A STAR DOMAIN WITH DENSEST ADMISSIBLE POINT SET NOT A LATTICE

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1. Introduction

A problem of wide interest in the geometry of numbers is the determination of the density of the closest packing of translates of a given body $\mathcal{K}$ in the plane. For $\mathcal{K}$ convex L. Fejes Toth [2] and C. A. Rogers [4] proved independently that any packing of translates of a convex body in the plane has a density not greater than the density of the best lattice packing.

The problem of packing convex bodies in the plane can be extended to the non-convex case in two distinct ways. One is to require that non-convex bodies be packed in a non-overlapping fashion. The other, the Minkowski-Hlawka type, allows overlapping of the bodies under conditions which highlight the relation between the critical lattice of a convex body and its best lattice packing. It is this latter type of packing which we consider in what follows.

Definitions

1. Let $\mathcal{S}$ be a star domain, symmetric about O. A set of points $\mathcal{D}$ is said to provide a *packing for* $\mathcal{S}$ if the domains $\{\mathcal{S}+P\}_{P\in\mathcal{D}}$ have the property that no domain $\mathcal{S}+P_0$ contains the center of another in its interior. We shall also say that $\mathcal{D}$ is an *admissible* point set for $\mathcal{S}$.

2. The *density* of a lattice, $\mathcal{D}(\mathcal{L})$, is the reciprocal of its determinant.

3. Consider the square $|x|<t$, $|y|<t$. Let $A(t)$ denote the number of points of a set $\mathcal{D}$ in the square; then the *density* of $\mathcal{D}$ (denoted $\mathcal{D}(\mathcal{D})$, is defined as $\overline{\lim_{t\to\infty}}\,\frac{A(t)}{4t^2}$ ([5], p. 5).

That the Rogers' theorem does not hold generally for non-convex figures is shown by following example of a bounded star domain for which the densest packing is not a lattice.

2. Description of $\mathcal{S}$

We take $\mathcal{S}$ to be the region defined in Fig. 1.

Let the point set $\mathcal{D}$ be the union of a lattice $\mathcal{L}_1$ of determinant 1 and $\mathcal{L}_2$, some trans-

(¹) This is part of the author's doctoral thesis at the University of Notre Dame (1961) under the direction of Professor Hans Zassenhaus.

lation of $\mathcal{L}_1$, such that $\mathcal{L}_1 \cup \mathcal{L}_2 = \mathcal{D}$ is not a lattice. We take as $\mathcal{L}_1$ the lattice of all points in the plane with both coordinates integers and define $\mathcal{L}_2$ as follows:

$$\mathcal{L}_2 = \{(x_2, y_2) \mid x_2 = x_1 + \tfrac{7}{16}, y_2 = y_1 + \tfrac{1}{2};\ (x_1 y_1) \in \mathcal{L}_1\}.$$

$\mathcal{L}_1 \cup \mathcal{L}_2$ is not a lattice. For let $L' = (\frac{7}{16}, \frac{1}{2})$ and consider $L = 2L' = (\frac{7}{8}, 1)$. $L \notin \mathcal{L}_1$, since $\frac{7}{8}$ is not an integer. $L \notin \mathcal{L}_2$ since it is not of the proper form. Hence, $L \notin \mathcal{D}$ and, therefore, $\mathcal{D}$ cannot be a lattice.

Note that the point set $\mathcal{D}$ has density equal to two.

The problem is then to construct a star S for which $\mathcal{D}$ is admissible and such that $\Delta(S)$ is strictly greater than $\frac{1}{2}$. Since S must be symmetric about the origin, we can include a point P in S only if we can also include $-P$, its reflection in O, in S. We assume further that if we include a point P in S, we include the whole segment OP in S as well, and hence this segment must not contain a point of $\mathcal{D}$. For the particular point set we have chosen these are easily seen to be the only restrictions. We note that our star S can contain a centrally symmetric rectangle of area two. (In Fig. 1 this rectangle is labelled $P_4 P_5 P_{11} P_{12} = \mathcal{R}$.) Now a critical lattice of a rectangle has mesh equal to one fourth the area of the rectangle and, therefore we can determine S so that it contains $\mathcal{R}$ and thus $\Delta(S) \geqslant \Delta(\mathcal{R}) = \frac{1}{2}$. By adding suitably to $\mathcal{R}$ to form S we can get $\Delta(S) > \frac{1}{2}$. We find that "adding suitably to $\mathcal{R}$' 'consists in no more than completing S in such a way as to include the point $(\frac{1}{2}, \frac{3}{2})$ as an interior point and (0,1) as a boundary point.

Fig. 2 shows that $\mathcal{D}$ is an admissible point set for S.

For S determined and defined as above we show first that $\Delta(S)$ is strictly greater than one half making S the desired example, and then we determine the precise value of $\Delta(S)$.

3. Critical lattice cannot have determinant $= \frac{1}{2}$

Since the rectangle $\mathcal{R}$ has critical lattice of determinant equal to one-half and $S \supset \mathcal{R}$ we have that $\Delta(S) \geqslant \frac{1}{2}$. If $\Delta(S) = \frac{1}{2}$, then every critical lattice for S is also a critical lattice for the rectangle $\mathcal{R}$. A critical lattice of a parallelogram has six or eight points on the boundary and at least a pair of these points are mid-points of opposite sides ([1], p. 160). The lattice can be generated by one of the mid-points and any one of the lattice points on an adjacent side. As the mid-point of $P_4 P_{12}$ (or $P_5 P_{11}$) is an interior point of S, we need only investigate the critical lattices of $\mathcal{R}$ generated by the mid-point of $P_4 P_5$, that is, $P' = (1,0)$ and some point P'' on $P_4 P_{12}$, in order to determine whether or not $\Delta(S)$ equals $\frac{1}{2}$. By symmetry we may further restrict P'' to the segment $P_3 P_4$. We then have the following conditions on the coordinates of $P' = (x', y')$, $P'' = (x'', y'')$:

$$x' = 1; \quad y' = 0,$$
$$\tfrac{7}{16} \leqslant x'' \leqslant 1; \quad y'' = \tfrac{1}{2}.$$

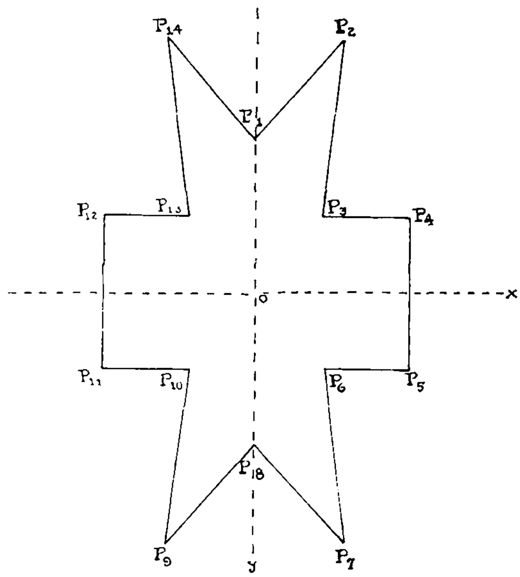

Fig. 1.

$P_1 = (0,1) = -P_8$
$P_2 = (\frac{7}{12}, \frac{5}{3}) = -P_9$
$P_3 = (\frac{7}{16}, \frac{1}{2}) = -P_{10}$
$P_4 = (1, \frac{1}{2}) = -P_{11}$
$P_5 = (1, -\frac{1}{2}) = -P_{12}$
$P_6 = (\frac{7}{16}, -\frac{1}{2}) = -P_{13}$
$P_7 = (\frac{7}{12}, -\frac{5}{3}) = -P_{14}$

P_1P_2: $8x - 7y = -7$
P_2P_3: $8x - y = 3$
P_3P_4: $y = \frac{1}{2}$
P_4P_5: $x = 1$
P_5P_6: $y = \frac{1}{3}$
P_6P_7: $8x + y = 3$
P_7P_8: $8x + 7y = -7$

P_8P_9: $-8x + 7y = -7$
P_9P_{10}: $-8x + y = 3$
$P_{10}P_{11}$: $y = -\frac{1}{2}$
$P_{11}P_{12}$: $x = -1$
$P_{12}P_{13}$: $y = \frac{1}{2}$
$P_{13}P_{14}$: $8x + y = -3$
$P_{14}P_1$: $8x + 7y = 7$

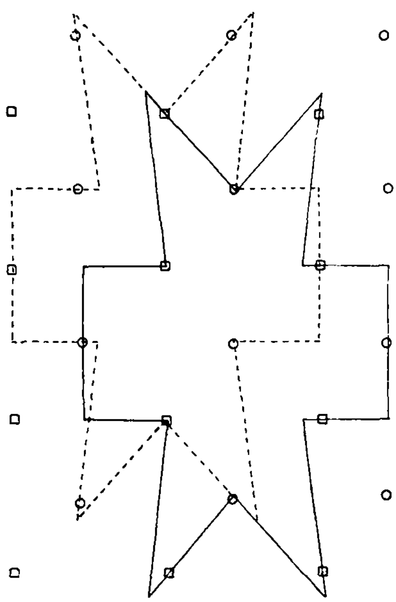

Fig. 2. $\mathcal{L}_1 \cup \mathcal{L}_2$ is an $\mathcal{S}$-admissible set. □ denotes point of $\mathcal{L}_1$, ○ denotes point of $\mathcal{L}_2$.

Consider the point $P^* = P'' - P'$. The coordinates of P^* satisfy the conditions

$$-\tfrac{9}{16} \leqslant x^* \leqslant 0; \quad y^* = \tfrac{1}{2},$$

and P^* is an interior point of S except when $-\tfrac{9}{16} \leqslant x^* \leqslant -\tfrac{7}{16}$. This last inequality implies that P^* is an *interior* point of except when $\tfrac{7}{16} \leqslant x'' \leqslant \tfrac{9}{16}$.

For x'' in this range, consider the point $P^{**} = 2P'' - P'$. We have

$$-\tfrac{1}{8} \leqslant x^{**} \leqslant \tfrac{1}{8}; \quad y^{**} = 1.$$

Hence, P^{**} is an interior point of S except when $x^{**} = 2x'' - x' = 0$, that is, when $x'' = \tfrac{1}{2}$; hence, when $P' = (1,0)$ and $P'' = (\tfrac{1}{2}, \tfrac{1}{2})$.

However, for this choice of P' and P'', $3P'' - P' = (\tfrac{1}{2}, \tfrac{3}{2})$ which is an interior point of S. This completes the proof that $\Delta(S) > \tfrac{1}{2}$ and shows that S is the desired example, since the density of the point set $\mathcal{D}$ is greater than the density of the critical lattice.

4. Determination of the Critical Lattice of S

Mahler ([3], p. 135) has proved that there must be a least two independent points of a critical lattice on the frontier of a bounded star S. Throughout we will denote by P' and P'' the independent points which satisfy the condition that

$$P' \text{ is on } P_j P_{j+1} \quad \text{and} \quad P'' \text{ on } P_k P_{k+1}$$

with $1 \leqslant j \leqslant k \leqslant n$. S has fourteen vertices but by symmetry we need only let $n = 8$.

We will show first that there must be at least three pairs of points of the critical lattice on the boundary of S and then we consider the possible number and placement of these points.

A singular critical lattice ([3] p. 135) of a star domain is one having only four points on the boundary of the star. Mahler ([3], p. 142) has proved that if $\pm P'$ and $\pm P''$ are the points of a singular lattice on the boundary then there is an inner tac-line(1) ([3], p. 141) through P' parallel to OP'' and an inner tac-line through P'' parallel to OP'. If S has a singular critical lattice, then his condition limits the choice of P' and P'' to the following:

(1) the inner vertices of S, namely P_1, P_3, P_6, P_8, P_{10}, and P_{13}, but from symmetry we really need to consider only P_1, P_3, and P_6; and

(2) the cases P' on $P_j P_{j+1}$, $P'' = P_k$, where $P_j P_{j+1}$ is parallel to OP_k.

Denote by $d(\mathcal{L})$ the determinant of the lattice generated by P' and P''.

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(1) An inner tac-line at P is a straight line such that all points of the line sufficiently near to P but distinct from P are in the star.

For case (1) we have the following possibilities:

	$P'=(x',y')$	$P''=(x'',y'')$	$d(\mathfrak{L})$
(a)	$P_3=(\frac{7}{16},\frac{1}{2})$	$P_1=(0,1)$	$\frac{7}{16}$
(b)	$P_6=(\frac{7}{16},-\frac{1}{2})$	$P_1=(0,1)$	$\frac{7}{16}$
(c)	$P_6=(\frac{7}{16},-\frac{1}{2})$	$P_3=(\frac{7}{16},\frac{1}{2})$	$\frac{7}{16}$

None of the lattices generated by these choices of P' and P'' can be critical, or even admissible, since $\Delta(S)>\frac{1}{2}>\frac{7}{16}$.

In case 2 we again have the number of possibilities reduced by symmetry and, therefore, have only the following two:

$$\text{(a)}\quad P'=P_1 \quad \text{and } P'' \quad \text{on } P_4P_5;$$

i. e., $P'=(0,1)$, $P''=(1,y)$ where $-\frac{1}{2}\leqslant y\leqslant\frac{1}{2}$, and $d(\mathfrak{L})=1$;

$$\text{(b)}\quad P' \text{ on } P_1P_2,\ P''=P_3.$$

By the tac-line condition, P' must satisfy

$$0\leqslant x'\leqslant\tfrac{7}{48};\quad 1\leqslant y'\leqslant\tfrac{7}{6}$$

in order that there be an inner tac-line through P_3 parallel to OP'. Consider than the point $P^*=P'-P''$. We have

$$-\tfrac{7}{16}\leqslant x^*\leqslant-\tfrac{7}{24}\quad\text{and}\quad\tfrac{1}{2}\leqslant y^*\leqslant\tfrac{2}{3}.$$

Thus P^* is interior to S except for $x^*=x'-x''=-\frac{7}{16}$. However, this implies thet $x'=0$; i. e., $P'=P_1$ which was considered in case 1.

In what follows it becomes clear that 2(a) does not yield a critical lattice and, therefore, S can have no singular critical lattice. Hence any critical lattice must have at least three pairs of points on the boundary of S. With this information we can find the critical lattices of S in the following way.

Let P', P'' be two independent points on the boundary of with P' on P_jP_{j+1}, P'' on P_kP_{k+1} for $1\leqslant j\leqslant k\leqslant 8$. If P' and P'' belong to a critical lattice, there must be at least one other pair of lattice points on the boundary. It is sufficient to consider the cases of one or two additional pairs of lattice points on the boundary of S. We have then only the following possibilities:

(1) P', P'' both vertices: $P'=P_j$, $P''=P_k$.

(2) P' a vertex, P'' a boundary point, but not a vertex and some lattice point Q on the boundary:

$$P'=Pj;\ P'' \text{ on } P_kP_{k+1};\quad Q=k_1P'+k_2P'' \text{ on } P_iP_{i+1},$$

where k_1, and k_2, are rational numbers.

(3) P', P'' boundary points, but not necessarily vertices, Q a vertex

$$P' \text{ on } P_j P_{j+1}; \quad P'' \text{ on } P_k P_{k+1}; \quad Q = P_i.$$

(4) P', P'' boundary points, but not necessarily vertices; Q_1, Q_2 lattice points on the boundary, but not vertices.

(5) P', P'' boundary points, but not necessarily vertices; Q not a vertex and no further lattice points on the boundary.

In cases (1) to (4) the conditions stated give four equations in four unknowns (the four coordinates of P' and P'') which in each instance yield either a unique solution for P' and P'' and a specific lattice generated by them or no solution. In case (5) we have only three equations in four unknowns and must deal with this case separately.

Since S is bounded, for the lattice points $Q = k_1 P' + k_2 P''$ which must be considered, we have, for $i = 1$ and 2, $|k_i| < c$ for some constant c depending on P' and P''. Further, for k_i rational the denominator is bounded as will be seen later, and, therefore, there are a finite number of lattice points of the form Q. The number of choices for P' and P'' is also finite and so the number of possibilities contained in cases (1) to (5) is finite.

For each system of equations in case (5) we proceed as follows:

First, find a solution to the system of equations and express the most general solution in terms of this solution and a single parameter, t. Since t is taken to be equal to one of the coordinates of either P' or P'', it ranges over a closed interval.

Next, write the expression for the determinant of the lattice generated by the general solution.

Then, minimize this expression as a function of t.

In all cases it was found that the minimum was attained at an end point of the interval. Hence, the smallest value for an admissible solution occurs at vertices thus giving an additional equation relating the unknowns and putting the system in one of the cases (1) to (4).

We note that P' and P'' may belong to a critical lattice without generating it; i. e., the lattice they generate could be a sublattice of the critical lattice. Since we know that $\frac{1}{2} < \Delta(S) \leqslant 1$ ($\mathfrak{L}_1$ is admissible), this could occur when the determinant of the lattice generated by P' and P'' is strictly greater than one. However, given two independent points of a lattice we can always use these two points to find a basis. That is, if P' and P'' are ndependent points of a lattice $\mathfrak{L}$ but do not generate $\mathfrak{L}$, they generate a sub-lattice $\mathfrak{L}'$ of $\mathfrak{L}$. Then there exists a basis Q_1 and Q_2 for $\mathfrak{L}$ such that

$$P' = v_{11} Q_1, \quad P'' = v_{21} Q_1 + v_{22} Q_2,$$

where the v_{ij} are integers and $v_{ii} \neq 0$ ([1], p. 12, Theorem I, B).

This theorem is used as follows. Suppose P' and P'' generate a lattice $\mathfrak{L}^*$ of determinant $a > 1$. Let $K = \left[\frac{a}{0.5}\right]$; then $K \geqslant 2$. Since S is a star domain there can be no lattice point on OP' between O and P' nor on OP'' between O and P''. Locate points

$$P = \frac{k}{N}P' + \frac{1}{N}P'',$$

where $\qquad 0 < k < N \leqslant K, \quad$ for k and N integral and $N \geqslant 2$.

Consider the lattice $\mathfrak{L}$ generated by P and P'. The determinant satisfies $\frac{1}{2} < d(\mathfrak{L}) \leqslant 1$ and the lattice generated by P' and P'' is a sub-lattice of $\mathfrak{L}$. We can check such lattices for admissibility and proceed as indicated below.

We note that by the construction of S, $d(\mathfrak{L})$ is greatest for $P' = P_2$ and $P'' = P_7$ when we have $d(\mathfrak{L}) = 35/18 < 2$. Hence we have that K is either 2 or 3.

Note also that the lattice points Q, above, are of the form

$$Q = k_1 P' + k_2 P'' = \frac{r_1}{k}P' + \frac{r_2}{k}P'',$$

when k_1 and k_2 are rational and r_1, r_2, and k are integers with $k = 1$, 2 or 3.

It is not difficult to show that we may even require k_1 and k_2 to be integral, since, for a suitable re-naming of the points, P' and P'' either generate a critical lattice containing Q or they generate a sub-lattice which also contains Q.

The finite number of possibilities in cases (1) to (4) is quite large since S has fourteen sides and we must consider all possible placements of P', P'', Q_1, Q_2. At this stage the problem was programmed and cases (1) to (4) were handled on a 610 IBM computer. We thus obtained solutions to the systems of equations (i. e., the points P' and P'')and in each case the determinant of the lattice generated by P' and P''. Solutions for which the determinant satisfied the condition $\frac{1}{2} < d(\mathfrak{L}) \leqslant 1$ or which could be sub-lattices of lattices satisfying this condition were checked for admissibility. Then, from the finitely many admissible lattices we chose those of least determinant; i. e., the critical lattices.

The minimum determinant was found to be $0.5234\overline{788359} = 1583/3024$. There are only two distinct admissible lattices having this determinant. One critical lattice can be generated by the points P', P'' satisfying the system:

$$\begin{array}{ll} P' \text{ on } P_1P_2 & P' - 3P'' \text{ on } P_{11}P_{12} \\ P'' \text{ on } P_3P_4 & 2P' - 3P'' \text{ on } P_{14}P_1 \end{array}$$

which gives the solution

$$P' = \left(\frac{23}{48}, \frac{65}{42}\right), \quad P'' = \left(\frac{71}{144}, \frac{1}{2}\right).$$

The other is the reflection of this in the y-axis.

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Received August 17, 1961