, it is not necessary to postulate the existence of an identity since this follows from the axiom 2.3. For in a semi-group with cancellation law the existence of an equation of the form $b = ab$ implies the existence of a unique identity.

Returning to general x-systems we shall now see how the condition that an x-system be of finite character is expressed in terms of the family $\mathfrak{X}$ of all x-ideals in S. By a *chain* of x-ideals we understand a family of x-ideals such that for any two of its members A_x and B_x we have either $A_x \subseteq B_x$ or $B_x \subseteq A_x$.

THEOREM 6. *An x-system is of finite character if and only if the set-theoretic union of any chain of x-ideals is an x-ideal.*

Proof: Let x be of finite character and let $\{A_x^{(\iota)}\}_{\iota \in I}$ be a chain of x-ideals. Since x is of finite character we only need to show that

$$N_x \subseteq \bigcup_{\iota \in I} A_x^{(\iota)}$$

whenever N is a finite set contained in the above union. This is obvious since N being a finite set is contained in one of the $A_x^{(\iota)}$. The converse can be shown in the following way. Let A be an arbitrary subset of S and let B be a subset of A such that any x-ideal of the form $(B \cup N)_x$, where N is a finite subset of A, is of finite character. The subsets B of A having this property form an inductive family $\mathfrak{B}$. For if $\{B^{(\iota)}\}_{\iota \in I}$ forms a chain in $\mathfrak{B}$, the union $\bigcup_{\iota \in I} B^{(\iota)}$ will again belong to $\mathfrak{B}$ since for any finite set $N \subseteq A$

$$(N \cup \bigcup_{i \in I} B^{(i)})_x = (\bigcup_{i \in I} (B^{(i)} \cup N))_x = \bigcup_{i \in I} (B^{(i)} \cup N)_x,$$

using here the condition that the union of a chain of x-ideals is an x-ideal. Any element in the latter union is contained in some $(B^{(i)} \cup N)_x$; hence in some N'_x where N' is a finite set contained in $B^{(i)} \cup N$. This shows that $\mathfrak{B}$ is inductive. By Zorn's lemma $\mathfrak{B}$ contains a maximal member B'. If $B' \neq A$ we would have $B' \cup N \in \mathfrak{B}$ for all finite $N \subseteq A - B'$ contradicting the maximality of B'. Thus $B' = A$ and x is of finite character.

Calling a family $\mathfrak{J}$ of subsets of S *chain-closed* if the set-theoretic union of the members of any chain in $\mathfrak{J}$ itself belongs to $\mathfrak{J}$, we get by combining Theorems 4 and 6.

THEOREM 7. *There is a one-to-one correspondence between the x-systems of finite character and the chain-closed families satisfying the conditions of Theorem* 4.

5. *(y, z)-homomorphisms and congruence modulo an x-ideal.* The usual congruence modulo an ideal in a ring is defined by a purely additive property and it seems therefore difficult to give a general definition of a congruence modulo an x-ideal which yields the usual notion of congruence when specialized to rings. We shall show, however, that it is possible to define a general notion of congruence which comes close to the usual one in the case of rings and which has similar properties.

Let us first state a simple lemma which will be used below.

LEMMA 1. *Let φ be a homomorphism of a semi-group S onto a semi-group T and let A and B be two subsets of T. We then have $\varphi^{-1}(A) \cdot \varphi^{-1}(B) \subseteq \varphi^{-1}(AB)$ and $\varphi^{-1}(A:B) = \varphi^{-1}(A): \varphi^{-1}(B)$.*

Proof. Let $a \in A$ and $b \in B$. Then $\varphi(\varphi^{-1}(a) \cdot \varphi^{-1}(b)) = a \cdot b$ since φ is a homomorphism. This means that $\varphi^{-1}(a)\varphi^{-1}(b) \subseteq \varphi^{-1}(ab) \subseteq \varphi^{-1}(A \cdot B)$. Therefore

$$\varphi^{-1}(A) \cdot \varphi^{-1}(B) = \bigcup_{a \in A} \varphi^{-1}(a) \cdot \bigcup_{b \in B} \varphi^{-1}(b) = \bigcup_{\substack{a \in A \\ b \in B}} \varphi^{-1}(a)\, \varphi^{-1}(b) \subseteq \varphi^{-1}(AB).$$

To prove the second half of the lemma let first $c \in \varphi^{-1}(A): \varphi^{-1}(B)$, i.e. $c\varphi^{-1}(B) \subseteq \varphi^{-1}(A)$. Applying φ on both sides of this inclusion we get $\varphi(c) \cdot B \subseteq A$ by using the fact that φ is a homomorphism onto T. Thus $\varphi(c) \in A:B$ and $c \in \varphi^{-1}(A:B)$. This last argument works equally well when applied backwards, showing that we have the desired equality.

Let now S and T each be equipped with an x-system denoted respectively by y and z. We shall say that a multiplicative homomorphism φ of S into T is a *(y,z)-homomorphism* if $\varphi(A_y) \subseteq (\varphi(A))_z$ for all subsets A of S, This means that if A is mapped into B by φ then

A_y is mapped into B_z. It is also clear that a homomorphism of S into T is a (y,z)-homomorphism if and only if the inverse image of a z-ideal in T is a y-ideal in S.

THEOREM 8. *Let φ be a multiplicative homomorphism of S onto T and let y be an x-system in S. Then the family of all sets $B \subseteq T$ such that $\varphi^{-1}(B)$ is a y-ideal in S defines an x-system in T denoted by y_φ. Relative to this x-system φ is a (y,y_φ)-homomorphism and y_φ is the finest x-system z in T such that φ is a (y,z)-homomorphism.*

Proof. It is clear that $B \to B_{y_\varphi}$ defines a closure operation in T since the family of all $B \subseteq T$ such that $\varphi^{-1}(B)$ is a y-ideal contains T and is closed under arbitrary intersections. Assume next that B is a y_φ-ideal, i.e. $\varphi^{-1}(B) = A_y$ is a y-ideal in S. Then $a\varphi^{-1}(B) \subseteq \varphi^{-1}(B)$ and applying φ on both sides we get $\varphi(a)B \subseteq B$. Since φ is supposed to be 'onto' this shows that axiom 1.3′ is satisfied for y_φ. Finally by Lemma 1 and the continuity axiom for y we get by using the same notations that $\varphi^{-1}(B:C) = \varphi^{-1}(B):\varphi^{-1}(C) = A_y:\varphi^{-1}(C)$ is a y-ideal for each $C \subseteq T$. Hence $B:C$ is a y_φ-ideal whenever B is a y_φ-ideal and this is the continuity axiom for y_φ. That φ is a (y, y_φ)-homomorphism follows from the definition of the y_φ-system. The maximality of y_φ is also clear.

We now define

$$b \equiv c \,(\mathrm{mod}\ A_x)$$

if and only if $(A_x,b)_x = (A_x,c)_x$, and we shall say that b and c are *x-congruent* mod A_x. (Here (A,a) means the set obtained by adjoining the element a to the set A.) Let us see what this means in the case of ordinary ideals in commutative rings. We shall refer to ordinary ideals in rings as d-ideals. From $(A_d,b)_d = (A_d,c)_d$ follows, in particular, that $b \in (A_d,c)_d$ and $c \in (A_d,b)_d$ and this amounts to the following two congruences now understood in the usual sense

$$b \equiv r_1 c + n_1 c \,(\mathrm{mod}\ A_d),$$
$$c \equiv r_2 b + n_2 b \,(\mathrm{mod}\ A_d),$$

where r_1, and r_2 are elements of the given ring R and n_1 and n_2 are integers. The terms $n_1 c$ and $n_2 b$ disappear if R has an identity element and we may write down the following immediate

PROPOSITION 4. *In a commutative ring R with an identity two elements b and c are d-congruent* mod A_d *if and only if the ordinary residue classes of b and c represent associate elements in the quotient ring R/A_d.*

That the residue classes $\bar{b}$ and $\bar{c}$ of b and c respectively are associate elements of R/A_d means as usual that $\bar{b} \mid \bar{c}$ and $\bar{c} \mid \bar{b}$.

THEOREM 9. *The relation $b \equiv c \,(\mathrm{mod}\, A_x)$ is a congruence relation in S. The x-ideal A_x forms an equivalence class such that the quotient semigroup S/A_x is a semi-group with zero element. The canonical homomorphism φ of S onto S/A_x establishes a one-to-one correspondence between the x-ideals of S containing A_x and the x_φ-ideals of S/A_x.*

Proof. That the given relation is an equivalence relation is clear. Suppose that $b \equiv c \,(\mathrm{mod}\, A_x)$, i.e. $(A_x, b)_x = (A_x, c)_x$. By the continuity axiom

$$db \in d(A_x, b)_x = d(A_x, c)_x \subseteq (A_x, dc)_x,$$

and therefore

$$(A_x, db)_x \subseteq (A_x, dc)_x.$$

Similarly $(A_x, db) \supseteq (A_x, dc)_x$ and $db \equiv dc$ (mod A_x). Two elements in A_x are clearly congruent mod A_x. On the other hand, if $a \in A_x$ and $b \notin A_x$ then $(A_x, a)_x \neq (A_x, b)_x$ and $a \not\equiv b$ (mod A_x). Thus A_x forms one of the equivalence classes and this class will be the zero element of S/A_x. For the last part of the theorem we only need to verify that $\varphi^{-1}(\varphi(B_x)) = B_x$ whenever $B_x \supseteq A_x$. The equality $\varphi^{-1}(\varphi(B_x)) = B_x$ means that B_x is a union of residue classes modulo A_x. If this were not the case there would exist elements $b, c \in S$ with $b \in B_x$ and $c \notin B_x$ such that $b \equiv c$ (mod A_x). But this is impossible since $(A_x, b)_x \subseteq B_x$ and $(A_x, c)_x \nsubseteq B_x$. We thus see that a subset of of S/A_x is an x_φ-ideal if and only if it is a direct image of an x-ideal in S.

In the case of groups and rings we have certain fundamental facts concerning homomorphisms which are no longer valid for the general case considered here. We have, for instance, no complete counterpart to the general homomorphism theorem for rings saying that any homomorphic image of a ring R is isomorphic to a certain quotient ring R/a. We can, however, get quite close to such a statement by making a couple of additional hypotheses.

THEOREM 10. *Let φ be a (y,z)-homomorphism of S onto T. We suppose that φ satisfies the identity $\varphi^{-1}(\varphi(B_y)) = B_y$ for any y-ideal B_y in S and that T has a zero element* 0 *such that $\{0\}$ forms a z-ideal O_z. Then* $\mathrm{Ker}\, \varphi = \varphi^{-1}(O_z)$ *is a y-ideal A_y in S such that $b \equiv c \,(\mathrm{mod}\, A_y)$ if and only if $\varphi(b) \equiv \varphi(c) \,(\mathrm{mod}\, O_z)$.*

Proof. Assume first that $b \equiv c \,(\mathrm{mod}\, A_y)$, i.e. $(A_y, b)_y = (A_y, c)_y$ and $\varphi(A_y, b)_y = \varphi(A_y, c)_y$. Since φ is a (y,z)-homomorphism this gives $\varphi(b) \in \varphi(A_y, c)_y \subseteq (O_z, \varphi(c))_z$ and consequently $(O_z, \varphi(b))_z \subseteq (O_z, \varphi(c))_z$. Similarly, $(0, \varphi(b))_z \supseteq (0, \varphi(c))_z$ showing that $\varphi(b) \equiv \varphi(c)$ (mod O_z). Conversely, if $b \not\equiv c \,(\mathrm{mod}\, A_x)$ we can, for instance, suppose that $b \notin (A_y, c)_y$. Applying φ and remembering the condition $\varphi^{-1}(\varphi(B)) = B_y$ we obtain $\varphi(b) \notin (O_z, \varphi(c))_z$ showing that $\varphi(b) \not\equiv \varphi(c)$ (mod O_z).

As in ring theory various properties of an x-ideal A_x are equivalent to corresponding properties of the quotient S/A_x. It is, for instance, obvious that an x-ideal P_x is prime if and only if S/P_x is without divisors of zero. Among other statements of this kind let us just mention the following theorem. The residue class containing a will be denoted by $\bar{a}$.

THEOREM 11. *The non-zero elements of S/A_x form a group if and only if the following two conditions are satisfied:* (1) *A_x is a maximal x-ideal in S;* (2) *$a^2 \in A_x$ implies $a \in A_x$. In general A_x is maximal if and only if S/A_x has two elements.*

Proof. Suppose first that $S/A_x - \{\bar{0}\} = S^*$ is a group.[1] Then S^* is in particular closed under multiplication and $\bar{a} \neq \bar{0}$ implies $\bar{a}^2 \neq \bar{0}$ and this is equivalent to 2). If $a \notin A_x$, i.e. $\bar{a} \neq \bar{0}$ the group-property assures the existence of a solution $\bar{y}_0$ of $\bar{a}\bar{y} = \bar{b}$. This means that $ay_0 \equiv b \pmod{A_x}$, i.e. $(A_x, ay_0)_x = (A_x, b)_x$. This gives $b \in (A_x, ay_0)_x \subseteq (A_x, a)_x$. Since b is arbitrary $(A_x, a)_x = S$ and A_x is maximal. Assuming conversely that (1) and (2) are satisfied we have to show that $\bar{a}\bar{y} = \bar{b}$ is solvable in $\bar{y}$ whenever $\bar{a} \neq \bar{0}$. Now $\bar{a} \neq \bar{0}$ means that $a \notin A_x$ and thus $a^2 \notin A_x$ by (2). A_x being maximal, this gives $(A_x, a^2)_x = S = (A_x, b)_x$. Thus $a^2 \equiv b \pmod{A_x}$ and $\bar{y} = \bar{a}$ gives a solution of $\bar{a}\bar{y} = \bar{b}$. The last statement of the theorem is obvious.

6. *Construction of x-systems from systems which do not satisfy the continuity axiom.* A system which satisfies 1.1, 1.2 and 1.3′ but not the continuity axiom 1.3″ will in this paragraph be termed an x^*-system. We shall now describe two general procedures which in a natural way permit us to associate an x-system to a given x^*-system. The first method is based on a retraction of the basic semi-group S while the second one is based on a retraction of the family of "x^*-ideals" in S. If S_0 is a subsemigroup of S and x^* is an x^*-system in S then the family of all intersections $A_{x^*} \cap S_0$ obviously defines an x^*-system in S_0. This x^*-system will be called the *trace* of x^* on S_0.

PROPOSITION 5. *Let S be a semi-group with a given x^*-system. The set of all elements $a \in S$ such that $A_{x^*}:a$ is an x^*-ideal for all x^*-ideals A_{x^*} in S forms a subsemigroup S^* of S and the trace of x^* on S^* is an x-system in S^*.*

Proof. If $A_{x^*}:a$ and $A_{x^*}:b$ are x^*-ideals for all A_{x^*} then $A_{x^*}:ab = (A_{x^*}:a):b$ is also an x^*-ideal for all A_{x^*} and S^* forms a subsemigroup of S. The traces $A_{x^*} \cap S^*$ obviously form an x^*-system in S^*. That these traces also satisfy the continuity axiom and thus define an x-system follows from

$$S^* \cap ((A_{x^*} \cap S^*):a) = S^* \cap (A_{x^*}:a) \cap (S^*:a) = S^* \cap (A_{x^*}:a).$$

We note that we can also define S^* as the set of all elements $a \in S$ such that $aB_{x^*} \subseteq (aB)_{x^*}$ for all subsets B of S.

[1] $\bar{0}$ denotes the residue class containing the elements of A_x.

The following proposition describes a dual procedure to obtain an x-system from a given x^*-system.

PROPOSITION 6. *Let $\mathfrak{X}^*$ denote the family of x^*-ideals in a given x^*-system. The subfamily $\mathfrak{X}$ of $\mathfrak{X}^*$ consisting of all $A_{x^*} \in \mathfrak{X}^*$ such that $A_{x^*}:a \in \mathfrak{X}^*$ for all $a \in S$ defines an x-system in S.*

Proof. We have to verify that the conditions 4.1 and 4.2 are satisfied for $\mathfrak{X}$. That $\mathfrak{X}$ is closed under arbitrary intersections is a consequence of

$$(\bigcap_{i \in I} A_{x^*}^{(i)}) : a = \bigcap_{i \in I} (A_{x^*}^{(i)} : a)$$

and the fact that $\mathfrak{X}^*$ is closed under arbitrary intersections. Assume that $A_{x^*} \in \mathfrak{X}$. The definition of $\mathfrak{X}$ gives $A_{x^*}:a \in \mathfrak{X}^*$ for all $a \in S$. Further $(A_{x^*}:a):b = A_{x^*}:ab \in \mathfrak{X}^*$ for all b showing that $A_{x^*}:a \in \mathfrak{X}$ for all $a \in S$.

7. *The lattice of x-systems in S.* Let $\mathfrak{L}_S$ denote the family of x-systems in S. We introduce a natural ordering in $\mathfrak{L}_S$ by the following definition: The x_1-system is said to be *finer* than the x_2-system if every x_2-ideal is an x_1-ideal. Denoting the family of x_i-ideals by $\mathfrak{X}_i$ this means that $\mathfrak{X}_2 \subseteq \mathfrak{X}_1$. We shall also denote this situation by $x_1 \succ x_2$. It is clear that we have $x_1 \succ x_2$ if and only if $A_{x_1} \subseteq A_{x_2}$ for all $A \subseteq S$. $\mathfrak{L}_S$ is a partially ordered set with respect to $\succ$ and has a greatest element $s \succ x$ for all $x \in \mathfrak{L}_S$ and a smallest element u satisfying $x \succ u$ for all $x \in \mathfrak{L}_S$. These two x-systems are explicitly defined by

$$A_u = S \quad \text{for all } A \subseteq S \quad \text{and} \quad A_s = SA \cup A.$$

PROPOSITION 7. *Every non-void subset $\{x_i\}_{i \in I}$ of $\mathfrak{L}_S$ has a least upper bound $x = \bigvee_{i \in I} x_i$ in $\mathfrak{L}_S$ and $A_x = \bigcap_{i \in I} A_{x_i}$.*

Proof. If $x = \bigvee_{i \in I} x_i$ exists it is clear that $A_x \subset \bigcap_{i \in I} A_{x_i}$ so that we only have to verify that $A \to \bigcap_{i \in I} A_{x_i}$ defines an x-system in S. The properties 1.1, 1.2 and 1.3′ are obvious. Moreover, $AB_{x_i} \subseteq (AB)_{x_i}$ for all $i \in I$ implies

$$AB_x = A \bigcap_{i \in I} B_{x_i} \subseteq \bigcap_{i \in I} AB_{x_i} \subseteq \bigcap_{i \in I} (AB)_{x_i} = (AB)_x.$$

COROLLARY. *$\mathfrak{L}_S$ forms a complete lattice with respect to the ordering $\succ$.*

PROPOSITION 8. *The family $\mathfrak{F}_S$ of all x-systems of finite character in S forms a complete sublattice of $\mathfrak{L}_S$, i.e. when $\{x_i\}_{i \in I}$ is a family of x-systems of finite character then $\bigwedge_{i \in I} x_i$ and $\bigvee_{i \in I} x_i$ are both x-systems of finite character.*

Proof. That $\bigwedge_{i\in I} x_i$ is of finite character follows from Theorem 6 since an intersection of chain-closed families is again chain-closed. Moreover, if $x=\bigvee_{i\in I} x_i$ we have

$$A_x=\bigcap_{i\in I} A_{x_i}=\bigcap_{i\in I}\bigcup_{N\subseteq A} N_{x_i}=\bigcup_{N\subseteq A}\bigcap_{i\in I} N_{x_i}=\bigcup_{N\subseteq A} N_x,$$

where N denotes a finite set. This shows that $x\in\mathfrak{J}_S$. Proposition 7 gave an explicit expression for A_x with $x=\bigvee_{i\in I} x_i$ in terms of the family $\{A_{x_i}\}_{i\in I}$. Within $\mathfrak{J}_S$ we can do something similar also in the case of a finite intersection $\bigwedge_{i\in I} x_i$. To this end we introduce the following notations. We write $A_{(x_1x_2)^n}$ for the set

$$\left(\left(\left(A_{x_1}\right)_{x_2}\right)_{x_1}\right)_{x_2\ldots},$$

where x_1 and x_2 are each repeated n times and put

$$A_{x_1 * x_2}=\bigcup_{n\geqslant 1} A_{(x_1x_2)^n}.$$

It is now easy to see that $x_1\wedge x_2=x_1 * x_2$.

Chapter 2

The Krull theory for x-systems of finite character

8. *The Krull-Stone theorem.* The purpose of the present chapter is to generalize Krull's ideal theory of commutative rings without finiteness assumptions, as developed in [16], to general commutative x-systems of finite character. We start with a proof of the fundamental Krull-Stone theorem concerning the representation of half-prime x-ideals as intersections of (minimal) prime x-ideals. This theorem was first proved by Krull in [16] in the case of ordinary ideals in commutative rings. The fundamental application of this theorem made by Stone in the case of Boolean algebras justifies the association of his name with the theorem.

We shall give two different proofs of the Krull-Stone theorem. The first one is identical whit Krull's original proof and the second one is modelled after the proof given by Ritt and Raudenbush in the case of perfect differential ideals.

Let S be a commutative semi-group in which we have fixed a certain x-system of finite character. An x-ideal P_x in S is said to be a prime x-ideal if $a\cdot b\in P_x$ and $a\notin P_x$ imply $b\in P_x$. The (*nilpotent*) *radical* of A_x, denoted by rad A_x, consists of all elements $b\in S$ such that $b^n \in A_x$ for some integer n. We shall say that A_x is *half-prime* if rad $A_x = A_x$. A subsemigroup of S will be called an *m-set*. In the following it will be convenient to consider the void set as an m-set.

PROPOSITION 9. *The x-ideal P_x is prime if and only if $A_x \circ B_x \subseteq P_x$ and $A_x \not\subseteq P_x$ imply $B_x \subseteq P_x$.*

Proof. Suppose that $A_x \circ B_x \subseteq P_x$, $A_x \not\subseteq P_x$ and $B_x \not\subseteq P_x$. We can then find elements $a \in A_x$ and $b \in B_x$ which do not belong to P_x such that $a \cdot b \in A_x \cdot B_x \subseteq A_x \circ B_x \subseteq P_x$. Conversely if P_x is not prime we have elements $a, b \notin P_x$ such that $a \cdot b \in P_x$. Then $(P_x \cup \{a\})_x \circ (P_x \cup \{b\})_x \subseteq P_x$ by Theorem 1 and the implication in the proposition is not satisfied.

COROLLARY. *The x-ideal A_x is non-prime if and only if there exist x-ideals B_x and C_x both properly containing A_x such that $B_x \circ C_x \subseteq A_x$.*

The following proposition is proved in exactly the same way as in ring theory.

PROPOSITION 10. *If M is a maximal m-set contained in $S - A_x$ and P_x is a maximal x-ideal containing A_x and being contained in $S - M$ then P_x is a minimal prime x-ideal containing A_x.*

COROLLARY 1. *Any prime x-ideal containing A_x contains at least one minimal prime x-ideal containing A_x.*

COROLLARY 2. *The complement of any maximal m-set contained in $S - A_x$ is a minimal prime x-ideal containing A_x.*

THEOREM 12. (The Krull-Stone theorem for x-systems of finite character.) *The nilpotent radical of the x-ideal A_x is equal to the intersection of all the minimal prime x-ideals containing A_x.*

Proof. According to Corollary 1 we only have to prove the equality

$$\text{rad } A_x = \bigcap_{A_x \subseteq P_x} P_x,$$

the intersection being extended over *all* prime x-ideals containing A_x. It is clear that the left hand side is contained in the right hand side. Let us suppose that this inclusion were a proper one. Then there would exist an element $a \notin \text{rad } A_x$ such that $a \in P_x$ for all $P_x \supseteq A_x$. The powers of a form an m-set M_a which does not meet rad A_x. We, therefore, have a maximal x-ideal P'_x containing A_x and contained in $S - M_a$. This P'_x must be prime, contradicting the fact that P'_x does not coincide with any of the P_x occurring in the intersection (1).

COROLLARY. *For an x-system of finite character the nilpotent radical of an x-ideal is again an x-ideal.*

We shall now give some simple properties of half-prime x-ideals which together with a direct proof of the above corollary will give a second proof of the Krull–Stone theorem. The above corollary may be proved directly in the following way. Let A_x be an x-ideal and let $b_1, \dots, b_n$ be a finite subset of rad A_x. Since x is supposed to be of finite character, it is sufficient to show that $\{b_1, \dots, b_n\}_x \subseteq \operatorname{rad} A_x$. If $b_i^{m_i} \in A_x$ for $i = 1, 2, \dots, n$ we put $m = m_1 + m_2 + \dots + m_n$ and get $\{b_1, \dots, b_n\}_x^m \subseteq (\{b_1, \dots, b_n\}^m)_x \subseteq A_x$.

PROPOSITION 11. *The x-ideal A_x is half-prime if and only if $a^2 \in A_x$ implies $a \in A_x$.*

Proof. Suppose that $a^n \in A_x$ with $n \geqslant 1$. Then also $a^{2^m} = a^{2^m - n} \cdot a^n \in A_x$ for $2^m > n$. By repeated application of the condition $a^2 \in A_x \Rightarrow a \in A_x$ we get $a \in A_x$.

PROPOSITION 12. *The x-ideal A_x is half-prime if and only if $B_x \circ C_x \subseteq A_x$ implies $B_x \cap C_x \subseteq A_x$.*

Proof. Suppose that A_x is half-prime and $B_x \circ C_x \subseteq A_x$. If $a \in B_x \cap C_x$ then $a^2 \in B_x \circ C_x \subseteq A_x$ and $a \in A_x$ by Proposition 11. Conversely if A_x is not half-prime there exists an element a such that $a \notin A_x$ and $a^2 \in A_x$. This gives $(A_x \cup \{a\})_x \circ (A_x \cup \{a\})_x \subseteq A_x$ while $(A_x \cup \{a\})_x \cap (A_x \cup \{a\})_x \nsubseteq A_x$.

We shall say that an x-system is *half-prime* if every x-ideal is half-prime. Ideals in distributive lattices and radical differential ideals form half-prime x-systems.

PROPOSITION 13. *For any half-prime x-system we have the identity $(A \cup B)_x \cap (A \cup C)_x = (A \cup BC)_x$.*

Proof. The inclusion $(A \cup BC)_x \subseteq (A \cup B)_x \cap (A \cup C)_x$ is obvious by observing that the operations of intersection and x-multiplication coincide within the family of x-ideals of a half-prime x-system. Conversely $(A \cup BC)_x$ being half-prime and $(A \cup B)_x \circ (A \cup C)_x = ((A_x \cup B)(A_x \cup C))_x \subseteq (A_x \cup BC)_x = (A \cup BC)_x$ we get $(A \cup B)_x \cap (A \cup C)_x \subseteq (A \cup BC)_x$ by using Proposition 12.

COROLLARY. *For a half-prime x-system we have $(A \cup \{b\})_x \cap (A \cup \{c\})_x = (A \cup \{bc\})_x$.*

An x-ideal A_x is said to be irreducible if $A_x = B_x \cap C_x$ implies $A_x = B_x$ or $A_x = C_x$.

PROPOSITION 14. *In a half-prime x-system an x-ideal is irreducible if and only if it is prime.*

Proof. That a prime x-ideal is irreducible is obvious. Suppose conversely that A_x is not prime. Then there exist elements b and c not contained in A_x such that $b \cdot c \in A_x$ and we get the proper decomposition $A_x = (A \cup \{bc\})_x = (A \cup \{b\})_x \cap (A \cup \{c\})_x$ by the above corollary.

Proposition 14, together with the following two propositions, give a second proof of the general Krull–Stone theorem.

PROPOSITION 15. *If x is of finite character. then any x-ideal is equal to the intersection of all the irreducible x-ideals containing it.*

Proof. Let A_x be an x-ideal $\neq S$ and let b be an element not in A_x. We consider the family of x-ideals which contain A_x but do not contain b. This family is inductive and hence contains at least one maximal member, say B_x. Then B_x is irreducible since any x-ideal which contains B_x properly also contains b.

PROPOSITION 16. *For a given x-system the family of half-prime x-ideals will also form an x-system.*

Proof. We only need to verify the continuity axiom, i.e. to show that $A_x:b$ is half-prime whenever A_x is half-prime. From $c^n \in A_x:b$ we obtain $c^n b \in A_x$ and $(cb)^n = (c^n b)\cdot b^{n-1} \in A_x$, showing that $c \in A_x:b$.

9. *A converse of the Krull–Stone theorem and other converses.* Though certain scattered results of ideal theory may be independent of one or more of the axioms 1, 2 and 3′ it is quite inconceivable that any larger and important parts of ideal theory may be developed without assuming at least these three conditions. As to the necessity of the fourth condition —the continuity axiom—this is perhaps a less transparent question. However, the equivalent forms of the continuity axiom, which were derived in Chapter 1, already give strong evidence for the necessity of this axiom in a large number of situations. In the present section we shall prove a few more converse results which strengthen the conviction that the continuity axiom is indispensable and that the present setting for a general ideal theory is the appropriate one.

We shall use the notation of Chapter 1 and refer to a generalized x-system which satisfies 1, 2 and 3′ but not necessarily 3″, as an x^*-system. By considering the passage $A \to \operatorname{rad} A_x$ instead of $A \to A_x$ it is clear that there is no loss of generality in formulating the Krull–Stone theorem for half-prime x-systems only. We now have the following converse result.

THEOREM 13. *Let x^* be a half-prime x^*-system of finite character in S. Then the necessary and sufficient condition for the validity of the Krull–Stone theorem for x^* (i.e. that any x^*-ideal in S is equal to the intersection of all the prime x^*-ideals containing it) is that x^* satisfies the continuity axiom and hence defines an x-system in S.*

Proof. If x^* satisfies the continuity axiom we have already proved that the Krull-Stone theorem holds. Suppose conversely that the Krull–Stone theorem holds for x^*, i.e. that any x^*-ideal A_{x^*} in S may be written as an intersection of prime x^*-ideals

$$A_{x^*} = \bigcap_{A_{x^*} \subseteq P_{x^*}} P_{x^*}$$

For an arbitrary element $b \in S$ we have

$$A_{x^*}:b = (\bigcap_{A_{x^*} \subseteq P_{x^*}} P_{x^*}):b = \bigcap_{A_{x^*} \subseteq P_{x^*}} (P_{x^*}:b).$$

Because of 3′ and the fact that the P_{x^*}'s are prime, $P_{x^*}:b$ is equal to S or P_{x^*} according to whether $b \in P_{x^*}$ or not. Since S and P_{x^*} are x^*-ideals and any intersection of x^*-ideals is an x^*-ideal we conclude that $A_{x^*}:b$ is an x^*-ideal and the continuity axiom is satisfied.

We now prove another converse of Proposition 13.

THEOREM 14. *If the identity*

$$(A \cup B)_{x^*} \cap (A \cup C)_{x^*} = (A \cup BC)_{x^*} \tag{2}$$

holds for x^* then x^* is half-prime and satisfies the continuity axiom.

Proof. We first show that x^* is half-prime. Putting $B = C = \{b\}$ and $A = A_{x^*}$ we get $(A_{x^*} \cup \{b\})_{x^*} = (A_{x^*} \cup \{b^2\})_{x^*}$. Thus if $b^2 \in A_{x^*}$ then $(A_{x^*} \cup \{b\})_{x^*} = A_{x^*}$ and $b \in A_{x^*}$. By Proposition 11 (which is independent of the continuity axiom) we conclude that x^* is half-prime. Since intersection and x^*-multiplication coincide for half-prime x^*-ideals (2) is equivalent to

$$(A \cup B)_{x^*} \circ (A \cup C)_{x^*} = (A \cup BC)_{x^*}.$$

Taking A to be the empty set we get $B_{x^*} \circ C_{x^*} = B \circ C$ which is one of the equivalent forms of the continuity axiom.

PROPOSITION 17. *Given a half-prime x^*-system, the family of x^*-ideals which can be written as an intersection of prime x^*-ideals will define an x-system and will form a distributive lattice under inclusion. In particular the family of all x-ideals in a half-prime x-system will form a distributive lattice under inclusion.*

Proof. If A_{x^*} is an intersection of prime x^*-ideals, $A_{x^*}:b$ will be of the same form according to the proof of Theorem 13. These intersections, therefore, define a half-prime x-system and will form a distributive lattice with respect to inclusion since the x-multiplication here coincides with the intersection. The second half of the proposition follows from this together with the Krull–Stone theorem.

PROPOSITION 18. *The identities of Proposition* 13 *and its corollary are equivalent for an x-system.*

Proof. As in the proof of Theorem 14 the identity $(A \cup \{b\})_x \cap (A \cup \{c\})_x = (A \cup \{bc\})_x$ implies that x is half-prime[1] and the lattice of x-ideals is hence distributive. We, therefore, have

$$(A \cup B)_x \cap (A \cup C)_x = (\bigcup_{b \in B}{}_x (A \cup \{b\})_x) \cap (\bigcup_{c \in C}{}_x (A \cup \{c\})_x) = \bigcup_{(b,c) \in B \times C}{}_x ((A \cup \{b\})_x \cap (A \cup \{c\})_x).$$

---

[1] *Added in proof.* By Proposition 13 we thus immediately infer that the general identity is valid and the rest of the present proof can be discarded.

Now, using the identity of the corollary, the right hand side is equal to

$$\bigcup_{(b,\,c)\in B\times C}{}_x\,(A\cup\{bc\})_x,$$

which, in turn, is equal to $(A\cup BC)_x$.

We shall say that the x^*-ideal P_{x^*} is *weakly prime* if $A_{x^*}\circ B_{x^*}\subseteq P_{x^*}$ is impossible whenever A_{x^*} and B_{x^*} both contain P_{x^*} properly. The following theorem gives some new properties which are also equivalent to the continuity axiom in the case of half-prime x^*-systems.

THEOREM 15. *If x^* is a half-prime x^*-system of finite character, then the following properties are equivalent:*

A. *x^* satisfies the continuity axiom.*

B. *The Krull–Stone theorem is valid for x^*.*

C. *Every irreducible x^*-ideal is prime.*

D. *Every weakly prime x^*-ideal is prime.*

Proof. The theorem is proved by verifying the following sequence of implications: $\mathbf{A}\Rightarrow\mathbf{D}\Rightarrow\mathbf{C}\Rightarrow\mathbf{B}\Rightarrow\mathbf{A}$. Since a weakly prime x^*-ideal is irreducible, everything follows from what we have already proved.

10. *The non-associative case.* We shall here establish an easy non-associative generalization of the Krull–Stone theorem. When trying to extend a theory to the non-associative case, one needs only to worry about those results which involve considerations of products containing more than two factors. Thus in our case the whole first chapter carries over to the non-associative case. In the present chapter, however, one needs a non-associative generalization of the notion of the nilpotent radical. In the commutative, but non-associative case, a^n does no longer represent a unique element of S when $n\geqslant 4$. In this case we shall let a^n denote the *set* of all elements obtained from the expression $a\cdot a\ldots a$ (n times) by putting parentheses in all possible ways. This suggests two natural non-associative generalizations of the radical: By the *strong radical* of A_x, denoted by $\mathrm{rad}_s A_x$, we shall understand the set of all elements $b\in S$ such that $b^n\cap A_x$ is non-void for some integer n. The *weak radical* of A_x, denoted by $\mathrm{rad}_w A_x$, consists of all $b\in S$ such that $b^n\subseteq A_x$ for some integer n. The following theorem, as well as other facts, show that the former definition should be adopted. The x^*-ideal A_{x^*} is called strongly half-prime if $\mathrm{rad}_s\, A_{x^*}=A_{x^*}$ and x^* is called strongly half-prime if every x^*-ideal is strongly half-prime.

THEOREM 16. *Let S be a commutative, but not necessarily associative, semi-group in which there is defined an x^*-system of finite character. The necessary and sufficient condition* *that any x^*-ideal can be written as an intersection of prime x^*-ideals is that x^* is strongly half-prime and satisfies the continuity axiom.*

The proof of this theorem is almost identical with the proof of the associative Krull–Stone theorem and its converse and can, therefore, be left to the reader. Denoting by $a_1 \cdot a_2 \dots a_n$ the set of all products obtained from this expression by putting parentheses in all possible ways we shall call a subset A of S *associatively closed* if $a_1 \dots a_n \cap A \neq \emptyset$ implies $a_1 \dots a_n \subseteq A$. Since a prime x-ideal is associatively closed we get the following corollaries:

COROLLARY 1. *Any strongly half-prime x-ideal is associatively closed.*

COROLLARY 2. *A weakly half-prime x-system is strongly half-prime if and only if every x-ideal is associatively closed.*

11. *Isolated primary components.* In the remaining sections of this chapter we shall show that almost all of the other results of the Krull theory carry over to general x-systems of finite character. There is one result, however, which is no longer valid in the general setting. It is not generally true that if a prime x-ideal is contained in a finite set-theoretic union of prime x-ideals it is contained in one of the given prime x-ideals. In fact a counter-example is given by the s-system defined by the mapping $A \rightarrow SA \cup A$. Here the set-theoretic union of a family of prime s-ideals is again a prime s-ideal, and this clearly contradicts the given assertion. Apart from this result (which at one place will be used in a weakened form as a postulate) all the basic results carry over to general x-ideals. Most of the proofs in the case of ordinary ideals carry over almost verbatim to the general case. A detailed checking of all the proofs is, of course, necessary, but since this checking is a routine matter and, in general, is of no interest, we shall mostly leave this to the reader. We give, however, a couple of samples of typical proofs which again will show the crucial role played by the continuity axiom. Besides Krull's paper [16] the reader can use [24] as a standard reference.

The x-ideal Q_x is said to be *primary* if $ab \in Q_x$ and $a \notin Q_x$ imply that $b^n \in Q_x$ for some positive integer n.

THEOREM 17. *To every minimal prime x-ideal P_x containing the x-ideal A_x there corresponds a uniquely determined minimal primary x-ideal Q_x, which contains A_x and which has P_x as its radical. This x-ideal Q_x is called the isolated primary x-component of A_x which belongs to P_x.*

Proof. As in the case of rings we prove this by explicit construction of Q_x. In fact Q_x will be identical with the set B of all elements q for which there exists an element $s \in S - P_x$ such that $qs \in A_x$. We first show that B is an x-ideal. Since x is of finite character, it is sufficient to show that $\{q_1, \ldots, q_n\}_x \subseteq B$ whenever $q_1, \ldots, q_n \in B$. By the definition of B we have $q_i s_i \in A_x$ for suitable $s_i \in S - P_x$. Putting $s = s_1 \ldots s_n$ we obtain $q_i s \in A_x$ with $s \in S - P_x$ This gives

$$s\{q_1, \ldots, q_n\}_x \subseteq (s\{q_1, \ldots, q_n\})_x \subseteq A_x$$

showing that $\{q_1, \ldots, q_n\}_x \subseteq B$ and B is an x-ideal. The rest of the proof is of a purely multiplicative character and therefore identical with the proof in the case of rings.

12. *Maximal prime x-ideals belonging to an x-ideal.* Let A_x be an x-ideal which can be represented as a finite intersection of primary x-ideals:

$$A_x = Q_x^{(1)} \cap \ldots \cap Q_x^{(n)}. \tag{1}$$

We shall always suppose that the decomposition (1) is irredundant in the sense that $Q_x^{(i)} \not\supseteq Q_x^{(1)} \cap \ldots \cap Q_x^{(i-1)} \cap Q_x^{(i+1)} \cap \ldots \cap Q_x^{(n)}$ for $i = 1, 2, \ldots, n$. Let $P_x^{(i)}$ denote the radical of $Q_x^{(i)}$. $P_x^{(i)}$ is clearly the unique minimal prime x-ideal containing $Q_x^{(i)}$. If M is an m-set we put $A(M) = \{c;\ cM \cap A_x \neq \emptyset\}$. If $M = S - P_x$ where P_x is a prime x-ideal we shall denote $A(S - P_x)$ by $A(P_x)$. An element $b \in S$ is said to be *non-prime to* A_x if there exists an element $c \notin A_x$ such that $bc \in A_x$. An x-ideal B_x is called non-prime to A_x if every element of B_x is non-prime to A_x. The elements which are prime to A_x form an m-set which does not meet A_x. Hence there will exist maximal x-ideals among the x-ideals which are non-prime to A_x and which contain A_x. The continuity axiom implies that these maximal x-ideals are prime. These maximal x-ideals are called *maximal prime x-ideals belonging to* A_x. It is easy to see that any minimal prime x-ideal containing A_x is non-prime to A_x and hence contained in at least one maximal prime x-ideal belonging to A_x.

The following proposition, which is valid for x-systems of finite character without further restrictions, already shows a part of the unicity we are aiming at.

PROPOSITION 19. *The set of elements non-prime to the x-ideal A_x represented in* (1) *is equal to the set-theoretic union of the maximal prime x-ideals belonging to A_x and is also equal to the set-theoretic union of the prime x-ideals $P_x^{(i)}$ attached to A_x by the decomposition* (1). *The latter union is, therefore, in particular independent of the given primary decomposition.*

Proof. The proof is the same as in the case of rings (see [24], p. 186). The continuity axiom is used when proving that if a is non-prime to A_x then a is contained in some maximal prime x-ideal belonging to A_x. For if $ab \in A_x$ with $b \notin A_x$, then $(A_x \cup \{a\})_x \cdot b \subseteq (A_x \cup \{ab\})_x \subseteq A_x$ and $(A_x \cup \{a\})_x$ is non-prime to A_x. The assertion then follows from Zorn's lemma.

We now generally say that a prime x-ideal P_x *belongs* to A_x if P_x is a maximal prime

x-ideal belonging to $A(P_x)$. The following proposition is also valid for general x-systems of finite character and the proof is identical with the one in case of ordinary ideals.

PROPOSITION 20. *Let M be an m-set in S and suppose that the decomposition* (1) *of A_x is such that $M \cap P_x^{(i)} = \emptyset$ for $i = 1,2,\dots,k$ and*(1) *$M \cap P_x^{(i)} \neq \emptyset$ for $i = k+1,\dots,n$. Then $A(M) = Q_x^{(1)} \cap \dots \cap Q_x^{(k)}$.*

In order to be able to prove the next theorem which is the main result of the Krull theory we must assume that the given x-system is a P-system in the sense of the following:

DEFINITION. An x-system is said to be a *P-system* if an irredundant set-theoretic union of a finite number of at least two prime x-ideals is never a prime x-ideal. A union is irredundant if no term in the union is contained in the union of the remaining terms. Most of the x-systems occurring in the literature are P-systems. In fact the s-system is the only natural x-system which is not a P-system, that comes to mind readily.

THEOREM 18. *Let x be a P-system of finite character and let A_x be an x-ideal admitting the finite primary decomposition* (1). *Then a prime x-ideal P_x is identical to one of the prime x-ideals $P_x^{(i)}$ attached to this decomposition if and only if P_x is a maximal prime x-ideal belonging to $A(P_x)$. In particular the family $P_x^{(1)},\dots,P_x^{(n)}$ is uniquely determined by A_x, and this is valid for any x-system of finite character.*

Proof. We shall establish one half of the theorem by showing that $P_x^{(i)}$ is the unique maximal prime x-ideal belonging to $A(P_x^{(i)})$. Assume that the numbering is chosen in such a way that $P_x^{(i)} \supseteq P_x^{(j)}$ for $j = 1, 2, \dots, i$ and $P_x^{(i)} \not\supseteq P_x^{(j)}$ for $j = i+1, \dots, n$. Proposition 20 then gives $A(P_x^{(i)}) = Q_x^{(1)} \cap \dots \cap Q_x^{(i)}$. It is clearly sufficient to show that $P_x^{(i)}$ just consists of all the elements of S which are non-prime to $A(P_x^{(i)})$. Any element outside $P_x^{(i)}$ is obviously prime to $A(P_x^{(i)})$. Suppose that $a \in P_x^{(i)}$ and, therefore, that $a^m \in Q_x^{(i)}$ for some integer m. Since the decomposition (1) is irredundant there is an element $b \in Q_x^{(1)} \cap \dots \cap Q_x^{(i-1)}$ such that $b \notin Q_x^{(i)}$. This gives $a^m b \in A(P_x^{(i)})$. If m_0 is chosen to be such that $a^{m_0} b \in A(P_x^{(i)})$ and $a^{m_0-1} b \notin A(P_x^{(i)})$ then we put $c = b$ if $m_0 = 1$ and $c = a^{m_0-1} b$ if $m_0 > 1$. In either case $ac \in A(P_x^{(i)})$ with $c \notin A(P_x^{(i)})$ and a is non-prime to $A(P_x^{(i)})$.

Suppose conversely that P_x is a maximal prime x-ideal belonging to $A(P_x)$. We must show that $P_x = P_x^{(i)}$ for some i. Certainly $P_x \supseteq P_x^{(i)}$ for at least one i. Assume now that the numbering in (1) is such that $P_x \supseteq P_x^{(j)}$ for $j = 1, 2, \dots, k$ and $P_x \not\supseteq P_x^{(j)}$ for $j = k+1, \dots, n$. Then $A(P_x) = Q_x^{(1)} \cap \dots \cap Q_x^{(k)}$ by Proposition 20.

(1) There are two obvious conventions to be made when one of these two possibilities does not occur.

Suppose that $P_x^{(j)}$ were properly contained in P_x for $j=1, 2, \dots, k$. Since x is a P-system this would imply the existence of an element $p \in P_x$ such that $p \notin P_x^{(j)}$ for $j=1, 2, \dots, k$. Since P_x is a maximal prime x-ideal belonging to $A(P_x)$ we would have an element $b \notin A(P_x)$ such that $bp \in A(P_x)$. By (1) $b \notin Q_x^{(i)}$ for some i and this together with $p \notin P_x^{(i)}$ contradicts the fact that $Q_x^{(i)}$ is primary.

The fact that the family $P_x^{(1)}, \dots, P_x^{(n)}$ is uniquely determined by A_x can also be proved in the same way as in [33, p. 76–77]. This proof is valid for any x-system of finite character and thus establishes the last statement of the theorem.

CHAPTER 3

x-Noetherian and x-Dedekindian semi-groups

13. *x-Noetherian semi-groups.* Let S be a commutative semi-group in which there is defined an x-system of finite character. We shall say that S is *x-Noetherian* if the following two conditions are satisfied:

I. S satisfies the ascending chain condition for x-ideals,

II. Every irreducible x-ideal is primary.

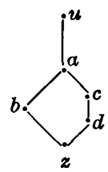
Since x is supposed to be of finite character, we can give various equivalent formulations of I which are well known from ordinary ideal theory in rings. In particular, I is equivalent to the fact that any x-ideal is finitely generated. By means of I one concludes that any x-ideal in S can be represented as a finite intersection of irreducible x-ideals—and hence by II as a finite intersection of primary x-ideals. Such a primary decomposition can be given a normal form called a *shortest* representation by first omitting any primary component containing the intersection of the other components and then grouping together primary x-ideals having one and the same radical. Indeed, as in ordinary ideal theory, one proves that a finite intersection of primary x-ideals is a primary x-ideal if and only if all the components have the same radical. From this and the last statement of Theorem 18 we obtain immediately the following:

THEOREM 19. *Every x-ideal A_x in an x-Noetherian semi-group has a shortest representation as an intersection of primary x-ideals. The family of the prime x-ideals consisting of the radicals of the primary x-ideals occurring in one such decomposition is independent of the given decomposition and hence is uniquely determined by the given x-ideal A_x.*

We know that for ordinary ideals in rings—here called d-ideals—assumption II is a consequence of I. It is clear, however, from the examples of half-prime x-systems that II may be satisfied also in cases where I fails to hold. We shall now show that II is, in general, not a consequence of I.

THEOREM 20. *There exists a semi-group S and an x-system of finite character in S such that the ascending chain condition for x-ideals is satisfied but S is not x-Noetherian.*

Proof. In order to prove this we shall use the m-system in a quasi-integral m-lattice L. By an m-lattice we mean a lattice with a binary multiplication satisfying $a(b \cup c) = ab \cup ac$. The multiplication is here supposed commutative. That L is quasi-integral means that $ab \leqslant b$ for all a and $b \in L$. Consider now a subset A of L and the mapping $A \to A_m$ where A_m consists of all elements c such that $c \leqslant a_1 \cup \dots \cup a_n$ with $a_1, \dots, a_n \in A$. It is easy to verify that the mapping $A \to A_m$ defines an x-system of finite character in L considered as a semi-group under the given multiplication. This x-system is called the m-system in L. (For more details see Chapter 5.) We now consider the following finite m-lattice (see [6], pp. 350–51),



where the multiplication is defined according to the following table:

	u	a	b	c	d	z
u	u	a	b	c	d	z
a	a	b	b	z	z	z
b	b	b	b	z	z	z
c	c	z	z	z	z	z
d	d	z	z	z	z	z
z	z	z	z	z	z	z

One verifies easily that this defines a quasi-integral m-lattice in which the ascending chain condition for m-ideals is satisfied. In fact, every m-ideal is principal in this case. The m-ideal $\{d\}_m$ is clearly irreducible. But it is not primary since $bc \in \{d\}_m$, $c \notin \{d\}_m$ and $b^n \notin \{d\}_m$ for all n.

We have not been able to find any simple and natural sufficient condition to impose on the given x-system such that I would imply II. To have such a condition is, however, not very essential since the verification of II is in most cases very simple and follows the lines of the proof in the classical case $x = d$. This is, for instance, the case for $x = s$.

Before proceeding with the discussion of the other decomposition theorems of E. Noether we ought to give some comments on the expression "prime x-ideal belonging to A_x". According to Theorem 18 it is possible that for a general x-system the family of prime x-ideals $P_x^{(1)},\ldots,P_x^{(n)}$ attached to A_x by any primary decomposition $A_x=Q_x^{(1)}\cap\ldots\cap Q_x^{(n)}$ with rad $Q_x^{(i)}=P_x^{(i)}$ might be different from the family of prime x-ideals belonging to A_x. What we have shown is that the two families coincide in the case of a P-system, but we have been unable to decide whether this is the case or not for general x-systems of finite character. From now on we shall have no more occasion to speak of "prime x-ideals belonging to A_x" in the intrinsic sense of Krull and we shall now by this expression always mean one of the $P_x^{(i)}$ attached to a shortest decomposition of A_x.

We shall say that the x-ideals A_x and B_x in the x-Noetherian semi-group S are *relatively prime* if $A_x:B_x=A_x$ and $B_x:A_x=B_x$. One proves that A_x and B_x are relatively prime if and only if there exists no inclusion relation between a prime x-ideal belonging to A_x and a prime x-ideal belonging to B_x. From a shortest primary decomposition we therefore obtain a decomposition by mutually relatively prime x-ideals by grouping together primary x-ideals whose corresponding prime x-ideals are related by an inclusion relation. It follows as in ordinary ideal theory that this decomposition is unique.

We shall now treat a stronger form of relative primeness which will lead to another decomposition theorem of E. Noether. In van der Waerden's book [33, pp. 80–83] this theory is developed in terms of the elements of the given ring and makes constant use of the property that the d-ideal generated by the two d-ideals A_d and B_d is the whole ring R if and only if there exist elements $a\in A_d$ and $b\in B_d$ such that $a+b=1$. (R is supposed to have an identity.) We shall here show that this theory may be freed from these special arguments and is valid for general x-systems of finite character.

We now suppose that S is an x-Noetherian semi-group with an identity element e. The x-ideals A_x and B_x are said to be *coprime* if $A_x\cup_x B_x=(A_x\cup B_x)_x=S$.

PROPOSITION 21. *Let $Q_x^{(1)}$ and $Q_x^{(2)}$ be two primary x-ideals such that $P_x^{(1)}=$ rad $Q_x^{(1)}$ and $P_x^{(2)}=$ rad $Q_x^{(2)}$. If $P_x^{(1)}$ and $P_x^{(2)}$ are coprime then $Q_x^{(1)}$ and $Q_x^{(2)}$ are also coprime.*

Proof. Since S is x-Noetherian we have $(P_x^{(1)})^{n_1}\subseteq Q_x^{(1)}$ and $(P_x^{(2)})^{n_2}\subseteq Q_x^{(2)}$ for suitable integers n_1 and n_2. Using the continuity axiom we obtain

$$S=S^{n_1+n_2}=(P_x^{(1)}\cup_x P_x^{(2)})^{n_1+n_2}=(P_x^{(1)})^{n_1+n_2}\cup_x(P_x^{(1)})^{n_1+n_2-1}\cdot P_x^{(2)}\cup_x\ldots\cup_x(P_x^{(2)})^{n_1+n_2}.$$

The right-hand side is contained in $Q_x^{(1)}\cup_x Q_x^{(2)}$ and we get $Q_x^{(1)}\cup_x Q_x^{(2)}=S$ as desired.

PROPOSITION 22. *If A_x and B_x are coprime then they are also relatively prime.*

Proof. Assume that $c\in A_x:B_x$ and that A_x and B_x are coprime. Then $cS=c(A_x\cup B_x)_x\subseteq (c(A_x\cup B_x))_x\subseteq A_x$ and $c\in A_x$ since S has an identity. In the same way $B_x:A_x=B_x$.

PROPOSITION 23. $(A_x \cap B_x)\circ(A_x \cup_x B_x) \subseteq A_x \circ B_x$.

Proof. $(A_x \cap B_x)\circ(A_x \cup_x B_x) = ((A_x \cap B_x)\circ A_x) \cup_x ((A_x \cap B_x)\circ B_x) \subseteq A_x \circ B_x$.

COROLLARY 1. *If A_x and B_x are coprime then $A_x \circ B_x = A_x \cap B_x$.*

PROPOSITION 24. *If A_x and B_x as well as A_x and C_x are coprime then also A_x and $B_x \circ C_x$ as well as A_x and $B_x \cap C_x$ are coprime.*

Proof. From $A_x \cup_x B_x = A_x \cup_x C_x = S$ we obtain

$$S = S^2 = (A_x \cup_x B_x)\circ(A_x \cup_x C_x) \subseteq (A_x \cup_x (B_x \circ C_x)),$$

showing that $A_x \cup_x (B_x \circ C_x) = S = A_x \cup_x (B_x \cap C_x)$.

COROLLARY 2. *If A_x and $B_x^{(i)}$ are coprime for each $i = 1, 2, \ldots, n$, then A_x and $B_x^{(1)}\circ \ldots \circ B_x^{(n)}$ are also coprime.*

PROPOSITION 25. *The intersection of a finite number of mutually coprime x-ideals is equal to their x-product.*

Proof. We use induction. In the case of two components the proposition is true by Corollary 1. Suppose, therefore, that $A_x^{(1)}, \ldots, A_x^{(n)}$ are n, $(n \geqslant 3)$ mutually coprime x-ideals such that

$$A_x^{(1)} \cap A_x^{(2)} \cap \ldots \cap A_x^{(n-1)} = A_x^{(1)}\circ A_x^{(2)}\circ \ldots \circ A_x^{(n-1)}.$$

By Corollaries 1 and 2 we obtain

$$A_x^{(1)} \cap \ldots \cap A_x^{(n)} = (A_x^{(1)}\circ \ldots \circ A^{(n-1)}) \cap A_x^{(n)} = A_x^{(1)}\circ \ldots \circ A_x^{(n)}.$$

showing that the proposition is valid for all $n \geqslant 2$.

In the following theorem we use the term *indecomposable* to mean that an x-ideal cannot be written as an intersection of two coprime x-ideals.

THEOREM 21. *Any x-ideal in an x-Noetherian semi-group with an identity element can be written uniquely as an x-product of a finite number of indecomposable and mutually coprime x-ideals.*

With the above preparations, the proof of this theorem goes exactly as in the case $x = d$; see [33, pp. 82–83]. The following theorem is now also an immediate consequence.

THEOREM 22. *Let S be an x-Noetherian semi-group with an identity element such that any prime x-ideal is maximal. Then any x-ideal in S can be written uniquely as an x-product of a finite number of primary x-ideals which are mutually coprime.*

14. *Fractionary x-ideals.* In order to generalize the theory of Dedekindian rings to x-systems we shall need the notion of a fractionary x-ideal.

Let S be a commutative semi-group with identity element e. The element $a \in S$ is said to be *regular* if $ab = ac$ implies $b = c$. The regular elements of S form a subsemigroup S_0 of S which contains e and in which the cancellation law holds. A subset of S is called *regular* if it contains at least one regular element. We now consider the set S' of formal quotients a/b where $b \in S_0$ and we define an equivalence relation $\sim$ in S' by putting $a/b \sim c/d$ whenever $ad = bc$. The relation $\sim$ is a congruence relation with respect to the multiplication $a_1/b_1 \cdot a_2/b_2 = a_1 a_2/b_1 b_2$ and the quotient $S^* = S'/\sim$ is a semi-group under the corresponding multiplication of equivalence classes. The semi-group S^* is called the semi-group of quotients of S and it contains a subsemigroup isomorphic with S. We note that a/b is regular in S^* if and only if both a and b are regular in S.

A subset A of S^* is called *fractionary* (or *bounded*) if there exists a regular element m such that $mA \subseteq S$. Such an element m is called a *multiplier* for A. If $A \subseteq S$ we call A *integral.* Any integral set is fractionary. We now say that there is defined a *fractionary x-system* in S (or in S^*) if there corresponds to any fractionary subset A of S^* a subset A_x of S^* such that

1. $A \subseteq A_x$.
2. $A \subseteq B_x \Rightarrow A_x \subseteq B_x$.

3′. $SB_x \subseteq B_x$.

3″. $aB_x \subseteq (aB)_x$.

4. $S_x = S$.

It is easy to verify that the operations we are going to perform within the family of fractionary regular sets, namely the closure operation $A \to A_x$, the multiplications $A \cdot B$ and $A \circ B$, and the residuation $A : B$ do not lead out of this family.

In Paragraph 6 we introduced the notion of trace of an integral x^*-system. If we have an (integral) x-system in S then its trace on a subsemigroup T will be an x-system x_T in T. In the case of fractionary x-systems we have the following:

THEOREM 23. *Let S_0 denote the semi-group of regular elements of S and let S_0^* and S^* denote the semi-group of quotients of S_0 and S respectively. For a given fractionary x-system x in S^* (relative to S) the family of all traces $A_x \cap S_0^*$ defines a fractionary x-system in S_0^* (relative to S_0). This x-system, which will be denoted by x_0 is called the regular fractionary x-system induced by x. The x_0-operation is given explicitly by $(A \cap S_0^*)_{x_0} = (A \cap S_0^*)_x \cap S_0^*$.*

Proof. We first note that the fractionary sets of S_0^* with respect to S_0 are just the sets of the form $A \cap S_0^*$ where A is a fractionary set of S^* with respect to S. Indeed, if $mA \subseteq S$ then $m(A \cap S_0^*) \subseteq mA \cap mS_0^* \subseteq S \cap S_0^* = S_0$. The explicit formula in the proposition is obvious and we use this formula in order to verify that x_0 defines a fractionary x-system in S_0^* relative to S_0. It is sufficient to verify the axioms 3′, 3″ and 4.

3'. $S_0(A \cap S_0^*)_{x_s} = S_0[(A \cap S_0^*)_x \cap S_0^*] \subseteq S(A \cap S_0^*)_x \cap S_0 S_0^* = (A \cap S_0)_{x_s}$.

3''. $(A \cap S_0^*)_{x_s} : c = [(A \cap S_0^*)_x \cap S_0^*] : c = ((A \cap S_0^*)_x : c) \cap (S_0^* : c)$
$= B_x \cap S_0^* = (B_x \cap S_0^*)_{x_s}$.

4. $(S_0)_{x_s} = (S_0)_x \cap S_0^* \subseteq S_x \cap S_0^* = S \cap S_0^* = S_0$.

We shall later see that this notion of induced regular x-system will give the link between the present theory and the more special one considered by Prüfer–Krull–Lorenzen.

15. *x-Dedekindian semi-groups.* We use the notations of the preceding paragraph. In particular S will denote a commutative semi-group with an identity element. We shall also suppose that all the x-ideals considered in this paragraph are regular, i.e. contain at least one regular element. An x-ideal is said to be *proper* if it is different from S. S is called *x-Dedekindian* if there is defined a fractionary x-system in S which satisfies the following conditions:

I. S satisfies the ascending chain condition for integral x-ideals

II. Every integral and proper prime x-ideal is maximal

III. S is integrally x-closed in the sense that $A_x : A_x = S$ for any integral x-ideal A_x.

It was shown by Prüfer [27] that III. coincides with the ordinary notion of integral closure in the case $x = d$.

We shall now prove the unique factorization theorem for x-ideals in x-Dedekindian semi-groups following closely the classical development of van der Waerden ([33], pp. 125–130). We note that assuming only the ascending chain condition for integral x-ideals and not that S is x-Noetherian we have to give a direct development independent of the theory of x-Noetherian semi-groups. We obtain, however, as a corollary, that an x-Dedekindian semi-group is x-Noetherian. From now on, all x-ideals which are not explicitly specified as fractionary will all be integral x-ideals, i.e. contained in S.

LEMMA 1. *To any x-ideal A_x in S we can find prime x-ideals $P_x^{(1)}, \dots, P_x^{(n)}$ such that $A_x \subseteq P_x^{(i)}$ for $i = 1, 2, \dots, n$ and $P_x^{(1)} \circ \dots \circ P_x^{(n)} \subseteq A_x$.*

Proof. We here use the divisor induction which is one of the equivalent forms of the ascending chain condition (see [33, p. 63]). S obviously has the property announced in the lemma. Let therefore $A_x \neq S$. If A_x is prime there is nothing to prove. If A_x is not prime there exist, according to the corollary of Proposition 9, two x-ideals B_x and C_x containing A_x properly and such that $B_x \circ C_x \subseteq A_x$. We now use the divisior induction in order to complete the proof of the lemma.

LEMMA 2. *If P_x is a prime x-ideal properly contained in S then $P_x^{-1} = S : P_x \nsubseteq S$.*

Proof. Let c be a regular element in P_x. By Lemma 1 there exists a product of prime x-ideals such that

$$P_x^{(1)} \circ \ldots \circ P_x^{(n)} \subseteq \{c\}_x \subseteq P_x \tag{1}$$

and we assume that this product is a shortest one, i.e. that no product of $n-1$ prime x-ideals is contained in $\{c\}_x$. (1) implies by a suitable numbering that $P_x^{(1)} \subseteq P_x$ and therefore $P_x^{(1)} = P_x$ because of II. Hence

$$P_x \circ P_x^{(2)} \circ \ldots \circ P_x^{(n)} \subseteq \{c\}_x$$

and

$$P_x^{(2)} \circ \ldots \circ P_x^{(n)} \nsubseteq \{c_x\}.$$

There exists, therefore, an element $b \in P_x^{(2)} \circ \ldots \circ P_x^{(n)}$ such that $b \notin \{c\}_x$ and $P_x \cdot b \subseteq \{c\}_x$. Since c is regular, we can here multiply on both sides with c^{-1} and obtain

$$P_x \cdot b/c \subseteq \{e\}_x = S_x = S.$$

This means that $b/c \in S:P_x$. If $b/c \in S$ then $b \in Sc \subseteq \{c\}_x$ contrary to the choice of b. Thus $b/c \notin S$ and $S:P_x \nsubseteq S$.

LEMMA 3. *For any prime x-ideal P_x in S we have $P_x \circ P_x^{-1} = S$.*

Proof. Since $S(S:S) = S$ we can suppose that $P_x \neq S$. We have $S \subseteq P_x^{-1}$ and therefore $P_x = S \cdot P_x \subseteq P_x \circ P_x^{-1}$. By II this gives two possibilities, either $P_x \circ P_x^{-1} = P_x$ or $P_x \circ P_x^{-1} = S$. The former possibility is, however, excluded since $P_x \circ P_x^{-1} = P_x$ is in contradiction with the conjunction of Lemma 2 and Axiom III.

From this point on, the proof of van der Waerden carries over verbatim to the case of x-ideals. We have thus established the first half of the following

THEOREM 24. *In an x-Dedekindian semi-group any regular x-ideal can be written as a finite x-product of regular prime x-ideals. If $A_x \subseteq B_x$ and $A_x = P_x^{(1)} \circ \ldots \circ P_x^{(n)}$, $B_x = Q_x^{(1)} \circ \ldots \circ Q_x^{(m)}$ are two such decompositions of A_x and B_x respectively then any prime x-ideal $Q_x^{(i)} \neq S$ occurring in the factorization of B_x will occur at least as many times in the factorization of A_x as it does in that of B_x. Conversely such an existence and unicity statement implies that S satisfies* I *and* II *and the following weakened form of* III: $(A_x : A_x) \cap S_0^* = S_0$ *(which reduces to* III *in case S is regular).*

For the proof of the second half of Theorem 24 it is convenient to use the following immediate result.

PROPOSITION 26. *Let S be a semi-group satisfying the existence and unicity statement of Theorem 24. Then the following properties hold in S.*

A. *Any regular x-ideal in an x-Dedekindian semi-group S can be written uniquely as a finite product of prime x-ideals.*

B. *The regular x-ideals in S form a semi-group with cancellation law with respect to x-multiplication.*

C. *The inclusion $A_x \subseteq B_x \subseteq S$ implies the existence of an x-ideal $C_x \subseteq S$ such that $A_x = B_x \circ C_x$.*

Proof of the second half of Theorem 24. This proof is the same as in [33, pp. 129–130] except for the weakened condition III which here may be proved as follows: Suppose that $a/b \in A_x : A_x$ where a and b are regular and A_x is integral. Thus $a/b \cdot A_x \subseteq A_x$ or

$$aA_x \subseteq bA_x. \tag{2}$$

Using the continuity axiom (Theorem 1) we obtain

$$\{a\}_x \circ A_x = \{a\} \circ A_x = (aA_x)_x$$

which combined with (2) gives

$$\{a\}_x \circ A_x = (aA_x)_x \subseteq (bA_x)_x = \{b\}_x \circ A_x.$$

By Proposition 26 **C** we therefore have an x-ideal $C_x \subseteq S$ such that

$$\{a\}_x \circ A_x = \{b\}_x \circ A_x \circ C_x$$

and we conclude according to Proposition 26 **B** that

$$\{a\}_x = \{b\}_x \circ C_x. \tag{3}$$

x-multiplication on both sides of (3) by the fractionary ideal $\{e/b\}_x$ gives

$$\{a/b\}_x = \{e\}_x \circ C_x = C_x \subseteq S.$$

This proves that $a/b \in S$ and S satisfies the weakened form of III.

We have above been able to prove the equivalence of the conjunction of the three classical Noetherian conditions and the strong unicity statement of Theorem 24 under assumptions which are somewhat weaker than those of the Prüfer–Krull–Lorenzen theory. In fact we have not assumed that their axiom 2.3 (Paragraph 2)

$$\{a\}_x = Sa$$

is satisfied. If x satisfies 2.3 for any regular element a we shall say that x is *principal*. Though we did not assume this property at the outset, we shall see that it is actually a consequence of x being fractionary. We shall say that x has the *group property* if the regular fractionary x-ideals form a group under x-multiplication.

PROPOSITION 27. *If S is a semi-group which satisfies the group-property for x, then the following statements are valid*

A. $A_x \subseteq B_x \subseteq S \Rightarrow A_x : B_x \subseteq S$.

B. $A_x : A_x = S$.

Proof. Since $A_x \subseteq B_x$ implies that $A_x : B_x \subseteq A_x : A_x$ it is sufficient to prove **B**. Put $A_x : A_x = C_x$. Then $A_x \circ C_x \subseteq A_x$ which together with $S \subseteq C_x$ and

$$A_x \circ S = A_x \tag{1}$$

imply that

$$A_x \circ C_x = A_x. \tag{2}$$

Using the group property on (1) and (2) we infer that $S = C_x$ and hence $A_x : A_x = S$.

For the proof of the next theorem we shall also need the following simple generalization of a well-known theorem of Krull.

PROPOSITION 28. *If x is of finite character, then any invertible fractionary x-ideal is finitely generated.*

Proof. Let A_x be an invertible x-ideal, i.e. $A_x \circ B_x = S$ for some fractionary x-ideal B_x. Thus $e \in (A \cdot B)_x$ and since x is of finite character we can find two finite subsets N and M of A and B respectively such that $e \in (N \cdot M)_x$. Hence $N_x \circ B_x = S$ and $A_x = N_x$ by the uniqueness of the inverse, showing that A_x is finitely generated.

THEOREM 25. *A semi-group is x-Dedekindian if and only if x has the group property.*

Proof. Suppose that S is x-Dedekindian and let A_x be a fractionary x-ideal of S such that $bA_x \subseteq S$ with $b \in S$. This gives $\{b\}_x \circ A_x = (bA_x)_x \subseteq S$ and $\{b\}_x \circ A_x = C_x$ is thus an integral x-ideal. If $C_x = P_x^{(1)} \circ \ldots \circ P_x^{(n)}$ we conclude by Lemma 3 that the fractionary x-ideal $\{b\}_x \circ (P_x^{(1)})^{-1} \circ \ldots \circ (P_x^{(n)})^{-1}$ is the inverse of A_x. Assume conversely that x has the group property. We shall prove that S has the three properties which define an x-Dedekindian semi-group. Condition I is a consequence of Proposition 28. Suppose next that P_x is a prime x-ideal and $P_x \subset A_x \subseteq S$ where the first inclusion is supposed to be proper. By the group property we have a fractional x-ideal C_x such that $P_x = A_x \circ C_x$ where C_x is integral according to Proposition 27 **A**. The primeness of P_x then implies that $C_x = P_x$ and $A_x = S$ which shows that P_x is maximal. That S is integrally x-closed is an immediate consequence of Proposition 27 **B**.

As promised at the end of section 14 we shall now prove a simple theorem which gives the link between the present theory and the more special one considered by Prüfer–Krull–Lorenzen.

THEOREM 26. *The regular fractionary x-system induced on the group S_0^* by a fractionary x-system in S is an r-system in the sense of Lorenzen.*

Proof. Since S_0^* is a group it follows from Proposition 3 that Lorenzen's axiom 2.4 is satisfied for the induced x-system in S_0^* (see also Theorem 23). Suppose next that $b \in \{a\}_x$ where a is a regular element of S^*, i.e. $a \in S_0^*$. Then $\{b\}_x \subseteq \{a\}_x$ and $\{b\}_x \circ \{a^{-1}\}_x \subseteq \{a\}_x \circ \{a^{-1}\}_x = \{e\}_x = S$. This implies $ba^{-1} \in S$ or $b \in Sa$ which means that x is principal and this proves the theorem.

CHAPTER 4

Structure spaces of x-ideals

16. *The Stone topology for maximal x-ideals.* It is rather remarkable that ideals in rings were not first studied as kernels of ring homomorphisms. This property, however, is as one should expect, fundamental in most of the applications of ideal theory, and accounts in particular for the success of various concepts of ideal in functional analysis. In fact the use of maximal ideals of various types in problems concerning functional representation stems from the fact that the quotient algebra modulo different kinds of maximal ideals takes on simple forms, such as the additive group of the reals, the algebra of complex numbers, the two-element Boolean algebra, etc. Because of the difficulty in defining a congruence and a quotient modulo a general x-ideal, it does not seem to be easy to develop a general representation theory based on x-ideals. There are, however, various basic questions in this domain which have been treated separately for several types of ideals, where the theory of x-ideals admits a simple unified treatment. We shall here content ourselves with illustrating this in connection with certain facts in the theory of structure spaces.

We suppose in this section that S is a commutative semi-group equipped with an x-system of finite character such that S has an *x-identity* satisfying the two conditions

$$(1)\ e \in S^2 \quad \text{and} \quad (2)\ \{e\}_x = S.$$

These two conditions are satisfied for a multiplicative identity in a ring or semi-group ($x = d$ and $x = s$ respectively) and for a positive order unit (archimedian element) in a lattice ordered algebraic system with respect to the semi-group operation $a \circ b = |a| \cap |b|$ (see the c-systems treated in section 22). In the family $\mathfrak{M}$ of maximal x-ideals in S we can introduce a closure operation by defining the closure $\overline{\mathfrak{N}}$ of $\mathfrak{N} \subseteq \mathfrak{M}$ by $\overline{\mathfrak{N}} = \{M_x, M_x \supseteq \bigcap_{M_x^{(i)} \in \mathfrak{N}} M_x^{(i)}\}$. We write $\ker \mathfrak{N} = \bigcap_{M_x^{(i)} \in \mathfrak{N}} M_x^{(i)}$ and hull $M_x^0 = \{M_x, M_x \supseteq M_x^0\}$, so that $\overline{\mathfrak{N}} = \text{hull}\ (\ker \mathfrak{N})$.

THEOREM 27. *The closure operation* $\mathfrak{N} \to \text{hull}(\ker \mathfrak{N}) = \overline{\mathfrak{N}}$ *defines a topology in* $\mathfrak{M}$.

Proof. The fact that we really have a closure operation is clear and so is the inclusion $\overline{\mathfrak{N}}_1 \cup \overline{\mathfrak{N}}_2 \subseteq \overline{\mathfrak{N}_1 \cup \mathfrak{N}_2}$. It is therefore enough to show that $\overline{\mathfrak{N}_1 \cup \mathfrak{N}_2} \subseteq \overline{\mathfrak{N}}_1 \cup \overline{\mathfrak{N}}_2$. Suppose that $M_x^0 \in \overline{\mathfrak{N}_1 \cup \mathfrak{N}_2}$, but $M_x^0 \notin \overline{\mathfrak{N}}_1 \cup \overline{\mathfrak{N}}_2$. Then we would have elements $a, b \in S$ such that $a \notin M_x^0, b \notin M_x^0$ and $a \in \ker \mathfrak{N}_1$, $b \in \ker \mathfrak{N}_2$. Since M_x^0 is maximal and S has an x-identity, we infer from the corollary of Proposition 9 that M_x is prime; hence $ab \notin M_x^0$. But, on the other hand, $ab \in \ker(\mathfrak{N}_1 \cup \mathfrak{N}_2)$, showing that ab belongs to every M_x in $\overline{\mathfrak{N}_1 \cup \mathfrak{N}_2}$. Thus also $ab \in M_x^0$, and this gives the desired contradication.

The above topology is called the *Stone topology*, and $\mathfrak{M}$ equipped with the Stone topology is called the *x-structure space* of S.

THEOREM 28. *The x-structure space of* S *is a compact* T_1*-space.*

Proof. That $\mathfrak{M}$ is a T_1-space is obvious. In order to show compactness, let $\{\mathfrak{F}_i\}_{i \in I}$ be a family of closed sets in $\mathfrak{M}$ with the finite intersection property. We must show that $\bigcap_{i \in I} \mathfrak{F}_i \neq \emptyset$. Let us assume that $\bigcap_{i \in I} \mathfrak{F}_i = \emptyset$, and put

$$A_x = \bigcup_{i \in I}{}_x \ker \mathfrak{F}_i. \tag{1}$$

If $A_x \neq S$ we must have $A_x \subseteq M_x^0$ for a suitable $M_x^0 \in \mathfrak{M}$, $\ker \mathfrak{F}_i \subseteq A_x \subseteq M_x^0$ for all $i \in I$, and therefore $M_x^0 \in \overline{\mathfrak{F}}_i = \mathfrak{F}_i$ for all i, a contradiction of our assumption. Therefore $A_x = S$ and the x-identity e belongs to A_x. Since x is of finite character we deduce from (1) that $e \in \bigcup_{i \in J}{}_x \ker \mathfrak{F}_i$ where J is a finite subset of I, say $J = \{i_1, \dots, i_n\}$. This implies $\bigcap_{k=1}^{n} \mathfrak{F}_{i_k} = \emptyset$. For if $M_x \in \bigcap_{k=1}^{n} \mathfrak{F}_{i_k}$, then $M_x \in \overline{\mathfrak{F}}_{i_k} = \mathfrak{F}_{i_k}$ for $k = 1, 2, \dots, n$, i.e. $M_x \supseteq \ker \mathfrak{F}_{i_k}$ for $k = 1, \dots, n$ and $S = \{e\}_x \subseteq \bigcup_{k=1}^{n}{}_x \ker \mathfrak{F}_{i_k} \subseteq M_x$, which establishes the desired contradiction.

17. *Characteristic function semi-groups.* It is well-known that the topology of a compact Hausdorff space X can be characterized in various ways up to a homeomorphism by the algebraic structure of subfamilies of the family $C(X)$ of all complex-valued continuous functions on X. Let us cite just two examples. Gelfand and Kolmogoroff [9] proved that if for two compact Hausdorff spaces X and Y the rings $C(X)$ and $C(Y)$ are isomorphic, then X and Y are homeomorphic. Stone [32] proved that if $C^R(X)$ and $C^R(Y)$ are isomorphic as lattice-ordered additive groups where $C^R(X)$ is the set of all real-valued functions in $C(X)$, then X and Y are homeomorphic. The operations and order referred to here are all the pointwise ones. We shall now show how these two theorems as well as other special cases can be derived from a general theorem by using the Stone topology for x-ideals.

Let X be a topological space and let $S(X)$ be a semi-group of complex-valued continuous functions on X, equipped with an x-system of finite character, such that $S(X)$ has an x-identity. For the moment the semi-group operation in $S(X)$ is completely unspecified. In special cases it can, for instance, be pointwise multiplication, pointwise addition, or the operation $f \circ g = |f| \cap |g|$. We now make the following additional assumptions about $S(X)$:

I. *$S(X)$ separates points in X.*

II. *$S(X)$ separates points and closed sets in X.*

III. *The set $\{f \mid f(a)=0\}$ is a maximal x-ideal $M_x^{(a)}$ in $S(X)$ for a given point $a \in X$, and there are no other maximal x-ideals in $S(X)$ if X is compact.*

I means that for $a \neq b$ there exists an $f \in S(X)$ such that $f(a)=0$ and $f(b) \neq 0$. The meaning of II is that for every closed set $F \subseteq X$ and every point a not in F, there exists $f \in S(X)$ such that $f \equiv 0$ on F and $f(a) \neq 0$. If $S(X)$ satisfies all the conditions mentioned so far, we shall call it a *characteristic function semi-group* for the space X. Condition I implies immediately that X is Hausdorff, and this latter property could have been used as an assumption instead of I. In fact it is sufficient to assume that X is a T_0-space in order that II imply I. But I is the relevant formulation for our purposes because of the following obvious

PROPOSITION 29. *The condition* I *is necessary and sufficient in order that the mapping $a \to M_x^{(a)}$ of X into M be injective.*

The two other conditions have a similar purpose with respect to the mapping $a \to M_x^{(a)}$, namely, to assure that it be a homeomorphism and surjective in case X is compact. Indeed, we have the following theorem where $\mathfrak{M}$ denotes the x-structure space of $S(X)$.

THEOREM 29. *A necessary and sufficient condition that the mapping $a \to M_x^{(a)}$ be a homeomorphism of X onto $\mathfrak{M}$ is that X be a compact Hausdorff space.*

Proof. Assume first that X is a compact Hausdorff space. From the above proposition and condition III we know that $a \to M_x^{(a)}$ is a bijection. The proof that we actually have a homeomorphism is identical with the proof of Theorem 19F, p. 57 in [20]. Conversely if $a \to M_x^{(a)}$ is a homeomorphism of X onto $\mathfrak{M}$, it follows from Theorem 28 that X is compact, and that it is Hausdorff is a consequence of I.

If S and T are two semi-groups each equipped with an x-system denoted respectively by y and z, we defined in section 5 a multiplicative homomorphism φ of S into T to be a

(y,z)-homomorphism if $\varphi(A_y) \subseteq (\varphi A)_z$, or equivalently, if the inverse image of a z-ideal in T is a y-ideal in S. A (y,z)-*isomorphism* is therefore a bijection of S onto T such that

$$\varphi(ab) = \varphi(a) \cdot \varphi(b) \tag{1}$$

and

$$\varphi(A_y) = (\varphi A)_z \quad \text{and} \quad \varphi^{-1}(B_z) = (\varphi^{-1} B)_y. \tag{2}$$

This means that there is really no difference between the y-system in S and the z-system in T, so that we can put $y = z = x$ and call the given mapping an x-*isomorphism*. In the following theorem x_i denotes the x-system defined in $S_i(X_i)$, $i = 1, 2$.

THEOREM 30. *Let X_1 and X_2 be two compact Hausdorff spaces and let $S_1(X_1)$ and $S_2(X_2)$ be two characteristic function semi-groups for the spaces X_1 and X_2 respectively. Then X_1 and X_2 are homeomorphic if $S_1(X_1)$ and $S_2(X_2)$ are (x_1, x_2)-isomorphic.*

Proof. If the x_i-structure space of X_i is $\mathfrak{M}_i$, $(i = 1, 2)$, then $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are homeomorphic since $S_1(X_1)$ and $S_2(X_2)$ are (x_1, x_2)-isomorphic, and the theorem follows from Theorem 29.

In order to derive the above-mentioned theorems of Kolmogoroff–Gelfand and Stone from Theorem 30, we only have to verify that the function semi-groups involved in these two cases—namely $C(X)$ with respect to multiplication and $C^R(X)$ with respect to the operation $f \circ g = |f| \cap |g|$—have the properties of a characteristic function semi-group with respect to the d-system and the c-system respectively. The d-system is the system of ordinary ideals in a ring, while the c-system is the system of convex, lattice-closed subgroups in a lattice-ordered abelian group (see section 22).

Concluding remarks. It goes without saying that the results of the preceding chapters should be considered just as samples of results which can be generalized to x-ideals and do not in this sense aim at any completeness. We believe, however, that the results derived so far should clearly indicate the possibilities of this general approach to ideal theory. There would be no point in trying to generalize blindly as many as possible of the existing results on special x-systems to general x-ideals. But we think that such a generalization could, in many cases, lead to interesting results. In fact, it seems often a fruitful procedure to try to generalize a special result to x-ideals and then apply the general theorem to *other* particular x-systems. In [3] we have, following the idea of [32], applied this procedure in order to derive a general embedding theorem for lattice-ordered algebraic structures. We there applied the general Krull–Stone theorem for x-ideals to a particular kind of x-systems consisting of certain types of convex subalgebras with respect to the semi-group operation $a \circ b = |a| \cap |b|$.

As to further generalizations of the present theory we here mention two possibilities: Generalization to the non-commutative case and generalization to what we could call *x-modules*. The formulation of the axioms in the non-commutative case does not offer any difficulty although one must be careful about the order of the factors. The axioms 1.3′ and 1.3″ for a *left x-system* should, for instance, read

1.3′ $AB_x \subseteq B_x$.

1.3″ $B_x A \subseteq (BA)_x$.

We here observe that the order of A and B are reversed in 1.3″ with respect to that in 1.3′. Much of non-commutative ideal theory of rings carries over to this general setting. As an example see, for instance [2], Chapter 2. Grundy [10] has generalized the ideal theory of Noetherian rings to modules. Something similar might be done in the case of general x-ideals to arrive at a notion of an x-module. We should remark that a similar general concept has been introduced for arithmetical purposes by Lorenzen in [23].

Chapter 5

Applications to particular x-systems

We shall in this chapter go into some more detail with respect to a few of the special ideal theories which are subsumed under the theory of x-ideals. Since the process of getting special results by putting $x = d$, s, m, δ, etc. in the general theorems on x-ideals is in most cases a trivial matter, we shall here consider only certain samples of this kind. We shall in particular choose examples where the theory of x-ideals throws new light on well-known special results—or where we actually can obtain new results by the specialization process.

18. *Lattices.* As developed thus far, the ideal theory of lattices constitutes a very elementary subject with few—if any—really deep results. The standard reference [4] is not very complete as far as ideal theory is concerned and contains only a few simple results. More details can be found in [13] and [25]. The essential results on the ideal theory of distributive lattices are nearly all easy consequences of theorems on x-ideals and appear to be best understood in this connection. In fact the theory of x-ideals clarifies completely the crucial role played by distributivity and gives incidentally various new characterizations of this property.

Let L be a lattice under the operations $\cup$ and $\cap$. A subset A of L is called a lattice ideal or simply an *l-ideal* in L if $a \cup b \in A$ whenever a and $b \in A$ and $a \cap b \in A$ whenever $a \in A$ and $b \in L$. It is easily verified that the family of l-ideals in L will define an x-system in L

if L is distributive and is considered as a semi-group under intersection (see below). This shows that the ideal theory of distributive lattices is subsumed under the theory of x-ideals. Our main objective in this paragraph is to establish a converse of this result which will give the full explanation why most of the important results on l-ideals hold just for distributive lattices.

THEOREM 31. *The l-ideals in L define an x-system in L if and only if L is distributive.*

Proof. Assume first that L is distributive. We must show that the continuity axiom

$$a \cap B_l \subseteq (a \cap B)_l \tag{1}$$

is satisfied. An element c belongs to B_l if and only if $c \leqslant b_1 \cup \ldots \cup b_n$ for a finite number of b_i belonging to B. Thus

$$a \cap c \leqslant a \cap (b_1 \cup \ldots \cup b_n) = (a \cap b_1) \cup \ldots \cup (a \cap b_n) \in (a \cap B)_l$$

so that (1) is satisfied. Assume conversely that the continuity axiom is satisfied. Then by Theorem 1 (or more directly by Proposition 17) the family $\mathfrak{C}(L)$ of l-ideals will form a distributive lattice under inclusion. L is isomorphic to the sublattice of $\mathfrak{C}(L)$ which consists of all the principal l-ideals of L and hence is distributive.

A direct proof of the fact that the continuity axiom implies distributivity can be given as follows. The continuity axiom states that

$$a \cap \{b_1, \ldots, b_n\}_l \subseteq \{a \cap b_1, \ldots, a \cap b_n\}_l. \tag{2}$$

The left-hand side of (2) consists of all elements d such that $d \leqslant a \cap (b_1 \cup \ldots \cup b_n)$, while the right hand side of (2) consists of all the elements f such that $f \leqslant (a \cap b_1) \cup \ldots \cup (a \cap b_n)$. The inclusion (2) is therefore equivalent to the implication

$$d \leqslant a \cap (b_1 \cup \ldots \cup b_n) \Rightarrow d \leqslant (a \cap b_1) \cup \ldots \cup (a \cap b_n).$$

Putting here $n=2$, $b_1=b$, $b_2=c$ and $d=a \cap (b \cup c)$ we obtain

$$a \cap (b \cup c) \leqslant (a \cap b) \cup (a \cap c)$$

which shows that L is distributive because the reverse inclusion is satisfied in any lattice. Theorem 31 combined with general theorems on x-systems like Theorems 1, 2, 3, 13, 14 and 15 gives a great number of different characterizations of distributive lattices among which several do not seem to have been observed earlier. We collect some of these in the following corollary. We recall that in a lattice L $A \cap B$ is the set of all intersections $a \cap b$ with $a \in A$ and $b \in B$, and $A : b = \{c, c \cap b \in A\}$.

COROLLARY. *The following conditions are equivalent for a lattice L:*

1. *L is distributive.*
2. *The lattice of l-ideals in L is distributive.*
3. *Every l-ideal in L can be written as an intersection of prime l-ideals.*
4. *Every irreducible l-ideal is prime.*
5. $(A \cap B)_l = A_l \cap B_l$.
6. *$A_l : b$ is an l-ideal in L for all $b \in L$.*
7. $A_l : B_l = A_l : B$.
8. $[(A \cap B)_l \cap C]_l = [A \cap (B \cap C)_l]_l$ (in case L has a greatest element).

In section 5 we defined a general notion of x-congruence modulo an x-ideal A_x by putting $b \equiv c \,(\mathrm{mod}\ A_x)$ whenever $(A_x, b)_x = (A_x, c)_x$. By using the continuity axiom we showed that this relation was a congruence relation with respect to the multiplication of the underlying semi-group, i.e. $b \equiv c \,(\mathrm{mod}\ A_x) \Rightarrow bd \equiv cd \,(\mathrm{mod}\ A_x)$. We did not there treat the converse problem, i.e. to what extent the continuity axiom is implied by this congruence property. We shall now see that this problem admits a simple solution in the case $x = l$.

In the case of l-ideals, $(A, b)_l = (A, c)_l$ is equivalent to the existence of two elements a_1 and a_2 in A_l such that

$$b \leqslant a_1 \cup c \quad \text{and} \quad c \leqslant a_2 \cup b. \tag{3}$$

In case A_l is a principal l-ideal $(a)_l$ it is clear that $b \equiv c (\mathrm{mod}\ (a)_l)$ if and only if $b \leqslant a \cup c$ and $c \leqslant a \cup b$. The conjunction of these two inequalities is equivalent to the equation $a \cup b = a \cup c$. We therefore have the following:

PROPOSITION 30. *The elements b and c are l-congruent modulo the principal l-ideal $(a)_l$ if and only if $a \cup b = a \cup c$.*

Since $a \cup b = a \cup c$ is equivalent to $(a)_l \cup b = (a)_l \cup c$ (using the convention succeeding the axiom 1.3 of section 1) we see that the l-congruence in this case can be defined formally in exactly the same way as the ordinary congruence in rings substituting $\cup$ for $+$.

The next theorem again ties up the connection between distributivity and the general theory of x-ideals. The notion of l-congruence can be defined in any lattice but the following theorem shows that the name l-congruence is really appropriate only in distributive lattices.

We remark that it is really only the congruence property with respect to the intersection operation which matters here, but since the relation $a \equiv b (\mathrm{mod}\ A_l)$ is always a congruence with respect to union, our terminology coincides with the one used in lattice theory where a congruence is an equivalence relation satisfying

$$[a \equiv b] \Rightarrow [a \cup c \equiv b \cup c \quad \text{and} \quad a \cap c \equiv b \cap c].$$

THEOREM 32. *The following properties are equivalent in a lattice L.*

1. *L is distributive.*
2. *$a \equiv b (\mathrm{mod}\ A_l)$ is a congruence relation for all l-ideals A_l in L.*
3. *$a \equiv b (\mathrm{mod}\ (c)_l)$ is a congruence relation for all principal l-ideals $(c)_l$ in L.*

Proof. We prove the theorem by establishing the following sequence of implications: $3 \Rightarrow 1 \Rightarrow 2 \Rightarrow 3$. According to Proposition 30, 3. asserts that

$$[a \cup b = a \cup c] \Rightarrow [a \cup (b \cap d) = a \cup (c \cap d)]. \tag{4}$$

If L were non-distributive it would contain either of the two lattices

as sublattices.

Both these two lattices violate (4) and hence $3 \Rightarrow 1$, $1 \Rightarrow 2$ follows from Theorems 9 and 31, and $2 \Rightarrow 3$ is obvious.

The decomposition theorems derived for general x-ideals in Chapters 2 and 3 take on a particularly simple form in the case of l-ideals in a distributive lattice. This is due to the fact that we have irreducible = primary = prime in the case $x = l$. In addition the family $\mathfrak{C}(L)$ of l-ideals satisfies the ascending chain condition (i.e. L is l-Noetherian) if and only if L itself satisfies the ascending chain condition. In this case every l-ideal in L is principal and L is isomorphic to the lattice $\mathfrak{C}(L)$ of all l-ideals in L by the mapping $a \rightarrow (a)_l$. Indeed if L is l-Noetherian, every l-ideal A_l is finitely generated. Thus $A_l = \{a_1, \ldots, a_n\}_l = (a_1 \cup \ldots \cup a_n)_l$, and A_l is principal. Conversely if L satisfies the ascending chain condition, it is clear that every l-ideal in L is finitely generated. In the case of a distributive lattice L with ascending chain condition for its elements Theorem 19 therefore only gives the simple fact that any element in L can be written uniquely as an irredundant intersection of a finite number of irreducible elements.

19. *Multiplicative lattices and semi-lattices.* The observation first made by Krull in [17], that many results of the ideal theory of rings can be formulated in terms of ideals alone without any reference to the elements of the underlying ring has led to the introduction of lattices and semi-lattices over which a multiplication is defined. Among the main contributions to the subject we can mention [5], [6], [7], [8] and [19]. Apart from being an axiomatic study developing ideal theory without reference to elements these papers also have an objective similar to the present one, namely, to subsume other ideal theories as well as that of ordinary d-ideals. We believe, however, that the theory of x-ideals has several advantages over the theory of multiplicative lattices and semi-lattices. Because of the presence of "elements", the theory of x-ideals leads to a richer and more flexible calculus. In fact, we shall show below that there are important ideal-theoretic notions which never can be formulated purely in terms of multiplicative lattices. This is for instance the case with the continuity axiom itself. The following representation theorem also shows that the part of the theory of m-lattices which has been developed most extensively, is subsumed under the theory of x-ideals.

We have already defined the notion of a quasi-integral m-lattice. By a *groupoid* we mean a set G with a binary operation $G \times G \to G$ which will be denoted multiplicatively. G is said to be a $\cup$-groupoid if there is also defined an associative, commutative and idempotent operation $\cup$ in G such that $a(b \cup c) = ab \cup ac$. If the groupoid-operation is associative we shall speak of a $\cup$*-semi-group*. We here only consider $\cup$-semi-groups which are commutative. The definition of the m-system given in section 13 carries immediately over to a $\cup$-semi-group.

THEOREM 33. *To every quasi-integral* $\cup$*-semi-group* L *satisfying the ascending chain condition we can find a semi-group* S *and an* x*-system of finite character in* S *such that* L *is isomorphic to the* $\cup$*-semi-group of all* x*-ideals in* S *under the operations of* x*-union and* x*-multiplication.*

Proof. We shall actually show that we can choose $S = L$ with the multiplication in L as semi-group operation and put $x = m$. Let us first verify that the m-system really is an x-system, i.e. satisfies the continuity axiom $AB_m \subseteq (AB)_m$. The m-ideal generated by B consists of all elements c such that $c \leqslant b_1 \cup \dots \cup b_n$ with $b_1, \dots, b_n \in B$. If $a \in A$, we therefore have

$$ac \leqslant a(b_1 \cup \dots \cup b_n) = ab_1 \cup \dots \cup ab_n \in (AB)_m,$$

and the continuity axiom is satisfied. If L satisfies the ascending chain condition every m-ideal in L will be principal, and it is obvious that the mapping $a \to \{a\}_m$ is an isomorphism carrying products into m-products and union into m-union. This proves the theorem.

COROLLARY. *As far as properties which can be expressed entirely in terms of* x*-ideals and the operations of* x*-union and* x*-multiplication are concerned, the following theories are equivalent under the assumption of the ascending chain condition.*

1. *The theory of* x*-ideals.*
2. *The theory of* m*-ideals.*
3. *The theory of quasi-integral* m*-lattices.*
4. *The theory of quasi-integral* $\cup$*-semi-groups.*

Any result on quasi-integral $\cup$-semi-groups with ascending chain condition gives trivially a result on x-ideals. Conversely the above representation theorem shows that all theorems on quasi-integral $\cup$-semigroups with ascending chain condition can be derived from similar theorems on x-ideals. For the proof of the next theorem it will be convenient to have the following

LEMMA. *For a given x^*-system the following conditions, which all represent a weakening of the continuity axiom, are equivalent.*

$$\text{I.}\quad A_{x^*}\cdot B_{x^*}\subseteq (A_{x^*}\cdot B)_{x^*}.$$
$$\text{II.}\quad A_{x^*}\circ B_{x^*}=A_{x^*}\circ B.$$
$$\text{III.}\quad (A_{x^*}:B_{x^*})_{x^*}=A_{x^*}:B_{x^*}.$$
$$\text{IV.}\quad A_{x^*}\circ(B\cup_{x^*}C)=(A_{x^*}\circ B)\cup_{x^*}(A_{x^*}\circ C).$$

We can leave the simple proof of this lemma to the reader.

THEOREM 34. *The continuity axiom cannot be expressed by x-ideals alone, i.e. it cannot be formulated as a property of the m-lattice of all x-ideals.*

Remark. In fact all the equivalent forms of the continuity axiom given in Theorems 1, 2 and 3 involve elements or subsets of S which are not x-ideals. We shall now show that this must be the case for any formulation of the continuity axiom. For certain special x-systems like the l-system we know, however, that the continuity axiom is just equivalent to the fact that the l-ideals form an m-lattice under l-union and l-product.

Proof. We shall prove the theorem by exhibiting two semi-groups S_1 and S_2 equipped with x^*-systems x_1^* and x_2^* respectively such that the m-lattice of all x_1^*-ideals of S_1 is isomorphic to the m-lattice of all x_2-ideals of S_2 and such that x_2^* satisfies the continuity axiom but x_1^* does not. Let S_1 be the multiplicative semi-group of the polynomial ring $Z[x]$ and let x_1^* denote the x^*-system consisting of all the differential ideals in $Z[x]$. An ideal ($=d$-ideal) A in $Z[x]$ is said to be a differential ideal if it contains the derivative δa of any of its polynomials a. Let us refer to this x^*-system as the δ^*-system. We first observe that the δ^*-system does not satisfy the continuity axiom. In fact $x\cdot\{x\}_{\delta^*}\nsubseteq\{x^2\}_{\delta^*}$ since the polynomial x is contained in the left-hand side but not in the right-hand side. Nevertheless the family of δ^*-ideals forms an m-lattice under inclusion and δ^*-multiplication. This follows from the above lemma (IV) together with the fact that $A_{\delta^*}:B_{\delta^*}$ is always a δ^*-ideal (III). Indeed if $cb\in A_{\delta^*}$ for all $b\in B_{\delta^*}$, then $\delta(cb)=\delta c\cdot b+c\cdot\delta b\in A_{\delta^*}$ and $\delta c\cdot b\in A_{\delta^*}$ for all $b\in B_{\delta^*}$ showing that $\delta c\in A_{\delta^*}:B_{\delta^*}$. We have therefore established that the family of all δ^*-ideals in $Z[x]$ forms an m-lattice L with ascending chain condition. Now, choose $S_2=L$ with the multiplication in L as the semi-group operation of S_2 and put $x_2^* = m$. Since the ascending chain condition is satisfied in L, L is isomorphic to the m-lattice of all m-ideals in L under the mapping $a \to \{a\}_m$, and the theorem is thereby proved since the m-system satisfies the continuity axiom.

In view of the fundamental role which is played by the continuity axiom in ideal theory, the above theorem indicates anew that several more refined ideal-theoretic facts cannot be expressed in terms of m-lattices alone.

Another example of an x-system which has been considered in a quasi-integral $\cup$-semi-group is the following. We shall call a subset A_u of a $\cup$-semi-group L a *u-ideal* if the following two conditions are satisfied: (1) $a, b \in A_u \Rightarrow ab \in A_u$. (2) $a \in A_u$ and $b \in L \Rightarrow a \cup b \in A_u$. The u-ideal generated by A consists of all elements $a \geqslant a_1 \cdot a_2, \ldots, a_n$ where $a_1, \ldots, a_n \in A$ and the u-ideals form an x-system in L considered as a semi-group with respect to union. Indeed,

$$b \cup a \geqslant b \cup a_1 \cdot a_2, \ldots, a_n \geqslant (b \cup a_1) \cdot (b \cup a_2) \ldots (b \cup a_n) \in (b \cup A)_u,$$

for any element $a \in A_u$. For this concept of ideal see [8].

20. *Radical differential ideals and perfect difference ideals.* In the case of ordinary d-ideals in (commutative) rings the continuity axiom is essentially a consequence of the distributivity of multiplication with respect to addition. In fact, any element $b \in B_d$ is of the form

$$r_1 b_1 + \ldots + r_k b_k + n_1 b_1 + \ldots + n_k b_k$$

and

$$ab = a \sum_{i=1}^{k} r_i b_i + a \sum_{i=1}^{k} n_i b_i = \sum_{i=1}^{k} r_i ab_i + \sum_{i=1}^{k} n_i ab_i \in (aB)_d,$$

showing that the continuity axiom holds. In case of lattices we had that even the reverse implication holds. The distributivity was there a consequence of the continuity axiom. In these two cases where the distributivity essentially accounts for the properties which in the general case follow from the continuity axiom, the importance of the continuity axiom has naturally not been clearly recognized. In various cases of rings with operators which we are now going to discuss it is interesting to note that a much more crucial role is played by a direct use of the continuity axiom in one or the other of its many disguises. In [15] for instance Lemma 1.4 and Lemma 1.5 are nothing but two of the most familiar formulations of the continuity axiom for δ-ideals, and in [28] and [30] an essential role is played by particular cases of the corollary of Proposition 13.

A differential ring R is a commutative ring with a derivation satisfying

$$\delta(a+b) = \delta a + \delta b.$$

$$\delta(ab) = \delta a \cdot b + a \cdot \delta b.$$

A d-ideal A_{δ^*} in R is called a differential ideal if $\delta a \in A_{\delta^*}$ whenever $a \in A_{\delta^*}$. These differential ideals define the x^*-system δ^* which was used in the preceding paragraph. We saw there that the continuity axiom was not satisfied for the δ^*-system, and this is the main reason for restricting our attention to the following class of δ^*-ideals: A differential ideal A_{δ^*} is called a *radical differential ideal* if rad $A_{\delta^*} = A_{\delta^*}$. In this case we put $\delta^* = \delta$ and speak of the *δ-system.*

PROPOSITION 31. *The radical differential ideals of a differential ring R define an x-system in the multiplicative semi-group of R.*

Proof. Obviously we need only verify that the set $A_\delta : b$ is closed under derivation. Assume therefore that $cb \in A_\delta$. This implies that $\delta(cb) = \delta c \cdot b + c \cdot \delta b \in A_\delta$. Multiplying by $\delta c \cdot b$ we obtain $(\delta c \cdot b)^2 + cb \cdot \delta c \cdot \delta b \in A_\delta$, showing that $(\delta c \cdot b)^2 \in A_\delta$ and thus $\delta c \cdot b \in A_\delta$. From this and the general Krull–Stone theorem on x-ideals we obtain the following

COROLLARY. *Any radical differential ideal in a differential ring can be written as an intersection of prime differential ideals.*

This result was first proved by Raudenbush in [28] in the special case of a so-called Ritt algebra (see [15, p. 12]). A *Ritt algebra* is a differential ring which contains the field of rational numbers and hence can be regarded as an algebra over the rational numbers. In a Ritt algebra the radical of a differential ideal is again a differential ideal, and it was the use of this fact which led Raudenbush to suppose that the given differential ring is a Ritt algebra. What is exactly needed in order to carry through Raudenbush's argument to show that $\delta(\text{rad } A_{\delta^*}) \subseteq \text{rad } A_{\delta^*}$ is that the additive group R/A_{δ^*} is without torsion. It is for instance not enough to suppose that R is of characteristic zero, i.e. contains a copy of Z. Indeed, in $Z[x]$ the radical of the differential ideal $(x^2, 2)$ is the d-ideal $(x, 2)$ which is not differential.

Another type of rings with operators with an ideal concept which nicely falls into the pattern of x-ideals is the difference rings with their perfect difference ideals. This ideal theory was considered by Ritt and Raudenbush in [30]. We here content ourselves with showing that their ideals really define an x-system.

A *difference ring* is a commutative ring R together with an operator Δ satisfying

$$\Delta(a+b) = \Delta a + \Delta b.$$

$$\Delta(ab) = \Delta(a) \cdot \Delta(b).$$

In [30] it is assumed that R contains a unity e such that $\Delta e = e$. As defined here a difference ring is nothing more than a commutative ring with a distinguished endomorphism. We shall write $\Delta(\Delta a)=\Delta^2 a$, $\Delta^3 a=\Delta(\Delta^2 a)$ and generally $\Delta^n a$ when the operator is repeated n times. Ritt and Raudenbush [30] call a d-ideal A_d in R a *difference ideal* if

$$a\in A_d \Rightarrow \Delta a\in A_d \quad \text{and} \quad \Delta a\in A_d \Rightarrow a\in A_d \tag{1}$$

A difference ideal A_{Δ^*} is said to be a *perfect difference ideal* or a Δ-ideal if

$$(\Delta^{n_1} a)^{\alpha_1}(\Delta^{n_2} a)^{\alpha_2}\dots(\Delta^{n_k} a)^{\alpha_k}\in A_{\Delta^*}\Rightarrow a\in A_{\Delta^*} \tag{2}$$

Here $n_1,\dots,n_k$ are distinct integers $\geqslant 0$ and $\alpha_1,\dots,\alpha_k$ are integers $\geqslant 1$. A_Δ denotes as usual the unique minimal Δ-ideal containing A.

PROPOSITION 32. *The family of Δ-ideals defines an x-system with respect to the multiplicative semi-group of R.*

Proof. We need only verify the continuity axiom, i.e. that $A_\Delta:b$ is a Δ-ideal. Let us first verify that $A_\Delta:b$ has the property (2). If

$$(\Delta^{n_1} c)^{\alpha_1}\dots(\Delta^{n_k} c)^{\alpha_k}\cdot b\in A_\Delta,$$

we obtain by multiplication with $c\,(\Delta^{n_1} b)^{\alpha_1}\dots(\Delta^{n_k} b)^{\alpha_k}$ that

$$(\Delta^{n_1}(cb))^{\alpha_1}\dots(\Delta^{n_k}(cb))^{\alpha_k}\cdot cb\in A_\Delta.$$

Since A_Δ satisfies (2) this means that $cb\in A_\Delta$ and $c\in A_\Delta:b$. We then show that $A_\Delta:b$ satisfies (1). Assume first that $c\in A_\Delta:b$, i.e. $cb\in A_\Delta$. Then $\Delta(cb)=\Delta c\cdot\Delta b\in A_\Delta$, and multiplication by $\Delta^2 c\cdot b$ gives $\Delta c\cdot\Delta b\cdot\Delta^2 c\cdot b=\Delta c\cdot b\cdot\Delta(\Delta c\cdot b)\in A_\Delta$ which implies $\Delta c\cdot b\in A_\Delta$ according to (2). If conversely $\Delta c\cdot b\in A_\Delta$, then also $c\Delta b\cdot\Delta c\cdot b=cb\Delta(cb)\in A_\Delta$ and $cb\in A_\Delta$ by (2).

COROLLARY. *All theorems valid for general x-systems of finite character are valid for Δ-ideals.*

21. *Rings with operators and monadic ideals.* Differential rings and difference rings are both examples of rings with operators and so are, for instance, algebras over a field. In the latter case the operators satisfy $\alpha(a+b)=\alpha(a)+\alpha(b)$ and $\alpha(ab)=\alpha(a)\cdot b(=a\cdot\alpha(b))$, and an algebra ideal or a-ideal in the algebra R is a d-ideal in R which is closed under scalar multiplication. The continuity axiom is again satisfied since $c\in A_a:b$ implies that $\alpha(cb)=(\alpha c)\cdot b\in A_a$ and $\alpha c\in A_a:b$. Many other types of rings with operators have been considered in particular in connection with Boolean algebras as, for instance, closure algebras, projective algebras and relation algebras. We shall here content ourselves with discussing one example of a Bolean algebra with operators which is basic in the investigations of Halmos on the algebra of the quantification calculus. This example is also of interest because it shows that certain useful results on ideals can be derived also in cases where the continuity axiom is not satisfied.

A *monadic algebra* is a Boolean algebra B together with an operator $\exists$ satisfying the following axioms

$$\exists\, 0 = 0. \tag{1}$$

$$a \leqslant \exists\, a. \tag{2}$$

$$\exists\,(a \cap \exists\, b) = \exists\, a \cap \exists\, b. \tag{3}$$

Due to its interpretation in logic $\exists$ is called a *quantifier*. Denoting the complement by a dash, one readily verifies the following relations $\exists\exists = \exists$, $a \leqslant b \Rightarrow \exists\, a \leqslant \exists\, b$, $\exists\,(\exists\, a)' = (\exists\, a)'$ and $\exists\,(a \cup b) = \exists\, a \cup \exists\, b$. The quantifier $\exists$ is called *discrete* if $\exists\, a = a$ for all $a \in B$. This amounts to saying that the range of $\exists$ is B. An ideal A_l in B, i.e. an l-ideal, is called a *monadic ideal* or $\exists$-ideal and is denoted by $A_\exists$ if $a \in A_\exists$ implies $\exists\, a \in A_\exists$. We consider the monadic ideals as an x^*-system with intersection as the semi-group operation.

PROPOSITION 33. *$A_\exists : b$ is an $\exists$-ideal for all $A_\exists$ if and only if $\exists\, b = b$.*

Proof. If $\exists\, b = b$, then $A_\exists : b = A_\exists : \exists\, b$, and $c \cap b \in A_\exists$ implies $c \cap \exists\, b \in A_\exists$ and $\exists\,(c \cap \exists\, b) \in A_\exists$. Using (3) we obtain the desired result $\exists\, c \in A_\exists : b$. If conversely $A_\exists : b$ is always an $\exists$-ideal, $0 : b = (b')_l$ is in particular an $\exists$-ideal, and this implies $\exists\, b' = b'$. According to the first half of the proposition $0 : b' = (b)_l$ must therefore be an $\exists$-ideal and $\exists\, b = b$.

We have here a particular case of the general situation described in the first half of paragraph 6. The subsemi-group S^* is here the range $\exists\, B$ of $\exists$ and the family of traces $A_\exists \cap \exists\, B$ forms an x-system, namely the l-system, on the Boolean sub-algebra $\exists\, B$. In fact we have a one-to-one inclusion-preserving correspondence $A_\exists \to \exists\, A_\exists$ between the $\exists$-ideals in B and the l-ideals in $\exists\, B$. It is essentially this correspondence which enables one to prove certain useful results about the $\exists$-system by reducing the problem to the l-system where the continuity axiom is available. In this way one can, for instance, prove the following

PROPOSITION 34 (Halmos). *Every $\exists$-ideal $A_\exists$ is equal to the intersection of all the maximal $\exists$-ideals containing $A_\exists$.*

Remark. It is clear from Proposition 33 that Proposition 34 is *not* a Krull–Stone theorem for $\exists$-ideals since a maximal $\exists$-ideal need not be maximal considered as an l-ideal and hence need not be prime. In order to clarify this point we give the following.

Theorem 35. *In a monadic Boolean algebra the following statements are equivalent:*

1. *The $\exists$-ideals verify the continuity axiom.*
2. *The quantifier $\exists$ is discrete.*
3. *Every l-ideal is an $\exists$-ideal.*
4. *A maximal $\exists$-ideal is a maximal l-ideal.*
5. *A maximal $\exists$-ideal is prime.*

Proof. $1 \Rightarrow 2$ follows directly from Proposition 33; $2 \Rightarrow 3$, $3 \Rightarrow 4$ and $4 \Rightarrow 5$ are all obvious. Finally $5 \Rightarrow 1$ follows from Proposition 34 and the converse of the Krull–Stone theorem for half-prime x^*-systems (Theorem 13).

22. *Convex subgroups of lattice-ordered groups and rings.* The application of the theory of x-ideals to lattice-ordered algebraic systems is among the more interesting applications. But since we have treated this matter in more detail in a separate paper [3] we shall here content ourselves with treating the case of lattice-ordered rings and refer the reader to [3] for a more general treatment. Let G be a lattice-ordered group, i.e. there is defined an order relation $\geqslant$ in G such that G is a lattice and $a \geqslant b \Rightarrow a+c \geqslant b+c$ for all c. Putting $a^+ = a \cup 0$ and $a^- = -a \cup 0$, we define a semi-group operation $\circ$ in G by $a \circ b = |a| \cap |b|$. An additive subgroup H of the group G is said to be *absolutely convex* if $|a| \leqslant |b|$ for $a \in G$ and $b \in H$ implies $a \in H$. This definition gives a link with ideal theory because it just expresses that H has the multiplicative ideal property 1.3′ with respect to the semi-group operation $a \circ b = |a| \cap |b|$. The absolutely convex subgroups are the natural distinguished subsets of a lattice-ordered group since they just form the kernels of the structure preserving maps, i.e. the homomorphisms with respect to the addition and the lattice operations. In order that this basic property is maintained in case G also has a multiplication making it into a ring R it is necessary and sufficient that H also is a d-ideal in R. This family $\mathfrak{J}$ of absolutely convex d-ideals in R is, however, not satisfactory from the point of view of the theory of x-ideals since it does not satisfy the continuity axiom with respect to the operation $a \circ b$. In order to get an x-system we must single out a subfamily $\mathfrak{J}_c$ which has the property that $A \in \mathfrak{J}_c \Rightarrow A : b \in \mathfrak{J}_c$ for all $b \in R$. We get the unique maximal subfamily $\mathfrak{J}_c$ of this type by using the general procedure described in Section 6. More explicitly we have the

Definition. An absolutely convex d-ideal A of R belongs to $\mathfrak{J}_c$ and is called a *c-ideal* if $|a| \cap |b| \in A \Rightarrow |a| \cap |cb| \in A$ for all $c \in R$.

One of the basic problems concerning ordered algebraic structures is their representation by real-valued functions and more generally their embedding in a direct product of linearly ordered algebraic structures of the same type. The following theorem shows that the theory of x-ideals has a fundamental bearing on the latter more general question.

THEOREM 36. *The family of c-ideals defines a half-prime x-system in the semi-group* $R(\circ)$. *The necessary and sufficient condition that* R *can be embedded in a direct product of linearly ordered rings is that* $\{0\}$ *constitutes a c-ideal in* R, *i.e. that* $a \cap b = 0 \Rightarrow a \cap |cb| = 0$ *for all* $c \in R$. *It is also necessary and sufficient for such an embedding that there exists an x-system in* $R(\circ)$ *consisting of a family of convex d-ideals such that* $\{0\}$ *is an x-ideal.*

For the proof of this theorem we refer the reader to [3]. In [3] one also finds a more general statement which gives necessary and sufficient conditions for similar embeddings of lattice-ordered groups and lattice-ordered vector-spaces and algebras over a linearly ordered field. An absolutely convex subgroup of a lattice-ordered abelian group will also be called a c-ideal. In fact a lattice-ordered abelian group G can always be considered as a lattice-ordered ring with respect to the trivial multiplication $ab = 0$. Furthermore, an absolutely convex subgroup of G is evidently the same as a c-ideal in this ring.

We then show that Theorem 30 applies to the c-system of a lattice-ordered group and thus gives us the following theorem of Stone.

THEOREM 37 (Stone). *Let* X_1 *and* X_2 *be two compact Hausdorff spaces. Then* X_1 *and* X_2 *are homeomorphic if* $C^R(X_1)$ *and* $C^R(X_2)$ *are isomorphic as lattice-ordered additive groups.*

Proof. We only need to verify that the conditions of Theorem 30 are satisfied for the c-system in $C^R(X)$ when X is a compact Hausdorff space. The c-system is of finite character and $f \equiv 1$ is a c-identity. The three conditions 1, 2 and 3 just express simple and well-known facts concerning $C^R(X)$.

In connection with the c-structure space of a lattice-ordered abelian group we could also mention that it yields a topological characterization of the important arithmetical notion of complete integral closure. Consider a lattice-ordered group G with an archimedian element e, i.e. a positive element e such that for every a there exists an $n > 0$ with $ne \geqslant a$. Evidently such an archimedian element (sometimes called an order unit) is the same thing as a positive c-identity. G is said to be *completely integrally closed* if $na \geqslant b$ for a fixed $b \in G$ and all $n > 0$ implies $a \geqslant 0$. This gives the usual notion of complete integral closure in case G is the divisibility group of an integral domain.

THEOREM 38. *A lattice-ordered abelian group* G *with an archimedian element is completely integrally closed if and only if the set of maximal* c-*ideals is dense in the* c-*structure space of prime* c-*ideals.*

Remark. The proof of the fact that the hull-kernel procedure defines a topology in the space of prime x-ideals is exactly the same as in Theorem 27.

Proof. A theorem in the theory of lattice-ordered groups says that a lattice-ordered abelian group G with an archimedian element e is completely integrally closed if and only if the intersection of all the maximal c-ideals of G is equal to $\{0\}$. By the Krull–Stone theorem for c-ideals, the intersection of all the prime c-ideals of G is equal to $\{0\}$ for arbitrary G. The theorem follows by combining these two results.

23. *Another characteristic function semi-group.* We should like to give one more application of Theorem 30 showing that this theorem can also be used to produce some special results which are less familiar than those obtained by Gelfand–Kolmogoroff and Stone.

Let us consider the family $C_+^R(X)$ of all real-valued, non-negative continuous functions on the compact Hausdorff space X. We shall here consider $C_+^R(X)$ as a lattice-ordered semi-group with respect to pointwise multiplication and pointwise ordering. We shall say that a subset A_σ of $C_+^R(X)$ is a *σ-ideal* if the following two conditions are satisfied:

$$f \in A_\sigma \quad \text{and} \quad g \in C_+^R(X) \Rightarrow f \cdot g \in A_\sigma \tag{1}$$

$$f \in A_\sigma \quad \text{and} \quad g \in A_\sigma \Rightarrow f \cup g \in A_\sigma. \tag{2}$$

Proposition 35. *The σ-ideals form an x-system of finite character with multiplication as semi-group operation and $C_+^R(X)$ is a characteristic function semi-group for X with respect to the σ-system.*

Proof. Using the fact that we have the distributive law $f \cdot (g \cup h) = fg \cup fh$ it is easily seen that the continuity axiom is satisfied. The function $f \equiv 1$ is a σ-identity in $C_+^R(X)$, and we need only to check the condition III of a characteristic function semi-group. That $M_\sigma^{(a)} = \{f,\ f \in C_+^R(X) \text{ and } f(a) = 0\}$ is a σ-ideal in $C_+^R(X)$ is clear. That it is maximal can be seen as follows: Assume that $f \notin M_\sigma^{(a)}$, i.e. $f(a) \neq 0$. This implies that $f(x) \neq 0$ for a certain open set O_a containing a. To every $b \neq a$ we can find a $g \in M_\sigma^{(a)}$ such that $g(b) \neq 0$ and hence $g(x) \neq 0$ in an open set O_b containing b. Because of the compactness we have a finite number of points $b_1, \ldots, b_n$ with corresponding functions $g_1, \ldots, g_n$ such that $O_a \cup O_{b_1} \cup \ldots \cup O_{b_n} = X$ and $h = f \cup g_1 \cup \ldots \cup g_n$ is bounded away from zero on X. This shows that $1 = 1/h \cdot h \in (M_\sigma^{(a)}, f)_\sigma$ and $M_\sigma^{(a)}$ is a maximal σ-ideal in $C_+^R(x)$. The same type of argument shows also that there are no other maximal σ-ideals in $C_+^R(X)$ than those of the form $M_\sigma^{(a)}$.

The above proposition immediately implies the following

Theorem 39. *Let X_1 and X_2 be two compact Hausdorff spaces. Then X_1 and X_2 are homeomorphic if $C_+^R(X_1)$ and $C_+^R(X_2)$ are isomorphic as lattice-ordered multiplicative semi-groups.*

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