ON THE POLYNOMIALS $R_\nu^{[\lambda]}(x)$, $N_\nu^{[\lambda]}(x)$ AND $M_\nu^{[\lambda]}(x)$.

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1. In a former paper[1] I have considered a class of polynomials, the poweroids, which may be defined by the relation

$$x^{\overline{\nu}|} = x\left(\frac{D}{\theta}\right)^{\nu} x^{\nu-1}, \qquad (1)$$

θ denoting the operator

$$\theta = \varphi(D) = \sum_{\nu=1}^{\infty} k_\nu D^\nu \qquad (k_1 \neq 0). \qquad (2)$$

The function $\varphi(t)$ is assumed to be analytical at the origin, and expansions in powers of D or any other theta-symbol are only permitted when the operation is applied to a polynomial.

A consideration of the form (1) leads to an examination of the polynomials

$$R_\nu^{[\lambda]}(x) = \left(\frac{D}{\theta}\right)^{\lambda} x^\nu, \qquad (3)$$

where ν is the degree of the polynomial, while λ can be any real or complex number.

These polynomials contain as particular cases several polynomials which have already proved useful in analysis. Thus, the Nörlund polynomials $B_\nu^{[\lambda]}(x)$ and $\mathcal{E}_\nu^{[\lambda]}(x)$, which again include the Bernoulli and Euler polynomials, are obtained for $\theta = \triangle$ and $\theta = \left(1 + \frac{\triangle}{2}\right) D$ respectively, see P. (105) and P. (118), and for $\theta = e^{\triangle} D$ the polynomial

[1] The Poweroid, an Extension of the Mathematical Notion of Power. Acta mathematica, Vol. 73 (1941), p. 333. This paper will be referred to below as »P».

$$G_\nu(\lambda, x) = e^{-\lambda \triangle} x^\nu$$
$$= \sum_{s=0}^{\nu} (-1)^s \frac{\lambda^s}{s!} \triangle^s x^\nu,$$

or P. (71), results. The poweroid $x^{\overline{\nu}|}$, expressed by the polynomials (3), is written

$$x^{\overline{\nu}|} = x R_{\nu-1}^{[\nu]}(x). \tag{4}$$

From (3) we obtain at once the two important relations

$$D R_\nu^{[\lambda]}(x) = \nu R_{\nu-1}^{[\lambda]}(x), \tag{5}$$

$$\theta R_\nu^{[\lambda]}(x) = \nu R_{\nu-1}^{[\lambda-1]}(x). \tag{6}$$

From these follow the expansions in powers and in poweroids

$$R_\nu^{[\lambda]}(x + y) = \sum_{s=0}^{\nu} \binom{\nu}{s} x^s R_{\nu-s}^{[\lambda]}(y), \tag{7}$$

$$R_\nu^{[\lambda]}(x + y) = \sum_{s=0}^{\nu} \binom{\nu}{s} x^{\overline{s}|} R_{\nu-s}^{[\lambda-s]}(y), \tag{8}$$

and, if we write

$$R_\nu^{[\lambda]} \equiv R_\nu^{[\lambda]}(0), \tag{9}$$

in particular

$$R_\nu^{[\lambda]}(x) = \sum_{s=0}^{\nu} \binom{\nu}{s} x^s R_{\nu-s}^{[\lambda]}, \tag{10}$$

$$R_\nu^{[\lambda]}(x) = \sum_{s=0}^{\nu} \binom{\nu}{s} x^{\overline{s}|} R_{\nu-s}^{[\lambda-s]}. \tag{11}$$

We shall presently occupy ourselves with the question of determining the coefficients $R_\nu^{[\lambda]}$, which can be done in several ways, but first we propose to find the generating function of the polynomials $R_\nu^{[\lambda]}(x)$. This is obtained by P. (37), or

$$\Phi(t) e^{xt} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \Phi(D) x^\nu, \tag{12}$$

which is valid if $\Phi(t)$ is analytical at the origin. In this formula we may, owing to the assumptions we have made about $\varphi(t)$, put

$$\Phi(t) = \left(\frac{t}{\varphi(t)}\right)^\lambda, \tag{13}$$

$\varphi(t)$ being the function defined by (2)

$$\varphi(t) = \sum_{\nu=1}^{\infty} k_\nu t^\nu \qquad (k_1 \neq 0). \tag{14}$$

We thus obtain from (12), by (3), the generating function of $R_\nu^{[\lambda]}(x)$

$$\left(\frac{t}{\varphi(t)}\right)^\lambda e^{xt} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} R_\nu^{[\lambda]}(x). \tag{15}$$

In particular, for $x = 0$, we have the generating function of $R_\nu^{[\lambda]}$

$$\left(\frac{t}{\varphi(t)}\right)^\lambda = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} R_\nu^{[\lambda]}. \tag{16}$$

These coefficients deserve to be considered separately on account of their application to certain summation problems. Thus, if we put

$$\varphi(t) = (1 + t)^{\frac{1}{h}} - 1,$$

we have $R_\nu^{[1]} = \nu!\, A_\nu$, the A_ν being the coefficients in Lubbock's summation formula.[1] If λ is any positive integer, we get the coefficients in the corresponding formula for repeated summation of any order.

2. In certain cases $\varphi(t)$ is such a simple function that $R_\nu^{[\lambda]}(x)$ can be obtained directly from (3) by expanding $\left(\frac{D}{\theta}\right)^\lambda$. But here we are chiefly concerned with the general case where $\varphi(t)$ is only known by its expansion (14), so that the main problem is to express $R_\nu^{[\lambda]}$, and hence $R_\nu^{[\lambda]}(x)$, by the coefficients k_ν. This may be done in several ways.

The first one that occurs is to derive a recurrence formula from (16), using as initial value

$$R_0^{[\lambda]} = k_1^{-\lambda} \tag{17}$$

which is obtained directly from (16) for $t = 0$. We take the logarithm on both sides of (16) and differentiate, the result being

$$\frac{\lambda}{t} - \lambda \frac{\varphi'(t)}{\varphi(t)} = \frac{\sum_{\nu=1}^{\infty} \frac{t^{\nu-1}}{(\nu-1)!} R_\nu^{[\lambda]}}{\sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} R_\nu^{[\lambda]}},$$

[1] J. F. STEFFENSEN: Interpolation § 15(5) and § 18(41), or the Danish edition (where m is written for h).

whence

$$\left[\lambda\frac{\varphi(t)}{t} - \lambda\varphi'(t)\right]\sum_{\nu=0}^{\infty}\frac{t^\nu}{\nu!}R_\nu^{[\lambda]} = \varphi(t)\sum_{\nu=1}^{\infty}\frac{t^{\nu-1}}{(\nu-1)!}R_\nu^{[\lambda]}.$$

By (14) this may be written

$$-\lambda\sum_{s=1}^{\infty} s\,k_{s+1}\,t^s \cdot \sum_{\nu=0}^{\infty}\frac{t^\nu}{\nu!}R_\nu^{[\lambda]} = \sum_{s=1}^{\infty} k_s\,t^s \cdot \sum_{\nu=0}^{\infty}\frac{t^\nu}{\nu!}R_{\nu+1}^{[\lambda]},$$

and if we now compare the coefficients of t^r on both sides, we find the required recurrence formula

$$\sum_{\nu=0}^{r} k_{r-\nu+1}\frac{r\lambda + \nu(1-\lambda)}{\nu!}R_\nu^{[\lambda]} = 0 \tag{18}$$

with the initial value (17).

3. A direct expression for $R_\nu^{[\lambda]}$ is obtained as follows. In order to expand the left-hand side of (16) we write

$$\left(\frac{t}{\varphi(t)}\right)^\lambda = \left(\frac{\varphi(t)}{t}\right)^{-\lambda}$$
$$= \left(k_1 + \sum_{\nu=1}^{\infty} k_{\nu+1}\,t^\nu\right)^{-\lambda}.$$

If, now, we put

$$\Psi = \sum_{\nu=1}^{\infty} k_{\nu+1}\,t^\nu \tag{19}$$

and expand in powers of Ψ, we find

$$\left(\frac{t}{\varphi(t)}\right)^\lambda = \sum_{n=0}^{\infty}\binom{-\lambda}{n} k_1^{-n-\lambda}\,\Psi^n. \tag{20}$$

Next, we put

$$\Psi^n = \sum_{\nu=n}^{\infty} a_\nu^{(n)}\,t^\nu, \tag{21}$$

where the coefficients $a_\nu^{(n)}$, which are independent of λ, satisfy the recurrence formula

$$\sum_{\nu=0}^{r} k_{r-\nu+2}\,[n\,r - \nu(n+1)]\,a_{\nu+n}^{(n)} = 0 \tag{22}$$

with the initial value

$$a_n^{(n)} = k_2^{\,n} \tag{23}$$

resulting from (21) and (19). We may derive (22) in the same way as (18), but it is easier to observe that (22) is really (18) with a change of notation. For, comparing (16), written in the form

$$\left(\sum_{\nu=1}^{\infty} k_\nu t^{\nu-1}\right)^{-\lambda} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} R_\nu^{[\lambda]},$$

with (21) written in the form

$$\left(\sum_{\nu=1}^{\infty} k_{\nu+1} t^{\nu-1}\right)^{n} = \sum_{\nu=0}^{\infty} a_{\nu+n}^{(n)} t^\nu,$$

it is seen at once that, if

$$k_\nu, \qquad \lambda, \qquad R_\nu^{[\lambda]}$$

are replaced respectively by

$$k_{\nu+1}, \qquad -n, \qquad \nu!\, a_{\nu+n}^{(n)},$$

then (18) is changed into (22).

If, now, we regard the coefficients $a_\nu^{(n)}$ as known and insert (21) in (20), we have

$$\left(\frac{t}{\varphi(t)}\right)^{\lambda} = \sum_{n=0}^{\infty} \binom{-\lambda}{n} k_1^{-n-\lambda} \sum_{\nu=n}^{\infty} a_\nu^{(n)} t^\nu$$

or, arranging in powers of t, taking into account that $a_\nu^{(n)} = 0$ for $n < \nu$,

$$\left(\frac{t}{\varphi(t)}\right)^{\lambda} = \sum_{\nu=0}^{\infty} t^\nu \sum_{n=0}^{\nu} \binom{-\lambda}{n} k_1^{-n-\lambda} a_\nu^{(n)},$$

so that comparison with (16) shows that

$$R_\nu^{[\lambda]} = \nu! \sum_{n=0}^{\nu} (-1)^n \frac{\lambda^{(-n)}}{n!} k_1^{-n-\lambda} a_\nu^{(n)}, \tag{24}$$

where $\lambda^{(-n)} = \lambda(\lambda+1)\cdots(\lambda+n-1)$, $\lambda^{(0)} = 1$.

It is seen that if $k_1 = 1$, as is frequently the case, then $R_\nu^{[\lambda]}$ is a polynomial in λ of degree ν.

4. A direct expression for $a_\nu^{(n)}$ is obtained from (21) by expanding the polynomial

$$(k_2 t + k_3 t^2 + \cdots + k_{\nu+1} t^\nu)^n,$$

viz.

$$a_\nu^{(n)} = n! \sum \frac{k_2^\alpha k_3^\beta k_4^\gamma \ldots}{\alpha!\, \beta!\, \gamma! \ldots}, \tag{25}$$

where the summation extends to all positive integers α, β, γ ... for which simultaneously

$$\alpha + \beta + \gamma + \cdots = n \tag{26}$$

and

$$\alpha + 2\beta + 3\gamma + \cdots = \nu. \tag{27}$$

We state below a few special results, found by (25) and checked by (22)

$$a_n^{(n)} = k_2^n.$$

$$a_{n+1}^{(n)} = n\, k_3 k_2^{n-1}.$$

$$a_{n+2}^{(n)} = n\, k_4 k_2^{n-} + \binom{n}{2} k_3^2 k_2^{n-2}.$$

$$a_{n+3}^{(n)} = n\, k_5 k_2^{n-1} + n^{(2)} k_4 k_3 k_2^{n-2} + \binom{n}{3} k_3^3 k_2^{n-3}.$$

$$a_{n+4}^{(n)} = n\, k_6 k_2^{n-1} + \binom{n}{2}(2 k_5 k_3 + k_4^2) k_2^{n-2} + \frac{n^{(3)}}{2} k_4 k_3^2 k_2^{n-3} + \binom{n}{4} k_3^4 k_2^{n-4}.$$

$$a_{n+5}^{(n)} = n\, k_7 k_2^{n-1} + n^{(2)}(k_6 k_3 + k_5 k_4) k_2^{n-2} + \frac{n^{(3)}}{2}(k_5 k_3 + k_4^2) k_3 k_2^{n-3} + \\ + \frac{n^{(4)}}{6} k_4 k_3^3 k_2^{n-4} + \binom{n}{5} k_3^5 k_2^{n-5}.$$

$$a_{n+6}^{(n)} = n\, k_8 k_2^{n-1} + \binom{n}{2}(2 k_7 k_3 + 2 k_6 k_4 + k_5^2) k_2^{n-2} + \binom{n}{3}(3 k_6 k_3^2 + 6 k_5 k_4 k_3 + k_4^3) k_2^{n-3} + \\ + \binom{n}{4}(4 k_5 k_3^3 + 6 k_4^2 k_3^2) k_2^{n-4} + 5\binom{n}{5} k_4 k_3^4 k_2^{n-5} + \binom{n}{6} k_3^6 k_2^{n-6}.$$

We further have, by (21) and (19)

$$a_0^{(0)} = 1, \qquad a_\nu^{(0)} = 0 \quad (\nu > 0), \tag{28}$$

$$a_0^{(\ )} = 0, \qquad a_\nu^{(1)} = k_{\nu+1} \quad (\nu > 0). \tag{29}$$

In the expression (24) for $R_\nu^{[\lambda]}$ we want $a_\nu^{(0)}$, $a_\nu^{(1)}$, $a_\nu^{(2)}$, ... $a_\nu^{(\nu)}$. These may be written down as far as $\nu = 8$ by the formulas given above. The results are, leaving out $a_\nu^{(0)}$ and $a_\nu^{(1)}$, given by (28) and (29),

$\nu = 2.$ $a_2^{(2)} = k_2^2.$

$\nu = 3.$ $a_3^{(2)} = 2 k_3 k_2.$ $a_3^{(3)} = k_2^3.$

$\nu = 4.$ $a_4^{(2)} = 2 k_4 k_2 + k_3^2.$ $a_4^{(3)} = 3 k_3 k_2^2.$ $a_4^{(4)} = k_2^4.$

$\nu = 5.$ $a_5^{(2)} = 2 k_5 k_2 + 2 k_4 k_3.$ $a_5^{(3)} = 3 k_4 k_2 + 3 k_3^2 k_2.$
$a_5^{(4)} = 4 k_3 k_2^3.$ $a_5^{(5)} = k_2^5.$

$\nu = 6.\quad a_6^{(2)} = 2\,k_6 k_2 + k_4^2 + 2\,k_5 k_3.\qquad a_6^{(3)} = 3\,k_5 k_2^2 + 6\,k_4 k_3 k_2 + k_3^3.$

$a_6^{(4)} = 4\,k_4 k_2^3 + 6\,k_3^2 k_2^2.\qquad a_6^{(5)} = 5\,k_3 k_2^4.\qquad a_6^{(6)} = k_2^6.$

$\nu = 7.\quad a_7^{(2)} = 2\,k_7 k_2 + 2\,k_5 k_4 + 2\,k_6 k_3.\qquad a_7^{(3)} = 3\,k_6 k_2^2 + 3\,k_4 k_3^2 + 3\,k_4^2 k_2 + 6\,k_5 k_3 k_2.$

$a_7^{(4)} = 4\,k_5 k_2^3 + 12\,k_4 k_3 k_2^2 + 4\,k_3^3 k_2.\qquad a_7^{(5)} = 5\,k_4 k_2^4 + 10\,k_3^2 k_2^3.$

$a_7^{(6)} = 6\,k_3 k_2^5.\qquad a_7^{(7)} = k_2^7.$

$\nu = 8.\quad a_8^{(2)} = 2\,k_8 k_2 + 2\,k_7 k_3 + 2\,k_6 k_4 + k_5^2.$

$a_8^{(3)} = 3\,k_7 k_2^2 + 6\,k_6 k_3 k_2 + 6\,k_5 k_4 k_2 + 3\,k_5 k_3^2 + 3\,k_4^2 k_3.$

$a_8^{(4)} = 4\,k_6 k_2^3 + 12\,k_5 k_3 k_2^2 + 12\,k_4 k_3^2 k_2 + 6\,k_4^2 k_2^2 + k_3^4.$

$a_8^{(5)} = 5\,k_5 k_2^4 + 20\,k_4 k_3 k_2^3 + 10\,k_3^3 k_2^2.$

$a_8^{(6)} = 6\,k_4 k_2^5 + 15\,k_3^2 k_2^4.\qquad a_8^{(7)} = 7\,k_3 k_2^6.\qquad a_8^{(8)} = k_2^8.$

By means of these results, $R_\nu^{[\lambda]}$ may be immediately written down by (24), and thereafter $R_\nu^{[\lambda]}(x)$ by (10) or, in terms of poweroids, by (11).

In the particular case where $\lambda = -1$ we have directly by (16) and (10)

$$R_\nu^{[-1]} = \nu!\,k_{\nu+1},\quad R_\nu^{[-1]}(x) = \nu!\sum_{s=0}^{\nu} k_{\nu-s+1}\frac{x^s}{s!}. \tag{30}$$

5. A formula of some generality, a sort of binomial theorem for the R-polynomials, is obtained as follows. We replace, in (3), λ by $\lambda + \mu$, and x by $x + y$, writing the result in the form

$$R_\nu^{[\lambda+\mu]}(x + y) = \left(\frac{D}{\theta}\right)^\lambda \left(\frac{D}{\theta}\right)^\mu (x + y)^\nu.$$

Here, it evidently does not matter whether $\left(\frac{D}{\theta}\right)$ acts on x or on y. We may, therefore, let $\left(\frac{D}{\theta}\right)^\lambda$ act on x, and $\left(\frac{D}{\theta}\right)^\mu$ on y. Expanding $(x+y)^\nu$ by the binomial theorem and performing the two operations, we find, by (3),

$$R_\nu^{[\lambda+\mu]}(x + y) = \sum_{s=0}^{\nu} \binom{\nu}{s} R_s^{[\lambda]}(x)\, R_{\nu-s}^{[\mu]}(y) \tag{31}$$

which is the binomial theorem for our polynomials.

Several particular cases of this formula are of interest. Thus, observing that, by (3)

$$R_\nu^{[0]}(x) = x^\nu, \tag{32}$$

we obtain, putting $\mu = -\lambda$ in (31),

$$(x+y)^{\nu} = \sum_{s=0}^{\nu} \binom{\nu}{s} R_s^{[\lambda]}(x)\, R_{\nu-s}^{[-\lambda]}(y), \tag{33}$$

and from this, for $y = 0$,

$$x^{\nu} = \sum_{s=0}^{\nu} \binom{\nu}{s} R_s^{[\lambda]}(x)\, R_{\nu-s}^{[-\lambda]}. \tag{34}$$

This may be looked upon either as the expansion of x^{ν} in R-polynomials, or as a recurrence formula for $R_{\nu}^{[\lambda]}(x)$. In the latter case we have as initial value

$$R_0^{[\lambda]}(x) = k_1^{-\lambda}, \tag{35}$$

resulting from (15) for $t = 0$.

Next, putting $\mu = 0$ in (31), we have, by (32),

$$R_{\nu}^{[\lambda]}(x+y) = \sum_{s=0}^{\nu} \binom{\nu}{s} R_s^{[\lambda]}(x)\, y^{\nu-s} \tag{36}$$

which is really only the Maclaurin expansion in y.

Putting $y = -x$ in (36) we find

$$R_{\nu}^{[\lambda]} = \sum_{s=0}^{\nu} (-1)^{\nu-s} \binom{\nu}{s} x^{\nu-s} R_s^{[\lambda]}(x), \tag{37}$$

another recurrence formula for $R_{\nu}^{[\lambda]}(x)$, which may also be obtained from (7).

We further note that, putting $y = 0$ in (31), we have

$$R_{\nu}^{[\lambda+\mu]}(x) = \sum_{s=0}^{\nu} \binom{\nu}{s} R_s^{[\lambda]}(x)\, R_{\nu-s}^{[\mu]}, \tag{38}$$

and putting $x = 0$ in this

$$R_{\nu}^{[\lambda+\mu]} = \sum_{s=0}^{\nu} \binom{\nu}{s} R_s^{[\lambda]}\, R_{\nu-s}^{[\mu]}, \tag{39}$$

or the binomial theorem for the R-coefficients.

These binomial theorems are evidently generalizations of corresponding theorems by Nörlund[1] (in the case where the intervals of differencing are identical).

6. The R-polynomials may be generalized considerably without losing their essential properties. We may, in fact, in (3) replace D by any theta-symbol, provided that x^{ν} is replaced by the corresponding poweroid. Let, therefore, θ

[1] N. E. Nörlund: Differenzenrechnung, chapter VI.

and θ_1 be any two theta-symbols, $x^{\overline{\nu}|}$ and $x_1^{\overline{\nu}|}$ the corresponding poweroids; we write then, instead of (3),

$$N_\nu^{[\lambda]}(x) = \left(\frac{\theta}{\theta_1}\right)^\lambda x^{\overline{\nu}|}. \tag{40}$$

It is seen at once that the N-polynomials satisfy the two fundamental relations

$$\theta N_\nu^{[\lambda]}(x) = \nu N_{\nu-1}^{[\lambda]}(x) \tag{41}$$

$$\theta_1 N_\nu^{[\lambda]}(x) = \nu N_{\nu-1}^{[\lambda-1]}(x), \tag{42}$$

corresponding to (5) and (6).

From these polynomials we obtain the R-polynomials by choosing $\theta = D$ $x^{\overline{\nu}|} = x^\nu$, but the N-polynomials contain many other interesting polynomials. Thus, for instance, if $\theta = \underset{\omega}{\triangle}$, $x^{\overline{\nu}|} = x_\omega^{(\nu)}$, where $x_\omega^{(\nu)}$ is the factorial

$$x_\omega^{(\nu)} = x(x-\omega)\ldots(x-\nu\omega+\omega), \qquad x_\omega^{(0)} = 1, \tag{43}$$

and $\theta_1 = \triangle$, we obtain the polynomial

$$x_{\omega\lambda}^\nu = \left(\frac{\underset{\omega}{\triangle}}{\triangle}\right)^\lambda x_\omega^{(\nu)}. \tag{44}$$

I have on a former occasion[1] dealt with this polynomial in the case where λ is a non-negative integer, n. In that case, the polynomial is completely determined by satisfying the two relations (41) and (42), or

$$\underset{\omega}{\triangle} x_{\omega n}^\nu = \nu x_{\omega n}^{\nu-1},$$

$$\triangle x_{\omega n}^\nu = \nu x_{\omega, n-1}^{\nu-1},$$

besides the initial conditions $x_{\omega n}^0 = 1$ and $x_{\omega 0}^\nu = x_\omega^{(\nu)}$. This proves that it can be represented in the convenient form (44), where λ may, however, be any real or complex number.

For $\omega \to 0$ we obtain from (44) $x_{0\lambda}^\nu = B_\nu^{[\lambda]}(x)$.

Related to (44) is the corresponding »central» polynomial

$$x_{\omega\lambda}^{[\nu]} = \left(\frac{\underset{\omega}{\delta}}{\delta}\right)^\lambda x_\omega^{[\nu]}, \tag{45}$$

[1] J. F. STEFFENSEN: On a Generalization of Nörlund's Polynomials. Det Kgl. Danske Videnskabernes Selskab. Mathematisk-fysiske Meddelelser, VII, 5 (1926). Referred to below as »G.N.P.».

where central differences and central factorials

$$\underset{\omega}{\delta} = \frac{1}{\omega}\left(E^{\frac{\omega}{2}} - E^{-\frac{\omega}{2}}\right), \quad x_{\omega}^{[\nu]} = x\left(x + \frac{\nu - 2}{2}\omega\right)_{\omega}^{(\nu-1)}$$

are employed.

We may further mention the polynomials

$$b_{\nu}^{[\lambda]}(x) = \left(\frac{\Delta}{D}\right)^{\lambda} x^{(\nu)} \tag{46}$$

and

$$e_{\nu}^{[\lambda]}(x) = \left(1 + \frac{\Delta}{2}\right)^{-\lambda} x^{(\nu)} \tag{47}$$

which are related to the Nörlund polynomials $B_{\nu}^{[\lambda]}(x)$ and $\mathcal{E}_{\nu}^{[\lambda]}(x)$. The case $\lambda = 1$ has been dealt with by Charles Jordan[1], who calls $\frac{1}{\nu!} b_{\nu}^{[1]}(x)$ the Bernoulli polynomial of the second kind, and $\frac{1}{\nu!} e_{\nu}^{[1]}(x)$ Boole's polynomial.

The corresponding central polynomials are

$$\beta_{\nu}^{[\lambda]}(x) = \left(\frac{\delta}{D}\right)^{\lambda} x^{[\nu]}, \tag{48}$$

$$\varepsilon_{\nu}^{[\lambda]}(x) = \Box^{-\lambda} x^{[\nu]}, \tag{49}$$

where $\Box = \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)$.

7. The theory of the N-polynomials runs parallel to that of the R-polynomials. Writing

$$N_{\nu}^{[\lambda]} \equiv N_{\nu}^{[\lambda]}(0), \tag{50}$$

we obtain from (41) and (42) the two expansions corresponding to (10) and (11)

$$N_{\nu}^{[\lambda]}(x) = \sum_{s=0}^{\nu} \binom{\nu}{s} x^{\overline{s|}} N_{\nu-s}^{[\lambda]} \tag{51}$$

and

$$N_{\nu}^{[\lambda]}(x) = \sum_{s=0}^{\nu} \binom{\nu}{s} x_{1}^{\overline{s|}} N_{\nu-s}^{[\lambda-s]} \tag{52}$$

in the poweroids $x^{\overline{s|}}$ and $x_{1}^{\overline{s|}}$ respectively.

[1] Charles Jordan: Calculus of Finite Differences, p. 265 and p. 317. The notation differs from ours.

More generally we have

$$N_\nu^{[\lambda]}(x+y) = \sum_{s=0}^{\nu} \binom{\nu}{s} x^{\overline{s}|} N_{\nu-s}^{[\lambda]}(y), \tag{53}$$

$$N_\nu^{[\lambda]}(x+y) = \sum_{s=0}^{\nu} \binom{\nu}{s} x_1^{\overline{s}|} N_{\nu-s}^{[\lambda-s]}(y). \tag{54}$$

In order to obtain the generating function of $N_\nu^{[\lambda]}(x)$, we must begin by *generalizing* (12). According to P. (34) and P. (33) we have

$$e^{xt} = \sum_{\nu=0}^{\infty} \frac{x^{\overline{\nu}|}}{\nu!} \zeta^\nu, \qquad \zeta = \varphi(t)$$

for sufficiently small $|\zeta|$ and all x. If now

$$\Phi(\zeta) = \sum_{\nu=0}^{\infty} c_\nu \zeta^\nu$$

means any function which is analytical at the origin, and we require the coefficient of ζ^ν in the expansion of $\Phi(\zeta) e^{xt}$, this coefficient is

$$\sum_{s=0}^{\nu} c_{\nu-s} \frac{x^{\overline{s}|}}{s!} = \frac{1}{\nu!} \Phi(\theta) x^{\overline{\nu}|}.$$

We therefore have

$$\Phi(\zeta) e^{xt} = \sum_{\nu=0}^{\infty} \frac{\zeta^\nu}{\nu!} \Phi(\theta) x^{\overline{\nu}|}, \tag{55}$$

where t is regarded as the function of ζ determined by $\zeta = \varphi(t)$.

This theorem contains (12), which is obtained for $\theta = D$, $x^{\overline{\nu}|} = x^\nu$, $\zeta = \varphi(t) = t$.

Since any theta-symbol may be expanded in powers of any other theta-symbol, we may, in extension of (2), assume that θ_1 is given in the form

$$\theta_1 = \varphi_1(\theta) = \sum_{\nu=1}^{\infty} h_\nu \theta^\nu \qquad (h_1 \neq 0). \tag{56}$$

Corresponding to this we write, when θ and θ_1 are replaced by numbers, ζ instead of θ, and ζ_1 instead of θ_1, thus

$$\zeta_1 = \varphi_1(\zeta) = \sum_{\nu=1}^{\infty} h_\nu \zeta^\nu \qquad (h_1 \neq 0). \tag{57}$$

We now put, in (55),

$$\Phi(\zeta) = \left(\frac{\zeta}{\varphi_1(\zeta)}\right)^\lambda, \qquad \Phi(\theta) = \left(\frac{\theta}{\varphi_1(\theta)}\right)^\lambda = \left(\frac{\theta}{\theta_1}\right)^\lambda \tag{58}$$

and find, by (40),

$$\left(\frac{\zeta}{\varphi_1(\zeta)}\right)^\lambda e^{xt} = \sum_{\nu=0}^{\infty} \frac{\zeta^\nu}{\nu!} N_\nu^{[\lambda]}(x). \tag{59}$$

Since t is a function of ζ, (59) represents the generating function of $N_\nu^{[\lambda]}(x)$, and is a generalization of (15).

Putting $x = 0$ in (59), we have the generating function of $N_\nu^{[\lambda]}$

$$\left(\frac{\zeta}{\varphi_1(\zeta)}\right)^\lambda = \sum_{\nu=0}^{\infty} \frac{\zeta^\nu}{\nu!} N_\nu^{[\lambda]}, \tag{60}$$

being an extension of (16).

It now appears that the results obtained for $R_\nu^{[\lambda]}$ can be utilized for $N_\nu^{[\lambda]}$ by a change of notation. Comparing, in fact, (60) with (16), and (56) with (2), we see that if t is replaced by ζ, and $\varphi(t)$ by $\varphi_1(\zeta)$, that is, k_ν by h_ν, then R is replaced by N. Hence, we may write down from (18) and (17) the recurrence formula

$$\sum_{\nu=0}^{r} h_{r-\nu+1} \frac{r\lambda + \nu(1-\lambda)}{\nu!} N_\nu^{[\lambda]} = 0 \tag{61}$$

with the initial value

$$N_0^{[\lambda]} = h_1^{-\lambda}. \tag{62}$$

Further, if we write

$$\Psi_1 = \sum_{\nu=1}^{\infty} h_{\nu+1} \zeta^\nu \tag{63}$$

and

$$\Psi_1^n = \sum_{\nu=n}^{\infty} b_\nu^{(n)} \zeta^\nu \tag{64}$$

instead of (19) and (21), we have instead of (22) and (23) the recurrence formula

$$\sum_{\nu=0}^{r} h_{r-\nu+2} [nr - \nu(n+1)] b_{\nu+n}^{(n)} = 0 \tag{65}$$

with the initial value

$$b_n^{(n)} = h_2^n.$$

From (24) we obtain the direct expression

$$N_\nu^{[\lambda]} = \nu! \sum_{n=0}^{\nu} (-1)^n \frac{\lambda^{(-n)}}{n!} h_1^{-n-\lambda} b_\nu^{(n)} \tag{66}$$

and from (25)

$$b_\nu^{(n)} = n! \sum \frac{h_2^\alpha h_3^\beta h_4^\gamma \ldots}{\alpha! \beta! \gamma! \ldots} \tag{67}$$

where α, β, γ, ... satisfy the simultaneous relations (26) and (27). A number of special values of $b_\nu^{(n)}$, expressed by h_ν, are obtained from the values of $a_\nu^{(n)}$ given above, if we replace a by b, and k by h; we need not write them down.

Finally we note the particular cases resulting from (30)

$$N_\nu^{[-1]} = \nu! \, h_{\nu+1}, \qquad N_\nu^{[-1]}(x) = \nu! \sum_{s=0}^{\nu} h_{\nu-s+1} \frac{x^s}{s!}. \tag{68}$$

8. A binomial theorem for the N-polynomials may be derived as follows. From (40) we obtain

$$N_\nu^{[\lambda+\mu]}(x+y) = \left(\frac{\theta}{\theta_1}\right)^\lambda \left(\frac{\theta}{\theta_1}\right)^\mu (x+y)^{\overline{\nu|}},$$

where we may let $\left(\frac{\theta}{\theta_1}\right)^\lambda$ act on x, and $\left(\frac{\theta}{\theta_1}\right)^\mu$ on y. Now we have, by P. (141),

$$(x+y)^{\overline{\nu|}} = \sum_{s=0}^{\nu} \binom{\nu}{s} x^{\overline{s|}} y^{\overline{\nu-s|}}, \tag{69}$$

and on inserting this above we find the desired theorem

$$N_\nu^{[\lambda+\mu]}(x+y) = \sum_{s=0}^{\nu} \binom{\nu}{s} N_s^{[\lambda]}(x) \, N_{\nu-s}^{[\mu]}(y), \tag{70}$$

which has the same form as (31).

From (70) we obtain formulas corresponding to (33), (34) and (36)—(39). Thus, since, by (40),

$$N_\nu^{[0]}(x) = x^{\overline{\nu|}}, \tag{71}$$

we find on putting $\mu = -\lambda$ in (70)

$$(x+y)^{\overline{\nu|}} = \sum_{s=0}^{\nu} \binom{\nu}{s} N_s^{[\lambda]}(x) \, N_{\nu-s}^{[-\lambda]}(y) \tag{72}$$

and from this for $y = 0$

$$x^{\overline{\nu|}} = \sum_{s=0}^{\nu} \binom{\nu}{s} N_s^{[\lambda]}(x) \, N_{\nu-s}^{[-\lambda]}, \tag{73}$$

being the expansion of $x^{\overline{\nu}|}$ in N-polynomials, or, if preferred, a recurrence formula for these. In the latter case we have the initial value

$$N_0^{[\lambda]}(x) = h_1^{-\lambda} \tag{74}$$

resulting from (59) for $\zeta = 0$, since t vanishes with ζ.

For $\mu = 0$, (70) yields, by (71),

$$N_\nu^{[\lambda]}(x+y) = \sum_{s=0}^{\nu} \binom{\nu}{s} N_s^{[\lambda]}(x)\, y^{\overline{\nu-s}|}, \tag{75}$$

and hence we find for $y = -x$

$$N_\nu^{[\lambda]} = \sum_{s=0}^{\nu} \binom{\nu}{s} (-x)^{\overline{\nu-s}|} N_s^{[\lambda]}(x)$$

which is another recurrence formula for the N-polynomials. A similar formula is obtained by putting $x = -y$ in (54) and writing thereafter x for y.

If, finally, we put $y = 0$ in (70), we find

$$N_\nu^{[\lambda+\mu]}(x) = \sum_{s=0}^{\nu} \binom{\nu}{s} N_s^{[\lambda]}(x)\, N_{\nu-s}^{[\mu]} \tag{76}$$

and, putting $x = 0$ in this,

$$N_\nu^{[\lambda+\mu]} = \sum_{s=0}^{\nu} \binom{\nu}{s} N_s^{[\lambda]} N_{\nu-s}^{[\mu]}, \tag{77}$$

being the binomial theorem for the N-coefficients. The two last formulas have the same form as (38) and (39), only with R instead of N.

9. As an application, we will consider the polynomials $x_{\omega\lambda}^{\nu}$ defined by (44). We have here

$$\theta = \underset{\omega}{\Delta} = \frac{e^{\omega D} - 1}{\omega}, \qquad \theta_1 = \Delta = e^D - 1,$$

so that

$$\zeta = \frac{1}{\omega}(e^{\omega t} - 1), \qquad \zeta_1 = e^t - 1 = (1 + \omega\zeta)^{\frac{1}{\omega}} - 1.$$

Hence

$$h_\nu = \frac{1_{\omega}^{(\nu)}}{\nu!} = \frac{1}{\nu!}(1-\omega)(1-2\omega)\ldots(1-\nu\omega+\omega).$$

The generating function is, therefore, by (59)

$$\left(\frac{\zeta}{(1+\omega\zeta)^{\frac{1}{\omega}} - 1}\right)^{\lambda} (1+\omega\zeta)^{\frac{x}{\omega}} = \sum_{\nu=0}^{\infty} \frac{\zeta^\nu}{\nu!} x_{\omega\lambda}^{\nu}. \tag{78}$$

For $x = 0$ we have the generating function of the coefficients $0^\nu_{\omega\lambda}$

$$\left(\frac{\zeta}{(1+\omega\zeta)^{\frac{1}{\omega}}-1}\right)^\lambda = \sum_{\nu=0}^{\infty}\frac{\zeta^\nu}{\nu!}0^\nu_{\omega\lambda}. \tag{79}$$

From (41) and (42) we find

$$\mathop{\triangle}_{\omega} x^\nu_{\omega\lambda} = \nu x^{\nu-1}_{\omega\lambda}, \tag{80}$$

$$\triangle x^\nu_{\omega\lambda} = \nu x^{\nu-1}_{\omega,\lambda-1}. \tag{81}$$

The binomial theorem is, by (70),

$$(x+y)^\nu_{\omega,\lambda+\mu} = \sum_{s=0}^{\nu}\binom{\nu}{s} x^s_{\omega\lambda} y^{\nu-s}_{\omega\mu}. \tag{82}$$

We note the following particular cases of (82). Putting $\mu = -\lambda$, we have, since, by (44), $x^\nu_{\omega 0} = x^{(\nu)}_\omega$,

$$(x+y)^{(\nu)}_\omega = \sum_{s=0}^{\nu}\binom{\nu}{s} x^s_{\omega\lambda} y^{\nu-s}_{\omega,-\lambda}, \tag{83}$$

and from this, for $y = 0$,

$$x^{(\nu)}_\omega = \sum_{s=0}^{\nu}\binom{\nu}{s} 0^{\nu-s}_{\omega,-\lambda} x^s_{\omega\lambda}, \tag{84}$$

a recurrence formula for $x^\nu_{\omega\lambda}$, the initial value being

$$x^0_{\omega\lambda} = 1, \tag{85}$$

resulting from (78) for $\zeta = 0$. We may also look upon (84) as the expansion of the factorial on the left in polynomials $x^\nu_{\omega\lambda}$.

Putting $\mu = 0$ in (82), we find

$$(x+y)^\nu_{\omega\lambda} = \sum_{s=0}^{\nu}\binom{\nu}{s} x^s_{\omega\lambda} y^{(\nu-s)}_\omega \tag{86}$$

and from this, for $y = -x$,

$$0^\nu_{\omega\lambda} = \sum_{s=0}^{\nu}\binom{\nu}{s} (-x)^{(\nu-s)}_\omega x^s_{\omega\lambda}, \tag{87}$$

another recurrence formula for $x^\nu_{\omega\lambda}$.

Finally, putting $y = 0$ in (82), we have

$$x^\nu_{\omega,\lambda+\mu} = \sum_{s=0}^{\nu}\binom{\nu}{s} 0^{\nu-s}_{\omega\mu} x^s_{\omega\lambda} \tag{88}$$

and, putting $x = 0$ in this, the binomial theorem for the coefficients $0^\nu_{\omega\lambda}$

$$0^\nu_{\omega,\lambda+\mu} = \sum_{s=0}^{\nu} \binom{\nu}{s} 0^{\nu-s}_{\omega\mu} 0^s_{\omega\lambda}. \tag{89}$$

By (80) and (81) we find the two expansions of $x^\nu_{\omega\lambda}$ in factorials

$$x^\nu_{\omega\lambda} = \sum_{s=0}^{\nu} \binom{\nu}{s} 0^{\nu-s}_{\omega\lambda} x^{(s)}_{\omega}, \tag{90}$$

$$x^\nu_{\omega\lambda} = \sum_{s=0}^{\nu} \binom{\nu}{s} 0^{\nu-s}_{\omega,\lambda-s} x^{(s)}. \tag{91}$$

More generally we have

$$(x+y)^\nu_{\omega\lambda} = \sum_{s=0}^{\nu} \binom{\nu}{s} y^{\nu-s}_{\omega\lambda} x^{(s)}_{\omega}, \tag{92}$$

$$(x+y)^\nu_{\omega\lambda} = \sum_{s=0}^{\nu} \binom{\nu}{s} y^{\nu-s}_{\omega,\lambda-s} x^{(s)}. \tag{93}$$

Several of these relations have been derived in G. N. P., but only for integral, non-negative values of λ.

10. Another application of the N-polynomials may be made to the generalized Laguerre polynomials $L^{[\alpha]}_\nu(x)$[1]. We put, in (40)[2],

$$\theta = \frac{D}{1-D}, \qquad x^{\overline{\nu|}} = q_\nu(x) = \sum_{s=0}^{\nu-1} (-1)^s \binom{\nu}{s} (\nu-1)^{(s)} x^{\nu-s}; \tag{94}$$

further $\theta_1 = D$, $x_1^{\overline{\nu|}} = x^\nu$. Hence

$$\left.\begin{aligned} N^{[\lambda]}_\nu(x) &= (1-D)^{-\lambda} q_\nu(x) \\ &= \sum_{s=0}^{\nu} \binom{\lambda+s-1}{s} q^{(s)}_\nu(x). \end{aligned}\right\} \tag{95}$$

In order to show that this polynomial, after multiplication by a suitable constant, is a (generalized) Laguerre polynomial, we observe that we have here $\theta_1 = \frac{\theta}{1+\theta}$, whence $\zeta_1 = \frac{\zeta}{1+\zeta}$, so that, since $\zeta = \frac{t}{1-t}$, $t = \frac{\zeta}{1+\zeta}$, (59) becomes

$$(1+\zeta)^\lambda e^{\frac{x\zeta}{1+\zeta}} = \sum_{\nu=0}^{\infty} \frac{\zeta^\nu}{\nu!} N^{[\lambda]}_\nu(x). \tag{96}$$

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[1] PÓLYA und SZEGÖ: Aufgaben und Lehrsätze aus der Analysis, II p. 294. These authors write $L^{(\alpha)}_\nu(x)$ while I prefer $L^{[\alpha]}_\nu(x)$.

[2] P. (98).

Comparison with the generating function of $L_\nu^{[\alpha]}(x)$ shows thereafter that

$$N_\nu^{[\lambda]}(x) = (-1)^\nu \nu!\, L_\nu^{[-\lambda-1]}(x). \tag{97}$$

We may now write down a number of results, several of them already known, for $L_\nu^{[\alpha]}(x)$.

From (95) we obtain

$$\left.\begin{aligned} L_\nu^{[\alpha]}(x) &= \frac{(-1)^\nu}{\nu!}(1-D)^{\alpha+1} q_\nu(x) \\ &= \frac{1}{\nu!}\sum_{s=0}^{\nu}(-1)^{\nu+s}\binom{\alpha+1}{s} q_\nu^{(s)}(x), \end{aligned}\right\} \tag{98}$$

and (96) is written

$$(1+\zeta)^{-\alpha-1} e^{\frac{x\zeta}{1+\zeta}} = \sum_{\nu=0}^{\infty}(-1)^\nu \zeta^\nu L_\nu^{[\alpha]}(x). \tag{99}$$

Putting $x = 0$ in this, we find, on expanding the left-hand side,

$$L_\nu^{[\alpha]} = \binom{\alpha+\nu}{\nu}. \tag{100}$$

From (42) we find

$$D\, L_\nu^{[\alpha]}(x) = -L_{\nu-1}^{[\alpha+1]}(x) \tag{101}$$

and from (41)

$$\frac{D}{1-D} L_\nu^{[\alpha]}(x) = -L_{\nu-1}^{[\alpha]}(x) \tag{102}$$

or

$$D\, L_\nu^{[\alpha]}(x) = D\, L_{\nu-1}^{[\alpha]}(x) - L_{\nu-1}^{[\alpha]}(x). \tag{103}$$

Hence, comparing (103) and (101), we have

$$L_\nu^{[\alpha+1]}(x) = L_\nu^{[\alpha]}(x) - D\, L_\nu^{[\alpha]}(x). \tag{104}$$

By (53) we obtain

$$L_\nu^{[\alpha]}(x+y) = \sum_{s=0}^{\nu}\frac{(-1)^s}{s!} q_s(x)\, L_{\nu-s}^{[\alpha]}(y), \tag{105}$$

whence, for $y = 0$, by (100),

$$L_\nu^{[\alpha]}(x) = \frac{1}{\nu!}\sum_{s=0}^{\nu}(-1)^s\binom{\nu}{s}(\alpha+\nu-s)^{(\nu-s)} q_s(x). \tag{106}$$

Similarly, we find, by (54),

$$L_\nu^{[\alpha]}(x+y) = \sum_{s=0}^{\nu}\frac{(-1)^s}{s!} x^s L_{\nu-s}^{[\alpha+s]}(y) \tag{107}$$

and, for $y = 0$, the well-known explicit expression

$$L_\nu^{[\alpha]}(x) = \frac{1}{\nu!}\sum_{s=0}^{\nu}(-1)^s\binom{\nu}{s}(\alpha+\nu)^{(\nu-s)}x^s. \tag{108}$$

The binomial theorem for thê Laguerre polynomials is[1], by (70)

$$L_\nu^{[\alpha+\beta+1]}(x+y) = \sum_{s=0}^{\nu} L_s^{[\alpha]}(x)\, L_{\nu-s}^{[\beta]}(y). \tag{109}$$

By (98) we have

$$L_\nu^{[-1]}(x) = \frac{(-1)^\nu}{\nu!} q_\nu(x), \tag{110}$$

showing that $q_\nu(x)$ is, apart from a constant factor, a special Laguerre polynomial.

Putting now $\beta = -\alpha - 2$, we find, from (109) and (110),

$$q_\nu(x+y) = (-1)^\nu \nu! \sum_{s=0}^{\nu} L_s^{[\alpha]}(x)\, L_{\nu-s}^{[-\alpha-2]}(y). \tag{111}$$

Putting $y = 0$ in (111), we obtain, by (100),

$$q_\nu(x) = \nu! \sum_{s=0}^{\nu}(-1)^s\binom{\alpha+1}{\nu-s} L_s^{[\alpha]}(x), \tag{112}$$

an expansion of $q_\nu(x)$ in Laguerre polynomials, which expansion may be regarded as the inversion of (106).

A similar expansion is found by (98), writing this formula

$$q_\nu(x) = (-1)^\nu \nu!\,(1-D)^{-\alpha-1} L_\nu^{[\alpha]}(x), \tag{113}$$

whence, on expanding and applying (101),

$$q_\nu(x) = \nu! \sum_{s=0}^{\nu}(-1)^{\nu+s}\binom{\alpha+s}{s} L_{\nu-s}^{[\alpha+s]}(x). \tag{114}$$

Putting $\alpha = 0$ and writing $\nu - s$ for s, we have the simpler expansion

$$q_\nu(x) = \nu! \sum_{s=0}^{\nu}(-1)^s L_s^{[\nu-s]}(x). \tag{115}$$

[1] This result shows that it would be more consistent to define the Laguerre polynomial as $\mathfrak{L}_\nu^{[\alpha]}(x) = L_\nu^{[\alpha-1]}(x)$.

If, in (109), we put $\beta = -1$, we find, by (110),

$$L_\nu^{[\alpha]}(x+y) = \sum_{s=0}^{\nu} \frac{(-1)^{\nu-s}}{(\nu-s)!} L_s^{[\alpha]}(x)\, q_{\nu-s}(y), \tag{116}$$

or (105) in a different notation, whence, for $y = -x$, by (100),

$$\binom{\alpha+\nu}{\nu} = \sum_{s=0}^{\nu} \frac{(-1)^{\nu-s}}{(\nu-s)!} L_s^{[\alpha]}(x)\, q_{\nu-s}(-x). \tag{117}$$

Finally, we obtain from (109), for $y = 0$,

$$L_\nu^{[\alpha+\beta+1]}(x) = \sum_{s=0}^{\nu} \binom{\beta+\nu-s}{\nu-s} L_s^{[\alpha]}(x). \tag{118}$$

For $\alpha = -1$ we have again (106), with β instead of α.

11. An extension of (1) is P. (23) which may be written, by a change of notation,

$$x_1^{\overline{\nu}|} = x\left(\frac{\theta}{\theta_1}\right)^{\nu} x^{\overline{1}|-1}. \tag{119}$$

A consideration of this formula, which allows to obtain one poweroid from another, leads to examining the polynomials $M_\nu^{[\lambda]}(x)$, defined by

$$M_\nu^{[\lambda]}(x) = \left(\frac{\theta}{\theta_1}\right)^{\lambda} x^{\overline{\nu+1}|-1} \tag{120}$$

and analogous to the N-polynomials defined by (40).

In this notation (119) may be written

$$x_1^{\overline{\nu}|} = x\, M_{\nu-1}^{[\nu]}(x), \tag{121}$$

in analogy with (4).

Owing to the relation P. (17), or

$$\theta' x^{\overline{\nu+1}|-1} = x^{\overline{\nu}|}, \tag{122}$$

where $\theta' = \dfrac{d\theta}{dD}$, a number of relations for $M_\nu^{[\lambda]}(x)$ may be obtained with great ease from those for $N_\nu^{[\lambda]}(x)$. We need only observe that, performing θ' on both sides of (120) and applying (122), we have

$$\theta' M_\nu^{[\lambda]}(x) = \left(\frac{\theta}{\theta_1}\right)^{\lambda} \theta' x^{\overline{\nu+1}|-1} = \left(\frac{\theta}{\theta_1}\right)^{\lambda} x^{\overline{\nu}|}$$

or

$$\theta' M_\nu^{[\lambda]}(x) = N_\nu^{[\lambda]}(x). \tag{123}$$

Since, now, θ' contains a constant term, the operation $\frac{1}{\theta'}$ is completely determined and may be performed on both sides of (123), the result being

$$M_\nu^{[\lambda]}(x) = \frac{1}{\theta'} N_\nu^{[\lambda]}(x). \tag{124}$$

This shows that relations implying $M_\nu^{[\lambda]}(x)$ may be obtained from those for $N_\nu^{[\lambda]}(x)$ simply by performing $\frac{1}{\theta'}$ on both sides. Thus, for instance, we obtain from (41) and (42)

$$\theta M_\nu^{[\lambda]}(x) = \nu M_{\nu-1}^{[\lambda]}(x), \tag{125}$$

$$\theta_1 M_\nu^{[\lambda]}(x) = \nu M_{\nu-1}^{[\lambda-1]}(x). \tag{126}$$

Further, since, by (122),

$$\frac{1}{\theta'} x^{\overline{\nu|}} = x^{\overline{\nu+1|}-1}, \tag{127}$$

we find, by (51) and (53),

$$M_\nu^{[\lambda]}(x) = \sum_{s=0}^{\nu} \binom{\nu}{s} x^{\overline{s+1|}-1} N_{\nu-s}^{[\lambda]}, \tag{128}$$

$$M_\nu^{[\lambda]}(x+y) = \sum_{s=0}^{\nu} \binom{\nu}{s} x^{\overline{s+1|}-1} N_{\nu-s}^{[\lambda]}(y). \tag{129}$$

If, in (53), we operate on y instead of on x, we find

$$M_\nu^{[\lambda]}(x+y) = \sum_{s=0}^{\nu} \binom{\nu}{s} x^{\overline{s|}} M_{\nu-s}^{[\lambda]}(y), \tag{130}$$

and if, in (54), we act on y, we have

$$M_\nu^{[\lambda]}(x+y) = \sum_{s=0}^{\nu} \binom{\nu}{s} x_1^{\overline{s|}} M_{\nu-s}^{[\lambda-s]}(y). \tag{131}$$

These are the expansions of $M_\nu^{[\lambda]}(x+y)$ in the poweroids $x^{\overline{s|}}$ and $x_1^{\overline{s|}}$.

For $y = 0$ we obtain the corresponding expansions of $M_\nu^{[\lambda]}(x)$, viz., writing $M_\nu^{[\lambda]} \equiv M_\nu^{[\lambda]}(0)$,

$$M_\nu^{[\lambda]}(x) = \sum_{s=0}^{\nu} \binom{\nu}{s} x^{\overline{s|}} M_{\nu-s}^{[\lambda]}, \tag{132}$$

$$M_\nu^{[\lambda]}(x) = \sum_{s=0}^{\nu} \binom{\nu}{s} x_1^{\overline{s|}} M_{\nu-s}^{[\lambda-s]}. \tag{133}$$

Since the definition (120) assumes that $x^{\overline{\nu+1}|-1}$ is known, the constants $0^{\overline{\nu+1}|-1}$ are also known, and we may express the M-polynomials by the N-polynomials if, in (129), we put $x = 0$, and, thereafter, replace y by x. The result is

$$M_\nu^{[\lambda]}(x) = \sum_{s=0}^{\nu} \binom{\nu}{s} 0^{\overline{s+1}|-1} N_{\nu-s}^{[\lambda]}(x). \tag{134}$$

Putting $x = 0$ in this, we have the constants $M_\nu^{[\lambda]}$ expressed by the constants $N_\nu^{[\lambda]}$.

From the binomial theorem for the N-polynomials, or (70), we find, performing $\frac{1}{\theta'}$ on both sides

$$M_\nu^{[\lambda+\mu]}(x+y) = \sum_{s=0}^{\nu} \binom{\nu}{s} M_s^{[\lambda]}(x) N_{\nu-s}^{[\mu]}(y). \tag{135}$$

This is, however, not strictly a binomial theorem, since both M- and N-functions enter on the right.

Since, by (120),

$$M_\nu^{[0]}(x) = x^{\overline{\nu+1}|-1}, \tag{136}$$

we obtain from (135), on putting $\mu = -\lambda$,

$$(x+y)^{\overline{\nu+1}|-1} = \sum_{s=0}^{\nu} \binom{\nu}{s} M_s^{[\lambda]}(x) N_{\nu-s}^{[-\lambda]}(y), \tag{137}$$

and hence, for $y = 0$,

$$x^{\overline{\nu+1}|-1} = \sum_{s=0}^{\nu} \binom{\nu}{s} M_s^{[\lambda]}(x) N_{\nu-s}^{[-\lambda]}. \tag{138}$$

If this is used as recurrence formula for $M_\nu^{[\lambda]}(x)$, we want $M_0^{[\lambda]}(x)$, which may be found by (124) and (74), thus:

$$M_0^{[\lambda]}(x) = \frac{1}{\theta'} N_0^{[\lambda]}(x) = \frac{1}{k_1 + 2k_2 D + \cdots} h_1^{-\lambda}$$

or

$$M_0^{[\lambda]}(x) = \frac{1}{k_1 h_1^\lambda}. \tag{139}$$

We may also note the formula obtained from (135) by putting $y = 0$, viz.

$$M_\nu^{[\lambda+\mu]}(x) = \sum_{s=0}^{\nu} \binom{\nu}{s} M_s^{[\lambda]}(x) N_{\nu-s}^{[\mu]}, \tag{140}$$

whence, for $x = 0$,

$$M_{\nu}^{[\lambda+\mu]} = \sum_{s=0}^{\nu} \binom{\nu}{s} M_{s}^{[\lambda]} N_{\nu-s}^{[\mu]}. \tag{141}$$

Recurrence formulas for $M_{\nu}^{[\lambda]}(x)$ are obtained from (130) and (131) by putting $x = -y$ and thereafter writing x for y. We need not write them down.

The question of the generating function of $M_{\nu}^{[\lambda]}(x)$ must be considered independently, because we may not apply $\frac{1}{\theta^{\nu}}$ to the two sides of (59), since they are not polynomials. We proceed as follows.

Differentiating the relation P. (34)

$$e^{xt} = \sum_{\nu=0}^{\infty} \frac{x^{\overline{\nu}|}}{\nu!} \zeta^{\nu} \tag{142}$$

with respect to ζ, the result may be written

$$e^{xt} \frac{dt}{d\zeta} = \sum_{\nu=0}^{\infty} \frac{x^{\overline{\nu+1}|-1}}{\nu!} \zeta^{\nu}. \tag{143}$$

We now find for the coefficient of ζ^{ν} in the expansion of $\Phi(\zeta) e^{xt} \frac{dt}{d\zeta}$, where $\Phi(\zeta) = \sum_{\nu=0}^{\infty} c_{\nu} \zeta^{\nu}$,

$$\sum_{s=0}^{\nu} c_{\nu-s} \frac{x^{\overline{s+1}|-1}}{s!} = \frac{1}{\nu!} \Phi(\theta)\, x^{\overline{\nu+1}|-1}.$$

Hence

$$\Phi(\zeta) e^{xt} \frac{dt}{d\zeta} = \sum_{\nu=0}^{\infty} \frac{\zeta^{\nu}}{\nu!} \Phi(\theta)\, x^{\overline{\nu+1}|-1}, \tag{144}$$

where t and $\frac{dt}{d\zeta}$ through the relation $\zeta = \varphi(t)$ are regarded as functions of ζ.

If, now, we choose

$$\Phi(\zeta) = \left(\frac{\zeta}{\varphi_1(\zeta)}\right)^{\lambda}, \qquad \Phi(\theta) = \left(\frac{\theta}{\theta_1}\right)^{\lambda},$$

we have the generating function of $M_{\nu}^{[\lambda]}(x)$

$$\left(\frac{\zeta}{\varphi_1(\zeta)}\right)^{\lambda} e^{xt} \frac{dt}{d\zeta} = \sum_{\nu=0}^{\infty} \frac{\zeta^{\nu}}{\nu!} M_{\nu}^{[\lambda]}(x). \tag{145}$$

Putting $x = 0$, we obtain the generating function of $M_{\nu}^{[\lambda]}$, or

$$\left(\frac{\zeta}{\varphi_1(\zeta)}\right)^{\lambda} \frac{dt}{d\zeta} = \sum_{\nu=0}^{\infty} \frac{\zeta^{\nu}}{\nu!} M_{\nu}^{[\lambda]}. \tag{146}$$

12. Examples of the polynomials $M_\nu^{[\lambda]}(x)$ may be obtained from (120) on inserting any poweroid, or from (124) when $N_\nu^{[\lambda]}(x)$ is given. In certain cases the M-polynomial is, however, only an N-polynomial in a different notation.

Thus, for instance, if we choose $\theta = \underset{\omega}{\triangle}$, $\theta_1 = \triangle$, we find, by (44) and (124), since $\theta' = e^{\omega D} = E^\omega$,

$$N_\nu^{[\lambda]}(x) = x_{\omega\lambda}^\nu, \qquad M_\nu^{[\lambda]}(x) = (x-\omega)_{\omega\lambda}^\nu,$$

so that $M_\nu^{[\lambda]}(x) = N_\nu^{[\lambda]}(x-\omega)$. These M-polynomials therefore only differ from the corresponding N-polynomials by a displacement of the variable, and several of the M-relations are, therefore, really N-relations. A noteworthy result is, however, obtained by (121) which shows that $x(x-\omega)_{\omega\nu}^{\nu-1}$ is the poweroid corresponding to the operator $\triangle$. We have, therefore

$$x^{(\nu)} = x(x-\omega)_{\omega\nu}^{\nu-1}, \tag{147}$$

which may also be written

$$x_{\omega,\nu+1}^\nu = (x+\omega-1)^{(\nu)}. \tag{148}$$

In the latter form the theorem was proved by a more elaborate method in G. N. P. (38).

Again, putting $\theta = \frac{D}{1-D}$, $\theta_1 = D$, we find, by (97), (95) and (124), since $\frac{d\theta}{dD} = (1-D)^{-2}$,

$$N_\nu^{[\lambda]}(x) = (-1)^\nu \nu!\, L_\nu^{[-\lambda-1]}(x), \qquad M_\nu^{[\lambda]}(x) = (-1)^\nu \nu!\, L_\nu^{[1-\lambda]}(x),$$

so that $M_\nu^{[\lambda]}(x) = N_\nu^{[\lambda-2]}(x)$. Here, too, there is therefore only a question of notation.

Since, then,

$$M_\nu^{[0]}(x) = (-1)^\nu \nu!\, L_\nu^{[1]}(x)$$

and, by (120),

$$M_\nu^{[0]}(x) = \frac{1}{x} q_{\nu+1}(x),$$

we have

$$q_\nu(x) = (-1)^{\nu-1}(\nu-1)!\, x L_{\nu-1}^{[1]}(x),$$

or P. (100).

But let us now consider the poweroid P. (44), putting

$$\theta = \frac{1}{\beta}(E^{\alpha+\beta} - E^\alpha), \qquad x^{\overline{\nu}|} = x(x-\nu\alpha-\beta)_\beta^{(\nu-1)}. \tag{149}$$

If

$$\theta_1 = \frac{1}{\beta}(E^\beta - 1), \quad x_1^{\overline{\nu}|} = x(x-\beta)_\beta^{(\nu-1)}, \tag{150}$$

we have $\dfrac{\theta}{\theta_1} = E^\alpha$, so that, by (40) and (149),

$$N_\nu^{[\lambda]}(x) = (x + \alpha\lambda)(x + \alpha\lambda - \nu\alpha - \beta)_\beta^{(\nu-1)}. \tag{151}$$

In this case

$$\theta' = \frac{1}{\beta}[(\alpha+\beta)E^{\alpha+\beta} - \alpha E^\alpha], \tag{152}$$

but the reciprocal of this operator is inconvenient, so that, instead of using (124), we apply (120), the result being

$$M_\nu^{[\lambda]}(x) = (x + \alpha\lambda - (\nu+1)\alpha - \beta)_\beta^{(\nu)}. \tag{153}$$

It is easy to ascertain that this polynomial together with (151) satisfies (123).

The *M*- and *N*-polynomials are here really distinct.