A SPECIAL CASE OF DIRICHLET'S PROBLEM FOR TWO DIMENSIONS

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In a posthumous paper [1] of Riemann some indications are given about the construction of a real harmonic function W in a plane with several circular holes, the function W taking assigned real values on each of the circular rims. Riemann's treatment of the problem is based on the theory of conformal representation. The given area is to be represented conformally on part of a Riemann surface, bounded by rectilinear rims, and then the desired function W can be readily found by means of an appropriate integral of the third kind. In 1877 the conformal representation of the plane with the holes was discussed anew by Schottky [2], who arrived at a solution, depending on certain transcendental functions, not altered by the linear substitutions of a special discontinuous group, that was afterwards called by Poincaré [3] the symmetrical Kleinian group of the third family. In a second memoir Schottky [4] returned to this class of Kleinian functions and gave a full and ample discussion of their properties. By their means it is possible to treat in a direct way, and without having recourse to a previous mapping, the original Dirichlet's problem for the perforated plane.

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[1] *Gleichgewicht der Electricität auf Cylindern mit kreisförmigem Querschnitt und parallelen Axen.* Ges. W., 2nd Ed., p. 440.

[2] *Über conforme Abbildungen mehrfach zusammenhangender ebener Flächen.* Journal f. r. u. a. Math., t. 83, p. 300.

[3] *Mémoire sur les groupes Kleinéens.* Acta Mathematica, t. 3, p. 49.

[4] *Über eine specielle Function, welche bei einer bestimmten linearen Transformation ihres Argumentes unverändert bleibt.* Journal f. r. u. a. Math., t. 101, p. 227.

In what follows I propose to show, that by the use of Schottky's functions we can obtain for the required potential function a determinate analytic expression, which even lends itself more or less to actual calculation. Moreover from the form the function W assumes, it will appear that, in order to solve the general problem, it suffices to consider two special cases only: 1° the case of a single hole, 2° the case, wherein on each rim the function W is equal to a determinate constant.

Before entering however into further developments, it will be necessary to make some statements about Schottky's results and to give a short description of some of the particular functions, he was the first to introduce into analysis [1].

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1. *Schottky's region T and the Kleinian group belonging to it.* In the plane of the complex variable x there are drawn p circumferences $K_1, K_2, \ldots, K_p$ with the radii $R_1, R_2, \ldots, R_p$ and the centres $a_1, a_2, \ldots, a_p$. No two of these circles must intersect and each of them must lie wholly above the axis XX of real quantities. Reflecting these p circles upon the axis XX, a further set of p circles $K_{1'}, K_{2'}, \ldots, K_{p'}$ is obtained, their centres $a_{1'}, a_{2'}, \ldots a_{p'}$ are conjugate to $a_1, a_2, \ldots a_p$. It is the part of the plane outside these $2p$ circles that formed the base of Schottky's investigations and which we designate henceforth as Schottky's region T. Occasionally we will have to regard as a circl the axis XX itself, and as such we shall call it the circle K_q, where q stands for $p+1$.

Associated with the region T, of connectivity $2p$, there is an infinite discontinous group Γ of linear substitutions, p of these being fundamental and each derived from one of the p pairs of conjugate circles K_k and $K_{k'}$. So, for instance, supposing x to be a point in T, the relation

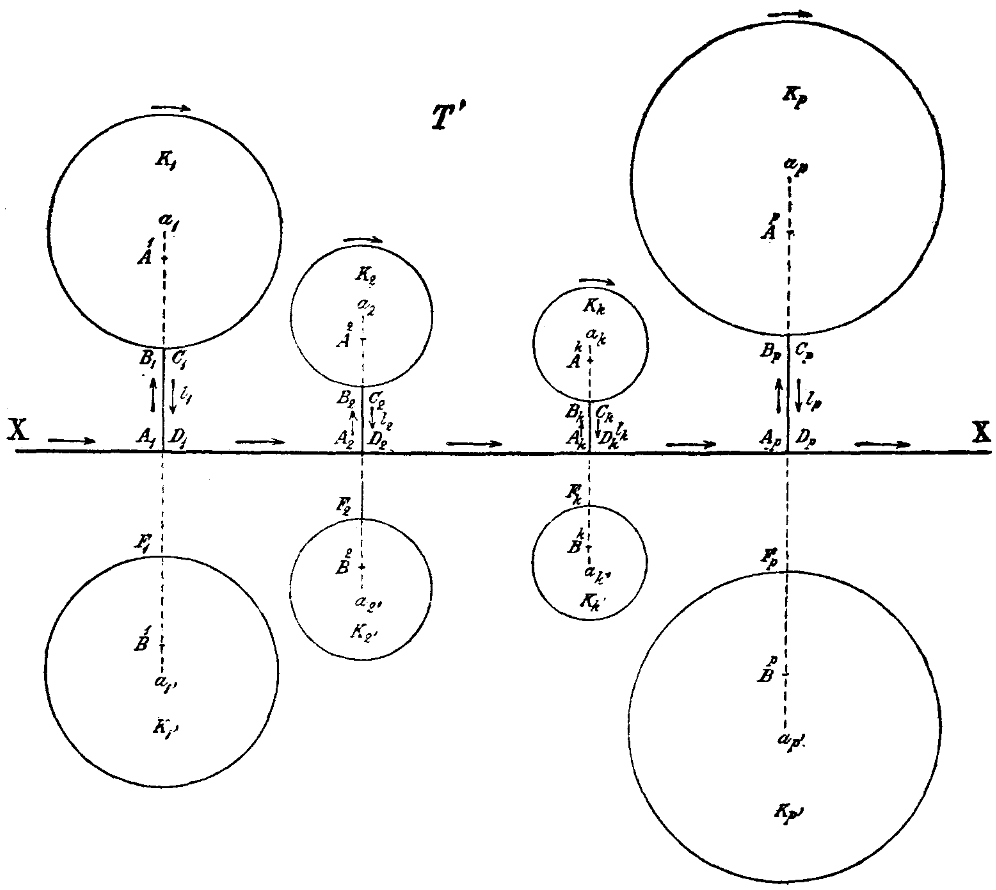
$$x_k = a_k + \frac{R_k^2}{x - a_{k'}} = f_k(x)$$

defines a point x_k, interior to the circle K_k, and by this substitution f_k

[1] Reference should be given here to the treatise of H. F. BAKER: *Abel's theorem and the allied theory, including the theory of the theta functions.* In Chapter XII of this volume the author gives an account of Schottky's investigations and explains the analogy between Schottky's theory and that of a Riemann surface.

the initial region T is transformed into another one T_k, of the same connectivity, and bounded by the same number of circular rims. One of these rims is the circumference K_k, along this boundary the regions T and T_k are contiguous.

It is evident that the thus defined hyperbolic substitution f_k is geometrically equivalent to a reflection upon XX or K_q, followed by a second reflection upon K_k, and it is also easily seen that this pair of reflections changes the circle $K_{k'}$ into the conjugate one K_k, so that corresponding points on these two circumferences have conjugate complex affixes. The inverse of the substitution f_k, which we shall denote by $f_{k'}$, has the effect of changing T_k again into T, thereby transforming the last named region into $T_{k'}$, a new region, wholly enclosed by $K_{k'}$ and contiguous to T along this circumference. By composition of a finite or infinite number of the foregoing fundamental substitutions $f_1, f_2, \ldots, f_p$, and of their inverses $f_{1'}, f_{2'}, \ldots, f_{p'}$, all the various substitutions f_α of the group Γ are obtained. We will call α the *mark* of the substitution f_α, employing always a greek letter when the substitution is not necessarily fundamental but perhaps composite. Thus then, α denotes an aggregate or symbolical product of the fundamental marks $1, 2, \ldots, p, 1', 2', \ldots, p'$, and if, for instance, we have $\alpha = 1'245'$, the loxodromic substitution f_α implies the successive application of the fundamental substitutions $f_{5'}, f_4, f_2$ and $f_{1'}$. In compliance with the order, in which these operations are to be performed, we will call $5'$ the first and $1'$ the last *factor* of the composite mark α. By inverting the order of the factors and by interchanging accented and non-accented marks, we obtain the mark $\alpha' = 54'2'1$ of the substitution $f_{\alpha'}$ that is the inverse of f_α. If we omitted however to invert the order of the factors, there would result the mark $\alpha_0 = 12'4'5$, which shall be called the *conjugate* of α, and we may obviously infer that conjugate substitutions, applied to a pair of conjugate points in T, change them again into a pair of conjugate points. All substitutions of the group Γ can be arranged by attending to the number of fundamental marks or factors, that enter into the composite mark α. First of all we have the identical substitution followed by the $2p$ substitutions $f_1, f_2, \ldots, f_p, f_{1'}, f_{2'}, \ldots, f_{p'}$ each with a single mark, then come the $2p(2p-1)$ substitutions of the second order, each compounded from two fundamental substitutions, and so on. Although it is scarcely possible to form a mental image of the geometrical configuration, generated by the group, it is analytically evident that all the regions T_a, derived from the fundamental region T, are bounded by $2p$ circumferences, and that no two of them will overlap. Together they cover the complete plane, we started with, with exception of certain limiting points, that are not reached as transformations of points in T, whatever finite series of substitutions we apply,



and which remain therefore always excluded from all the regions, whereinto the plane is divided. Every substitution f_a gives rise to a pair of such points, for if we agree to call $\overset{a}{A}$ and $\overset{a}{B}$ its double points, that is, if we define $\overset{a}{A}$ and $\overset{a}{B}$ by the equations

$$\operatorname*{Lim}_{n=\infty} f_{a^n}(x) = \overset{a}{A}, \qquad \operatorname*{Lim}_{n=\infty} f_{a'^n}(x) = \operatorname*{Lim}_{n=\infty} f_{a^{-n}}(x) = \overset{a}{B},$$

it is at once apparent, that no point whatever in T is changed into one of them, by subjecting it to a finite number of substitutions.

Of particular importance are the double points $\overset{k}{A}, \overset{k}{B}$ of the fundamental substitutions f_k. The three circles $K_k, K_{k'}$ and K_q belong to a system of circles having a common radical axis, and the limiting points or foci of this system are precisely the points A and B. Hence, the latter are each other's inverses with regard to K_q and their affixes are conjugate complexes.

We have already remarked that every fundamental substitution f_k is equivalent to a pair of inversions, the first with respect to K_q, the second with respect to K_k, and from this remark it is at once apparent, that any composite substitution f_a can always be replaced by an *even* number of inversions with regard to the $p+1$ circles $K_1, K_2, \ldots, K_p, K_q$. For our purpose it will be convenient occasionally to resolve the substitution f_a into its component inversions, therefore we will represent such an inversion by a distinct symbol. As such we choose doubly-accented marks, to prevent confusion with the substitutions of Γ. So, for instance, we will denote by $x_{1''3''4''6''}$ the point derived from x by four successive inversions with regard to the circles K_6, K_4, K_3, K_1. On the other hand, if we madé use of the hitherto employed notation, the same point $x_{1''3''4''6''}$ would be designated by $x_{13'46'}$, for the four pairs of inversions $q''6''$, $4''q''$, $q''3''$, $1''q''$ give rise successively tho the four substitutions $6', 4, 3', 1$.

1. *Functions existing in Schottky's region.* We proceed to give a short description of some of Schottky's functions, existing in the region T. In the first place we mention the expression

$$(xy\,;\xi\eta) = \prod_a \frac{x-\xi_a}{x-\eta_a} : \frac{y-\xi_a}{y-\eta_a},$$

the multiplication extending over all the substitutions of Γ, fundamental and composite. It was proved by Schottky, though his proof, as he himself points out, is still liable to some limitations, that the above infinite product is really convergent and that in T it can be considered as an analytic function $\varphi(x)$ of the variable x. From the form of the primary factor we conclude, that $(xy\,;\xi\eta)$ obeys the equations

$$(xy\,;\xi\eta) = (yx\,;\eta\xi) = (\xi\eta\,;xy),$$

moreover it is not difficult to see, that $\log(xy;\xi\eta)$ or, as we shall write it, $\log\varphi(x)$ possesses in T only the two logarithmic infinities ξ and η, the function $\log\,\varphi(x)$ increasing with $\pm\,2\pi i$, each time the variable x describes a circuit enclosing either ξ or η.

Intimately connected with $\varphi(x)$ is the function

$$E_\mu(x) = \prod_a \frac{x - \overset{\mu}{A}_a}{x - \overset{\mu}{B}_a},$$

not depending upon some parameter. It must be noted, that in this expression the variable mark α does not refer to all the substitutions of Γ without exception, excluded are all marks α, that are of the form $\beta\mu$ or $\beta\mu'$. The function $E_\mu(x)$ has neither zero's nor infinities in T, its essential property consists in the multiplication-theorem:

$$E_{a,\beta}(x) = E_{\beta a}(x) = E_a(x)\,.\,E_\beta(x),$$

from which it is immediately inferred that only the p fundamental functions $E_1(x), E_2(x), \ldots, E_p(x)$ need be considered, since, by arranging these in products, all similar functions with composite marks can be constructed.

Reverting again to logarithms, it can be shewn, that $\log E_k(x)$, everywhere finite in T, has its value increased by $\pm\,2\pi i$, whenever the variable x describes a closed path round one of the circles K_k and $K_{k'}$.

If we subject the argument of the foregoing functions $\varphi(x)$ and $E_k(x)$ to any substitution of Γ the result is very remarkable. So it is found that after the substitution the function $E_k(x)$ is reproduced, save as to a determinate constant factor. Otherwise expressed we have

$$E_k(x_a) = E_k(x)\,.\,E_{k,a}.$$

As for the constant $E_{k,a}$, introduced here, supposing $\alpha = \beta\gamma$, it satifies the relation

$$E_{k,a} = E_{a,k} = E_{k,\beta}\,.\,E_{k,\gamma}\,.$$

Again it becomes apparent that we can disregard the composite marks and that all constants $E_{a,\beta}$ are simply products of similar quantities $E_{h,k}$, each of the latter corresponding to a pair of fundamental substitutions. By differentiating the relation between $E_h(x_k)$ and $E_h(x)$ we get

$$\mathrm{dlog}\, E_h(x_k) = \mathrm{dlog}\, E_h(x),$$

hence, with respect to Γ, the differential $\mathrm{dlog}\, E_h(x)$ is automorphic. A similar result holds for the function $\varphi(x)$. Application of the substitution f_a gives

$$\varphi(x_a) = (x_a y; \xi\eta) = \varphi(x) \cdot \frac{E_a(\xi)}{E_a(\eta)},$$

and therefore again

$$\mathrm{dlog}\, \varphi(x_a) = \mathrm{dlog}\, \varphi(x).$$

3. *Rim values of $E_k(x)$ and $\varphi(x)$.* It is necessary, at the present stage, to make some statements about the nature of the values, the functions $E_k(x)$ and $\varphi(x)$ acquire on the rims of the region T. Commencing with $E_k(x)$, we observe that in the infinite product

$$E_k(x) = \prod_a \frac{x - \overset{k}{A}_a}{x - \overset{k}{B}_a}$$

we can combine the primary factors, due to every pair of conjugate substitutions f_a and f_{a_0}, so that we have, writing down separately the leading factor corresponding to the identical substitution,

$$E_k(x) = \frac{x - \overset{k}{A}}{x - \overset{k}{B}} \prod_a \left[\frac{x - \overset{k}{A}_a}{x - \overset{k}{B}_a} \cdot \frac{x - \overset{k}{A}_{a_0}}{x - \overset{k}{B}_{a_0}} \right].$$

Now remembering that the conjugate points $\overset{k}{A}$ and $\overset{k}{B}$, subjected to conjugate substitutions, transform again into conjugate points, it plainly appears that, for real values of the variable x, the function $E_k(x)$ is of modulus unity. Hence on the axis XX, otherwise said on the circle K_q, the function $\log E_k(x)$, and also its differential $\mathrm{dlog}\, E_k(x)$, is purely imaginary. As for the rims of the region T, we may draw a similar conclusion in the following manner. Supposing x and x_0 to be conjugate complexes, what we shall indicate by writing $x \mathrel{+\!\!\!\!=} x_0$, we have in general

$$i \log E_k(x) \mathrel{+\!\!\!\!=} i \log E_k(x_0), \ldots \text{(mod. } 2\pi)$$

since $i \log E_k(x)$ is real for all real values of x. Now, if we make x describe the rim K_h, x_0 moves on $K_{h'}$, and both variables are connected by the relation $x = f_h(x_0)$, hence we have simultaneously

$$i \operatorname{dlog} E_k(x) \doteq i \operatorname{dlog} E_k(x_0),$$

$$\operatorname{dlog} E_k(x) = \operatorname{dlog} E_k(x_0),$$

and these equations can not be satisfied, unless $\operatorname{dlog} E_k(x)$ is purely imaginary on the rim K_h.

Another fact of equal importance should be noticed here. Taking again x on K_h, and therefore x_0 on $K_{h'}$, it follows from the simultaneous relations

$$i \log E_k(x) \doteq i \log E_k(x_0) \; ; \ldots \text{ (mod. } 2\pi)$$

$$\log \frac{E_k(x)}{E_k(x_0)} = \log E_{h,k},$$

that the p^2 constants $E_{h,k}$ have real and positive values. Hence one of the values of $\log E_{h,k}$ is purely real, we shall denote it henceforth by $2\tau_{h,k} = 2\tau_{k,h}$; and it is not difficult to prove, that the complete set of the p^2 constants $\tau_{h,k}$ may serve as a system of moduli for a p-tuple theta-function.

Quite the same reasoning does apply to the function $\varphi(x) = (xy; \xi\eta)$, if only the parameters y, ξ, η are fixed in a particular manner. Supposing y to be real, ξ and η to be conjugate complexes, we can easily see that, for real values of x, we have always

$$\text{mod. } \varphi(x) = 1.$$

For in writing down the infinite product represented by $\varphi(x)$, we may again combine the factors corresponding to a pair of conjugate substitutions, and having

$$\varphi(x) = \left[\frac{x-\xi}{x-\eta} : \frac{y-\xi}{y-\eta}\right] \prod_a \left[\frac{x-\xi_a}{x-\eta_{a_0}} \cdot \frac{x-\xi_{a_0}}{x-\eta_a}\right] : \left[\frac{y-\xi_a}{y-\eta_{a_0}} \cdot \frac{y-\xi_{a_0}}{y-\eta_a}\right],$$

the validity of the above assertion is obvious. Accordingly the differential $\operatorname{dlog} \varphi(x)$ takes only purely imaginary values as x moves on the axis XX, and the same conclusion holds, when x describes one of the rims. For, in the latter case, we have at the same time

$$i \operatorname{dlog} \varphi(x) = i \operatorname{dlog} \varphi(x_0),$$

and

$$\operatorname{dlog} \varphi(x) = \operatorname{dlog} \varphi(x_0),$$

and these equations necessarily involve a purely imaginary value of $\operatorname{dlog} \varphi(x)$.

4. *Integration along the rims.* The solution of the proposed Dirichlet's problem will be found to depend mainly on the value of certain curvilinear integrals, taken along the different circumferences K_k, therefore it will be useful to deduce some inferences concerning these integrals.

We assume that with every point x on the rim K_k there is associated a determinate real and finite quantity, and though this quantity is in the ordinary sense not a function of the variable x, it will lead to no misconception, if we denote the succession of these real values on the rim K_k by the symbol $\psi_k(x)$.

We now consider the integral

$$J_k = \frac{1}{2\pi i} \int\limits_{\widehat{K}_k} \psi_k(x) \operatorname{dlog} \varphi(x) = \frac{1}{2\pi i} \int\limits_{\widehat{K}_k} \psi_k(x) dx \sum_a \left[\frac{1}{x - \xi_a} - \frac{1}{x - \eta_a}\right],$$

taken along K_k in that direction, that leaves the region T to the left. If the parameters ξ and η of the function $\varphi(x)$ are chosen quite arbitrarily, the integral J_k is a complex quantity, its real part however is in all cases capable of an easy interpretation.

In order to obtain this real part we substitute into the integral

$$x = a_k + R_k e^{i\theta}, \quad \xi_a = a_k + r_a e^{iu_a}, \quad \eta_a = a_k + s_a e^{iv_a},$$

and so we get without difficulty

$$\Re J_k =$$

$$\sum_a \left[\frac{1}{2\pi} \int_0^{2\pi} \frac{(r_a^2 - R_k^2)\psi_k(x) d\theta}{r_a^2 + R_k^2 - 2r_a R_k \cos(\theta - u_a)} - \frac{1}{2\pi} \int_0^{2\pi} \frac{(s_a^2 - R_k^2)\psi_k(x) d\theta}{s_a^2 + R_k^2 - 2s_a R_k \cos(\theta - v_a)}\right].$$

Now to the integrals occurring here we can attach a definite meaning. In fact, supposing the circle K_k to be the only hole in the plane of the variable x, there exists in that plane a real uniform and finite potential function U_k with the boundary values $\psi_k(x)$ on the rim K_k. In case ξ_a lies outside K_k, it follows from the ordinary theory that the value $U_k(\xi_a)$ of U_k at the point ξ_a is equal to

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{(r_a^2 - R_k^2)\,\psi_k(x)\,d\theta}{r_a^2 + R_k^2 - 2r_a R_k \cos(\theta - u_a)},$$

whereas the same integral indicates in case of an interior point ξ_a the value

$$-\,U_k(\xi'_a),$$

ξ'_a being the inverse of ξ_a with regard to K_k. Hence we may write

$$\Re J_k = \frac{1}{2}\sum_a \left[U_k(\xi_a) - U_k(\eta_a)\right],$$

if we only agree to replace in the above series every interior point by its reflection upon K_k, changing thereby at the same time the sign of the corresponding value of U_k.

The same reasoning can be applied to the integral

$$J_q = \frac{1}{2\pi i}\int_{K_q} \psi_q(x)\,\mathrm{dlog}\,\varphi(x),$$

taken along the axis XX, the positive halfplane lying to the left. In the positive halfplane without any holes we can imagine the real potential function U_q, taking on the axis XX the assigned rim values $\psi_q(x)$, and by introducing this function U_q, we shall find as before

$$\Re J_q = \frac{1}{2}\sum_a \left[U_q(\xi_a) - U_q(\eta_a)\right].$$

The function U_q being however only defined in the upper halfplane, every point in the lower halfplane must be replaced by the conjugate one, and the corresponding value of U_q must have its sign changed.

5. *Dirichlet's problem for the upper half T' of Schottky's region.* By the preceding deductions we are now enabled to treat Dirichlet's problem for the upper half T' of Schottky's region with its q circular boundaries $K_1, K_2, \ldots, K_p, K_q$, that is, we can construct in this area a real, uniform and finite potential function W, satisfying given boundary conditions. This function W we assume to be the real part of an unknown function $V(x)$ of the complex variable x, everywhere finite in T'. Now as W must be uniform in T', the moduli of periodicity of $V(x)$ must be either zero or purely imaginary quantities, otherwise stated, if the variable x describes a circuit enclosing one of the holes, say K_k, the initial and the final value of $V(x)$ can only differ by an imaginary constant S_k.

Starting with the thus characterised function $V(x)$, the potential function W can be obtained, as in the case of a single hole, in the form of a definite integral. In fact, it will be found that the construction of the required potential can be based upon the consideration of the integral

$$J = \frac{1}{2\pi i}\int V(x)[d\log\varphi(x) - h_1\, d\log E_1(x) - h_2\, d\log E_2(x) - \ldots - h_p\, d\log E_p(x)]$$
$$= \frac{1}{2\pi i}\int V(x)\, dF(x),$$

wherein $h_1, h_2, \ldots, h_p$ denote certain real coefficients, depending upon the parameters ξ and η of the function $\varphi(x)$, it being moreover understood that ξ and η are to be conjugate points, the former belonging to T'. In order to fix a suitable path of integration, we draw from each of the p rims $K_1, K_2, \ldots, K_p$ in T' a rectilinear cross-cut l_k (se the figure) to the axis XX. So the resolved region T' becomes simply connected and throughout this region the one-valuedness of the subject of integration is secured. Hence integrating along the complete rim: $XA_1B_1C_1D_1A_2B_2C_2D_2, \ldots, D_pX$, we get, since ξ is the only pole of the integrand in T',

$$\begin{aligned} J = V(\xi) = \quad & J_{XA_1} + J_{D_1A_2} + J_{D_2A_3} + \ldots + J_{D_pX} \\ & + J_{B_1C_1} + J_{B_2C_2} + J_{B_3C_3} + \ldots + J_{B_pC_p} \\ & + (J_{A_1B_1} - J_{D_1C_1}) + (J_{A_2B_2} - J_{D_2C_2}) + \ldots + (J_{A_pB_p} - J_{D_pC_p}). \end{aligned}$$

Now, for our purpose, the real parts at both sides of this equation need only be considered, and as such we find at the left hand side the value $W(\xi)$, the function W assumes at the point ξ. At the right-hand side we must consider separately the parts contributed by the rims of the unresolved area T', and those relative to the cross-cuts l_k. Commencing with the axis XX, we remark that along that rim the differential $dF(x)$ is imaginary, hence only the real part of $V(x)$ must be retained, that is, integrating along XX we must replace the function $V(x)$ by the assigned rim values $\phi_q(x)$ of the potential W. Thus then, contracting the sum

$$J_{XA_1} + J_{D_1A_2} + J_{D_2A_3} + \ldots + J_{D_pX}$$

into a unique integral, we may write

$$\Re[J_{XA_1} + J_{D_1A_2} + J_{D_2A_3} + \ldots + J_{D_pX}] = \frac{1}{2\pi i}\int\limits_{\widehat{K}_q} \phi_q(x)\,dF(x).$$

The same argument holds for the integrals $J_{B_kC_k}$ contributed by the circumferences K_k. Again we shall have

$$\Re[J_{B_kC_k}] = \frac{1}{2\pi i}\int\limits_{\widehat{K}_k} \phi_k(x)\,dF(x)$$

and so there remain only the integrals along the cross-cuts. Now along the cross-cut l_k the values of the integrand at opposite places have a difference equal to

$$S_k \frac{dF(x)}{dx}$$

and hence we have

$$J_{A_kB_k} - J_{D_kC_k} = \frac{S_k}{2\pi i}\left[F(x)\right]_{x=A_k}^{x=B_k}.$$

At the lower limit A_k the function $F(x)$ has been shewn to have an imaginary value, therefore we may put

$$\Re[J_{A_kB_k} - J_{D_kC_k}] = 0,$$

on condition that we subject the as yet undetermined coefficients λ to the relation

$$0 = \Re[\log(xy;\xi\eta) - \lambda_1 \log E_1(x) - \lambda_2 \log E_2(x) - \ldots - \lambda_p \log E_p(x)]_{x=B_k}.$$

In all we get p of such relations; supposing them satisfied, the value $W(\xi)$ of the potential function takes the form

$$W(\xi) =$$
$$\sum_{k=1}^{k=q} \frac{1}{2\pi i}\int_{\widehat{K_k}} \psi_k(x)[\mathrm{dlog}(xy;\xi\eta) - \lambda_1\,\mathrm{dlog}\,E_1(x) - \lambda_2\,\mathrm{dlog}\,E_2(x) - \ldots - \lambda_p\,\mathrm{dlog}\,E_p(x)].$$

This expression can be transformed in the following manner. From the equation

$$(xy;\xi\eta_k) = (xy;\xi\eta)\cdot\frac{E_k(y)}{E_k(x)},$$

or as we write it

$$\varphi_k(x) = \varphi(x)\cdot\frac{E_k(y)}{E_k(x)},$$

we deduce

$$\mathrm{dlog}\,E_k(x) = \mathrm{dlog}\,\varphi(x) - \mathrm{dlog}\,\varphi_k(x).$$

This relation enables us to eliminate from the foregoing expression of $W(\xi)$ the functions $E_k(x)$, and in this way we get

$$W(\xi) =$$
$$\sum_{k=1}^{k=q} \frac{1}{2\pi i}\int_{\widehat{K_k}} \psi_k(x)[(1-\lambda_1-\lambda_2-\ldots-\lambda_p)\,\mathrm{dlog}\,\varphi(x) + \lambda_1\,\mathrm{dlog}\,\varphi_1(x) + \lambda_2\,\mathrm{dlog}\,\varphi_2(x) +$$
$$\ldots + \lambda_p\,\mathrm{dlog}\,\varphi_p(x)].$$

Here, making use of the results established in art. 4, we can introduce the auxiliary potentials U_k considered there. U_k is a real uniform and finite potential function existing in the simply connected area outside the circular hole K_k, and fully determined in this region by the rim values $\psi_k(x)$, it takes on the only rim. For arbitrary values of the parameter ξ and η of $\varphi(x)$ we have established the relation

$$\Re\frac{1}{2\pi i}\int_{\widehat{K_k}} \psi_k(x)\,\mathrm{dlog}\,\varphi(x) = \frac{1}{2}\sum_a\{U_k(\xi_a) - U_k(\eta_a)\},$$

hence we may now affirm that

$$W(\xi) =$$

$$\frac{1}{2}(1-\lambda_1-\lambda_2-\ldots-\lambda_p)\sum_{k=1}^{k=q}\Big[\sum_a\{U_k(\xi_a)-U_k(\eta_a)\}\Big]+\frac{1}{2}\lambda_1\sum_{k=1}^{k=q}\Big[\sum_a\{U_k(\xi_a)-U_k(\eta_{a1})\}\Big]$$

$$+\frac{1}{2}\lambda_2\sum_{k=1}^{k=q}\Big[\sum_a\{U_k(\xi_a)-U_k(\eta_{a2})\}\Big]+\ldots+\frac{1}{2}\lambda_p\sum_{k=1}^{k=q}\Big[\sum_a\{U_k(\xi_a)-U_k(\eta_{ap})\}\Big].$$

The above expression acquires a perfect symmetry, if we resolve every substitution f_a of the group Γ into its component inversions with regard to the rims of T', remembering at the same time that, ξ and η being conjugate points, we may write $\xi_{q'}$ for η, $\xi_{k'}$ for η_k.

Let the mark α'' denote a succession of an *even* number (zero included) of inversions with respect to $K_1, K_2, \ldots, K_p, K_q$, then using λ_q to denote

$$1-\lambda_1-\lambda_2-\ldots-\lambda_p,$$

we shall have finally

$$W(\xi)=\sum_{h=1}^{h=q}\Big[\frac{1}{2}\lambda_h\sum_{k=1}^{k=q}\Big[\sum_{\alpha''}\{U_k(\xi_{\alpha''})-U_k(\xi_{\alpha''h''})\}\Big]\Big].$$

Meanwhile it is to be distinctly understood that when a point $\xi_{\beta''}$, occurring in the above serie, is interior to the hole K_k, the symbol $U_k(\xi_{\beta''})$ denote the value of

$$-U_k(\xi_{k''\beta''}),$$

where $\xi_{k''\beta''}$ is the reflection of $\xi_{\beta''}$ upon K.

6. *The coefficients* λ. Before we proceed to examine in what manner the values of the coefficients λ may be obtained, we wish to shew that they are in a simple and characteristic way related to the region T'.

To this end we will consider the case that the given rim values of W were zero on all rims but one, say K_s, and that the rim value on K_s was throughout equal to unity.

First, we have now

$$\sum_{\alpha''}\{U_s(\xi_{\alpha''})-U_s(\xi_{\alpha''h''})\}=0,$$

when h is distinct from s. In fact, whatever α'' may be, the points $\xi_{\alpha''}$ and $\xi_{\alpha''h''}$ are always simultaneously within or without the circle K_s, and $U_s(\xi_{\alpha''})$ and $U_s(\xi_{\alpha''h''})$ are therefore at the same time equal to -1 or to $+1$.

Next, we have

$$\sum_{\alpha''}\{U_s(\xi_{\alpha''}) - U_s(\xi_{\alpha''s''})\} = 2.$$

For, as before, each term of the series, save the first, vanishes, whereas we obtain for the first term, corresponding to $\alpha'' = 0$,

$$U_s(\xi) - U_s(\xi_{s''}) = 2\,U_s(\xi) = 2.$$

In the remaining series

$$\sum_{\alpha''}\{U_k(\xi_{\alpha''}) - U_k(\xi_{\alpha''h''})\},$$

where k is distinct from s, all the potentials U_k are separately zero, hence for the very special case under consideration we find

$$W(\xi) = \lambda_s.$$

Thus then, we may enunciate that the coefficient λ_s indicates the value of the potential function W, whenever W is zero on all the rims, except on K_s, whereon it is equal to unity.

Moreover this interpretation of the λ's implies that the system of linear equations

$$\Re[\log(xy;\xi\eta) - \lambda_1 \log E_1(x) - \lambda_2 \log E_2(x) - \ldots - \lambda_p \log E_p(x)]_{x=B_k} = 0,$$

$$(k = 1, 2, \ldots, p)$$

$$1 = \lambda_1 + \lambda_2 + \ldots + \lambda_p + \lambda_q,$$

which served originally to define them, is always capable of a definite solution. To bring these equations in a form somewhat better fitted for actual calculation, we proceed as follows.

Let, in the diagram of art. 5, F_k be the reflection of B_k upon XX, then we have simultaneously

$$i\log(xy;\xi\eta)_{x=B_k} \mp i\log(xy;\xi\eta)_{x=F_k}, \ldots \text{(mod. } 2\pi)$$

and

$$\log(xy\,;\,\xi\eta)_{x=B_k} = \log(xy\,;\,\xi\eta)_{x=F_k} + \log\frac{E_k(\xi)}{E_k(\eta)}$$

whence it is inferred that

$$\Re\log(xy\,;\,\xi\eta)_{x=B_k} = \frac{1}{2}\log\frac{E_k(\xi)}{E_k(\eta)}.$$

Since the points ξ and η are conjugate, the value of the right hand side is depending upon ξ alone, accordingly we will henceforth represent it by $L_k(\xi)$.

Reverting to the points B_k and F_k and the corresponding values of $\log E_k(x)$, we have in the same way

$$i\log E_h(x)_{x=B_k} \mp i\log E_h(x)_{x=F_k}\,.\;.\;.\;.\;(\text{mod. } 2\pi)$$

and

$$\log E_h(x)_{x=B_k} = \log E_h(x)_{x=F_k} + \log E_{h,k}\,,$$

so that

$$\Re\log E_h(x)_{x=B_k} = \frac{1}{2}\log E_{h,k} = \tau_{h,k}.$$

Consequently the equations, from which the λ's are to be solved, may be written in the form

$$L_k(x) = \tau_{1,k}\,\lambda_1 + \tau_{2,k}\,\lambda_2 + \tau_{p,k}\,\lambda_p\,,$$

$$(k = 1\,,\,2\,,\,.\,.\,.\,,\,p)$$

$$1 = \lambda_1 + \lambda_2 + .\,.\,. + \lambda_p + \lambda_q.$$

The solution is possible as soon as the values of the L's and of the τ's are known, and we will now indicate how these values can be found by means of convergent infinite products.

Owing to the definition of $L_k(\xi)$, it follows that

$$e^{2L_k(\xi)} = \frac{E_k(\xi)}{E_k(\eta)} = \left[\frac{\xi-\overset{k}{A}}{\xi-\overset{k}{B}} : \frac{\eta-\overset{k}{A}}{\eta-\overset{k}{B}}\right]\prod_\alpha\left[\frac{\xi-\overset{k}{A}_\alpha}{\xi-\overset{k}{B}_\alpha} : \frac{\eta-\overset{k}{A}_\alpha}{\eta-\overset{k}{B}_\alpha}\right],$$

the primary factor of the infinite product taking the form of an anharmonic ratio $\left[\xi\eta\,;\,\overset{k}{A}_\alpha\,\overset{k}{B}_\alpha\right]$, and the variable mark α indicating all possible substistutions of Γ, save those that have a mark of the form βk or $\beta k'$. Again, since ξ and η are conjugate complexes, the two factors $\left[\xi\eta\,;\overset{k}{A}_\alpha\overset{k}{B}_\alpha\right]$ and $\left[\xi\eta\,;\overset{k}{A}_{\alpha_0}\overset{k}{B}_{\alpha_0}\right]$, corresponding to a pair of conjugate marks α and α_0, are also conjugate complex quantities, hence, if we agree to denote henceforth by $\left[\xi\eta\,;\overset{k}{A}_\alpha\overset{k}{B}_\alpha\right]$ the absolute value of the anharmonic ratio, we arrive, by extracting the square root, to a result of the form

$$e^{L_k(\xi)} = \left[\xi\eta\,;\overset{k}{A}\overset{k}{B}\right]^{\frac{1}{2}}\prod_\beta\left[\xi\eta\,;\overset{k}{A}_\beta\overset{k}{B}_\beta\right],$$

the product extending over all marks β, the first factor of which is either $1, 2, 3, \ldots, (k-1), (k+1), \ldots, (p-1)$ or p.

The last step is to introduce the inversions instead of the substitutions, and so we find finally

$$e^{L_k(\xi)} = \left[\xi\xi_{q''}\,;\overset{k}{A}\overset{k}{B}\right]^{\frac{1}{2}}\prod_{\gamma''}\left[\xi\xi_{q''}\,;\overset{k}{B}_{\gamma''}\overset{k}{A}_{\gamma''}\right],$$

where γ'' designates a product of an *odd* number of the marks $1'', 2'', \ldots, p'', q''$, the first factor being neither k'' nor q''.

The constants $\tau_{h,k}$ are expressible by a similar expansion. In fact, whenever the point ξ approaches indefinitely the rim K_h, we have

$$L_k(\xi) = \frac{1}{2}\log\frac{E_k(\xi)}{E_k(\eta)} = \frac{1}{2}\log E_{h,k} = \tau_{h,k},$$

and so perhaps the easiest way to evaluate $\tau_{h,k}$ is to evaluate $L_k(\xi)$ for some point ξ arbitrarily chosen on the rim K_h.

7. *Summary of results obtained for the region T'.* The following is a summary of the results that have been obtained in the preceding articles, relative to Dirichlet's problem for the upper half T' of Schottky's region:

I. The required potential function W is given by the general formula

$$W(\xi) = \sum_{h=1}^{h=q}\left[\frac{1}{2}\lambda_h\sum_{k=1}^{k=q}\left[\sum_{a''}\left\{U_k(\xi_{a''}) - U_k(\xi_{a''h''})\right\}\right]\right],$$

the mark α'' designating an *even* number of reflections, the potential function U_k being defined as in art. 4.

II. The q coefficients λ, entering in the above formula, are determined by the p linear equations

$$L_k(\xi) = \tau_{1,k}\lambda_1 + \tau_{2,k}\lambda_2 + \ldots + \tau_{p,k}\lambda_p,$$
$$(k = 1, 2, \ldots, p)$$

and by the supplementary condition

$$1 = \lambda_1 + \lambda_2 + \ldots + \lambda_p + \lambda_q.$$

III. The value of $L_k(\xi)$ is given by the equation

$$e^{L_k(\xi)} = \left[\xi\xi_{q''}; \overset{k}{A}\overset{k}{B}\right]^{\frac{1}{2}} \prod_{\gamma''} \left[\xi\xi_{q''}; \overset{k}{B}_{\gamma''}\overset{k}{A}_{\gamma''}\right],$$

where γ'' is compounded from an *odd* number of inversions, its first factor being neither k'' nor q''. In order to find $\tau_{h,k}$, we take ξ to be a point on K_h and have then $\tau_{h,k} = L_k(\xi)$.

An additional remark suggests itself. The value of $W(\xi)$ has been found to involve solely the λ's and the auxiliary potentials U_k. Hence, remembering the definition of λ_k and of U_k, we have made good, as far as concerns the region T', the assertion, made in the beginning, that Dirichlet's problem can be completely solved, when a solution is found: 1° in case the rim values for each rim reduce themselves to a constant, 2° in case there is but one hole.

8. *Verification of the preceding solution.* In establishing a definite expression for $W(\xi)$, we took it for granted that there really existed in the region T' a potential function, obeying given boundary conditions. Therefore a verification of our result is necessary, in other words, we have to shew that, as soon as the point ξ approaches indefinitely a point x on one of the rims, say K_m, the value of $W(\xi)$ tends to the corresponding rim value $\phi_m(x)$.

Now, considering the quantities $L_k(\xi)$, we have immediately

$$\lim_{\xi=x} L_k(\xi) = \tau_{m,k}$$
$$(k = 1, 2, \ldots p)$$

and for this special value of $L_k(x)$ it is inferred from the equations II, art. 7, that $\lambda_1, \lambda_2, \ldots, \lambda_{m-1}, \lambda_{m+1}, \ldots, \lambda_p, \lambda_q$ become vanishing quantities and that λ_m tends to unity. Hence the general formula I, art. 7, is somewhat simplified, we may conclude

$$\operatorname*{Lim}_{\xi=x} W(\xi) = \operatorname*{Lim}_{\xi=x} \frac{1}{2} \sum_{k=1}^{k=q} \left[\sum_{\alpha''} \{ U_k(\xi_{\alpha''}) - U_k(\xi_{\alpha'' m''}) \} \right].$$

It is easily proved that in this aggregate of infinite series every series

$$\sum_{\alpha''} \{ U_k(\xi_{\alpha''}) - U_k(\xi_{\alpha'' m''}) \},$$

where k is distinct from m, will ultimately vanish. For as ξ approaches x from the outside of K_m, the point $\xi_{m''}$ will tend to the same point x from the inside of K_m. Hence the points $\xi_{\alpha''}$ and $\xi_{\alpha'' m''}$ ultimately will unite, so that each term of the above series vanishes separately.

We may deal in the same way with the remaining series

$$\sum_{\alpha''} \{ U_m(\xi_{\alpha''}) - U_m(\xi_{\alpha'' m''}) \}.$$

Again the values of $U_m(\xi_{\alpha''})$ and $U_m(\xi_{\alpha'' m''})$ will tend to the same limit. An exception occurs however. According to the definition of the potential U_k, we find for the leading term, corresponding to the identical substitution,

$$U_m(\xi) - U_m(\xi_{m''}) = 2\, U_m(\xi)$$

and hence we have

$$\operatorname*{Lim}_{\xi=x} W(\xi) = \operatorname*{Lim}_{\xi=x} U_m(\xi).$$

But from the ordinary theory of Dirichlet's problem for the plane with a single circular hole, it is known that $U_m(\xi)$ changes continuously into the boundary value $\psi_m(x)$, therefore we have also

$$\operatorname{Lim} W(\xi) = \psi_m(x),$$

and it is proved that the potential function W, as defined in I, art. 7, satisfies indeed the assigned boundary conditions.

9. *Dirichlet's problem for a plane with $q = p + 1$ circular holes.* In the preceding investigations one of the rims K_q was a circle of infinite radius, there remains to shew that this circumstance is totally irrelevant. In fact, when we have to solve Dirichlet's problem for a plane S with q circular holes, it is always possible, by means of a proper linear substitution, that changes one of the rims into a right line, to represent the area S conformally upon the region T', and as we are able to solve the problem for T', we can get in this way the solution for S. However it is easily seen that the previous mapping of S on T' is entirely superfluous, in as much the quantities, entering into our formulae, are either potential functions or anharmonic ratios, not altered by linear transformation. Thus then, if among the given circumferences in S we have chosen one as K_q, we have only to construct the p pairs of limiting points $\overset{k}{A}$ and $\overset{k}{B}$, each pair belonging to one of the p systems of circles K_k, K_q, and we may use directly all the formulae of art. 7 without the slightest modification.

Merely by way of illustration, and also in order to shew that with the aid of our formulae even numerical approximation is not wholly impracticable, we finally will consider a very special case. Let K_1, K_2, K_3 be three equal circular holes made in a plane, the centres a_1, a_2, a_3 of which form the vertices of an equilateral triangle, and let the common diameter of the holes be one third of the side of the triangle. As to the rim values of the potential function W, existing in the space outside the holes, we assume that W is equal to unity on that half of each rim, that is turned towards the centre ξ of the triangle, and equal to zero on the other half. We will now ask for the value $W(\xi)$ the function W takes at the centre ξ.

The first step is the construction of the two pairs of limiting points $\overset{1}{A}, \overset{1}{B}$ and $\overset{2}{A}, \overset{2}{B}$. They are readily found as the points of intersection of the sides $a_1 a_3$ and $a_2 a_3$ with the orthogonal circle of K_1, K_2, K_3 (so that $\overset{1}{B}$ and $\overset{2}{B}$ lie within K_3). Then we proceed to calculate $L_1(\xi)$ and $L_2(\xi)$, necessarily equal to each other from reasons of symmetry. Now as with respect to their mutual distances the diameter of the holes is comparatively small, we may regard as practically coïncident two points $x_{\mu''\alpha''}$

and $x_{\mu''\beta''}$, whenever the common factor μ'' contains three fundamental marks at least. So the general formula III, art. 7.

$$e^{L_1(\xi)} = \left[\xi\xi_{3''}; \overset{1}{A}\overset{1}{B}\right]^{\frac{1}{2}} \prod_{\gamma''} \left[\xi\xi_{3''}; \overset{1}{B}_{\gamma''}\overset{1}{A}_{\gamma''}\right]$$

becomes simply

$$e^{L_1(\xi)} = \left[\xi\xi_{3''}; \overset{1}{A}\overset{1}{B}\right]^{\frac{1}{2}} \left[\xi\xi_{3''}; \overset{1}{B}_{2''}\overset{1}{A}_{2''}\right] \left[\xi\xi_{3''}; \overset{1}{B}_{3''1''2''}\overset{1}{A}_{3''1''2''}\right],$$

or, by a slight transformation of the last factor,

$$e^{L_1(\xi)} = \left[\xi\xi_{3''}; \overset{1}{A}\overset{1}{B}\right]^{\frac{1}{2}} \left[\xi\xi_{3''}; \overset{1}{B}_{2''}\overset{1}{A}_{2''}\right] \left[\xi_{3''}\xi; \overset{1}{B}_{1''2''}\overset{1}{A}_{1''2''}\right].$$

In this form the above equation may be used to evaluate $L_1(\xi)$. From it we shall find

$$L_1(\xi) = L_2(\xi) = -1{,}740.$$

Similarly we obtain, by considering, instead of ξ, a point on the rim K_1 and a point on the rim K_2,

$$\tau_{11} = \tau_{22} = -3{,}474, \qquad \tau_{12} = \tau_{21} = -1{,}736.$$

Substituting these results in the equations

$$L_1(\xi) = \tau_{11}\lambda_1 + \tau_{12}\lambda_2,$$
$$L_2(\xi) = \tau_{21}\lambda_1 + \tau_{22}\lambda_2,$$
$$1 = \lambda_1 + \lambda_2 + \lambda_3,$$

we get approximatively

$$\lambda_1 = \lambda_2 = 0{,}334, \qquad \lambda_3 = 0{,}332,$$

the exact result being of course

$$\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{3}.$$

Employing the latter value of the coefficients λ, symmetry again permits to write the formula I, art. 7, in the simplified form

$$W(\xi) = \frac{1}{2}\sum_{k=1}^{k=3}\left[\sum_{\alpha''}\{U_k(\xi_{\alpha''}) - U_k(\xi_{\alpha''3''})\}\right].$$

In expanding the right-hand side still further simplification is possible from the same reason, moreover a very few terms of the infinite series need only be retained, because we agree to consider as practically coïncident two points $x_{\mu''\alpha''}$ and $x_{\mu''\beta''}$, as soon as μ'' contains three or more fundamental marks.

In fact, we shall find

$$W(\xi) = 3\,U_1(\xi) - 6\,U_1(\xi_{2''}) + 6\,U_1(\xi_{2''3''}) + 6\,U_1(\xi_{2''1''}) - 2\,U_1({}_{2''3''2''})$$
$$- 2\,U_1(\xi_{2''1''2''}) - 2\,U_1(\xi_{2''1''3''}) - 2\,U_1(\xi_{2''3''1''})$$

Substituting in this expression the values of the potential U_1 at the points ξ, $\xi_{2''}$, $\xi_{2''3''}$, etc., determined beforehand by the usual method, we arrive at the final result

$$W(\xi) = 0{,}534\,.$$