PERIODIC ORBITS

BY

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§ 1. *Introduction.*

The existing methods of treating the Problem of the three Bodies are only applicable to the determination, by approximation, of the path of the third body when the attraction of the first largely preponderates over that of the second. A general solution of the problem is accordingly not to be obtained by these methods.

In the Lunar and Planetary theories it has always been found necessary to specify the motion of the perturbed body by reference to a standard curve or intermediate orbit, of which the properties are fully known. The degree of success attained by any of these methods has always depended on the aptness of the chosen intermediate orbit for the object in view. It is probable that future efforts will resemble their precursors in the use of standard curves of reference.

Mr G. W. HILL's papers on the Lunar Theory[1] mark an epoch in the history of the subject. His substitution of the Variational Curve for the ellipse as the intermediate orbit is not only of primary importance in the Lunar Theory itself, but has pointed the way towards new fields of research.

The variational curve may be described as the distortion of the moon's circular orbit by the solar attraction. It is one of that class

[1] American Journal of Mathematics, Vol. I pp. 5—29, 129—147, 245—260 and Acta Mathematica, T. 8 pp. 1—36.

of periodic solutions of the Problem of the three Bodies which forms the subject of the present paper.

Of these solutions M. POINCARÉ writes:

»Voici un fait que je n'ai pu démontrer rigoureusement, mais qui me paraît pourtant très vraisemblable.»

»Etant données des équations de la forme définie dans le n° 13 et une solution particulière de ces équations, on peut toujours trouver une solution périodique (dont la période peut, il est vrai, être très longue), telle que la différence entre les deux solutions soit aussi petite qu'on le veut, pendant un temps aussi long qu'on le veut. D'ailleurs, ce qui nous rend ces solutions périodiques si précieuses, c'est qu'elles sont, pour ainsi dire, la seule brèche par où nous puissions essayer de pénétrer dans une place jusqu'ici réputée inabordable.» [1]

He tells us that he has been led to distinguish three kinds of periodic solutions. In those of the first kind the inclinations vanish and the eccentricities are very small; in those of the second kind the inclinations vanish and the eccentricities are finite; and in those of the third kind the inclinations do not vanish. [2]

If I understand this classification correctly the periodic orbits, considered in this paper, belong to the first kind, for they arise when the perturbed body has infinitely small mass, and when the two others revolve about one another in circles.

M. POINCARÉ remarks that there is a quadruple infinity of periodic solutions, for there are four arbitrary constants viz. the period of the infinitesimal body, the constant of energy, the moment of conjunction, and the longitude of conjunction. [3] For the purpose of the present investigation this quadruple infinity may however be reduced to a single infinity, for the moment and longitude of conjunction need not be considered; and the scale on which we draw the circular orbit of the second body round the first is immaterial. Thus we are only left with the constant of relative energy of the motion of the infinitesimal body as a single arbitrary.

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[1] Mécanique Céleste, T. 1, p. 82.
[2] » » T. 1, p. 97 and Bull. Astr., T. 1, p. 65.
[3] » » T. 1, p. 101.

Notwithstanding the great interest attaching to periodic orbits, no suggestion has, up to the present time, been made by any writer for a general method of determining them. As far as I can see, the search resolves itself into the discussion of particular cases by numerical processes, and such a search necessarily involves a prodigious amount of work. It is not for me to say whether the enormous labour I have undertaken was justifiable in the first instance; but I may remark that I have been led on, by the interest of my results, step by step, to investigate more and again more cases. Now that so much has been attained I cannot but think that the conclusions will prove of interest both to astronomers and to mathematicians.

In conducting extensive arithmetical operations, it would be natural to avail oneself of the skill of professional computers. But unfortunately the trained computer, who is also a mathematician, is rare. I have thus found myself compelled to forego the advantage of the rapidity and accuracy of the computer, for the higher qualities of mathematical knowledge and judgment.

In my earlier work I received the greatest assistance from Mr J. W. F. Allnutt; his early death has deprived me of a friend and of an assistant, whose zeal and care were not to be easily surpassed. Since his death Mr J. I. Craig (of Emmanuel College) and Mr M. J. Berry (of Trinity College) have rendered and are rendering valuable help. I have besides done a great deal of computing myself.[1]

The reader will see that the figures have been admirably rendered by Mr Edwin Wilson of Cambridge, and I only regret that it has not seemed expedient to give them on a larger scale.

The first part of the paper is devoted to the mathematical methods employed, the second part contains the discussion of the results, and the tables of numerical results are relegated to an Appendix.

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[1] About two thirds of the expense of these computations have been met by grants from the Government Grant and Donation Funds of the Royal Society.

PART. I.

§ 2. *Equations of motion.*

The particular case of the problem of the three bodies, considered in this paper, is where the mass of the third body is infinitesimal compared with that of either of the two others which revolve about one another in circles, and where the whole motion takes place in one plane.

For the sake of brevity the largest body will be called the Sun, the planet which moves round it will be called Jove, and the third body will be called the planet or the satellite, as the case may be.

Jove J, of unit mass, moves round the Sun S, of mass ν, in a circle of unit radius SJ, and the orbit to be considered is that of an infinitesimal body P moving in the plane of Jove's orbit.

Let S be the origin of rectangular axes; let SJ be the x axis, and let the y axis be such that a rotation from x to y is consentaneous with the orbital motion of J. Let x, y be the heliocentric coordinates of P, so that $x - 1, y$ are the jovicentric coordinates referred to the same x axis and a parallel y axis.

Let r denote SP, and θ the angle JSP; let ρ denote JP, and let the angle SJP be $180° - \phi$. Thus r, θ are the polar heliocentric coordinates, and ρ, ϕ the polar jovicentric coordinates of P.

Let n denote Jove's orbital angular velocity, so that in accordance with KEPLER's law

$$n^2 = \nu + 1.$$

The equations of motion of a particle referred to axes rotating with angular velocity ω, under the influence of forces whose potential is U, are

$$\frac{d}{dt}\left(\frac{dX}{dt} - \omega Y\right) - \omega\left(\frac{dY}{dt} + \omega X\right) = \frac{\partial U}{\partial X},$$
$$\frac{d}{dt}\left(\frac{dY}{dt} + \omega X\right) + \omega\left(\frac{dX}{dt} - \omega Y\right) = \frac{\partial U}{\partial Y},$$

where t is the time.

Now in the present problem, if the origin be taken at the centre of inertia of the Sun and Jove with SJ for the X axis, the coordinates of P are $X = x - \frac{1}{\nu + 1}$, $Y = y$. Also the potential function is $\frac{\nu}{r} + \frac{1}{\rho}$. Hence the equations of motion are

$$\frac{d^2x}{dt^2} - 2n\frac{dy}{dt} - (\nu + 1)\left(x - \frac{1}{\nu + 1}\right) = \frac{\partial}{\partial x}\left(\frac{\nu}{r} + \frac{1}{\rho}\right),$$

$$\frac{d^2y}{dt^2} + 2n\frac{dx}{dt} - (\nu + 1)y \qquad\qquad = \frac{\partial}{\partial y}\left(\frac{\nu}{r} + \frac{1}{\rho}\right).$$

But $r^2 = x^2 + y^2$, $\rho^2 = (x - 1)^2 + y^2$. Hence if we put

$$(1) \qquad 2\Omega = \nu\left(r^2 + \frac{2}{r}\right) + \left(\rho^2 + \frac{2}{\rho}\right),[1]$$

the equations of motion may be written

$$(1) \qquad \begin{cases} \dfrac{d^2x}{dt^2} - 2n\dfrac{dy}{dt} = \dfrac{\partial\Omega}{\partial x}, \\[2ex] \dfrac{d^2y}{dt^2} + 2n\dfrac{dx}{dt} = \dfrac{\partial\Omega}{\partial y}, \end{cases}$$

where $n^2 = \nu + 1$.

Let the second of (1) be multiplied by $2\frac{dx}{dt}$, and the third by $2\frac{dy}{dt}$, let the two be added together and integrated, and we have Jacobi's integral

$$(2) \qquad V^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2\Omega - C,$$

where C is a constant, and V denotes the velocity of P relatively to the rotating axes.

Let s be the arc of the planet's relative orbit measured from any fixed point, and let φ be the inclination to the x axis of the outward normal of the orbit. Then

$$\frac{dx}{ds} = -\sin\varphi, \qquad \frac{dy}{ds} = \cos\varphi.$$

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[1] It is perhaps worth noting that 2Ω may be written in the form

$$\nu(r - 1)^2\left(1 + \frac{2}{r}\right) + (\rho - 1)^2\left(1 + \frac{2}{\rho}\right) + 3(\nu + 1).$$

Hence if P be the component of inward effective force,

(3) $$P = -\frac{\partial \Omega}{\partial x}\cos\varphi - \frac{\partial \Omega}{\partial y}\sin\varphi.$$

Therefore

$$PV = -\frac{\partial \Omega}{\partial x}\frac{dy}{dt} + \frac{\partial \Omega}{\partial y}\frac{dx}{dt}.$$

Now if R denotes the radius of curvature at the point x, y, of the relative orbit of P,

$$\frac{1}{R} = \frac{\frac{d^2y}{dt^2}\frac{dx}{dt} - \frac{d^2x}{dt^2}\frac{dy}{dt}}{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{\frac{3}{2}}}.$$

On substituting for the second differentials from (1), we have

$$\frac{V^3}{R} = \frac{\partial \Omega}{\partial y}\frac{dx}{dt} - \frac{\partial \Omega}{\partial x}\frac{dy}{dt} - 2n\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right].$$

Hence by means of (2) and (3)

(4) $$\frac{1}{R} = \frac{P}{V^2} - \frac{2n}{V}.$$

If the value of Ω in (1) be substituted in (3) we easily find

(4) $$\left\{\begin{array}{l} P = \nu\left(\frac{1}{r^2} - r\right)\cos(\varphi - \theta) + \left(\frac{1}{\rho^2} - \rho\right)\cos(\varphi - \psi), \\ \text{and} \\ V^2 = \nu\left(r^2 + \frac{2}{r}\right) + \left(\rho^2 + \frac{2}{\rho}\right) - C. \end{array}\right.$$

Thus the curvature at any point of the orbit is expressible in terms of the coordinates and of the direction of the normal. If $s_0, \varphi_0, x_0, y_0, t_0$ be the initial values of the same quantities, it is clear that

$$
(5)\qquad \begin{cases} \varphi = \varphi_0 + \int\limits_{s_0}^{s} \frac{ds}{R}, \\ x = x_0 - \int\limits_{s_0}^{s} \sin\varphi \, ds, \\ y = y_0 + \int\limits_{s_0}^{s} \cos\varphi \, ds, \\ n(t - t_0) = \int\limits_{s_0}^{s} \frac{n}{V} ds. \end{cases}
$$

Also the polar coordinates of P relatively to axes fixed in space with heliocentric origin are $r, \theta + n(t - t_0)$, and with jovicentric origin are $\rho, \psi + n(t - t_0)$.

Hence the determination of x and y involves in each case two integrations, and another integration is necessary to find the time, and the orbit in space.

§ 3. *Partition of relative space according to the value of the relative energy.*[1]

It may be easily shown that the function Ω arises from three sources, and that it is the sum of the rotation potential, the potential of the Sun and the disturbing function for motion relatively to the Sun. Hence Ω is the potential of the system, inclusive of the rotation potential. Thus the equation $V^2 = 2\Omega - C$ may be called the equation of relative energy.

For a real motion of the planet V^2 must be positive, and therefore 2Ω must be greater than C. Accordingly the planet can never cross the curve represented by $2\Omega = C$, and if this curve has a closed branch

[1] A somewhat similar investigation is contained in a paper by M. BOHLIN, Acta Math. T. 10, p. 109 (1887). The author takes the Sun as a fixed centre, which is equivalent to taking the Sun's mass as very large compared with that of Jove; he thus fails to obtain the function Ω in the symmetrical form used above.

with P inside, it must always remain inside; or if P be outside, it must always remain so.

This is Mr Hill's result in his celebrated memoir[1] on the Lunar Theory, save that the value of Ω used here has not been reduced to an approximate form.

We shall now proceed to a consideration of the family of curves $2\Omega = C$. That is to say we shall find, for a given value of C, the locus of points for which the three bodies may move for an instant as parts of a single rigid body. We are clearly at the same time finding the curves of constant velocity relatively to the moving axes for other values of C.

For any given value of ρ, the values of r are the roots of the cubic equation

$$r^2 + \frac{2}{r} = \frac{1}{\nu}\left(C - \rho^2 - \frac{2}{\rho}\right).$$

If C' be written for the value of the right hand of this equation, the cubic becomes

$$r^3 - C'r + 2 = 0.$$

The solution is

$$r = 2\sqrt{\frac{1}{3}C'}\cos\alpha, \quad \text{where} \quad \cos 3\alpha = -C'^{-\frac{3}{2}}\sqrt{27}.$$

In order that α may be a real angle, such a value of ρ must be assumed that C' may be greater than 3, or $\rho^2 + \frac{2}{\rho}$ less than $C - 3\nu$. The limiting form of this last inequality is $\rho^2 + \frac{2}{\rho} = C - 3\nu$, a cubic of the same form as before. Hence it follows that $C - 3\nu$ must be greater than 3. Thus the minimum value of C is $3(\nu + 1)$.

With C greater than $3(\nu + 1)$, let β be the smallest positive angle such that $\cos 3\beta = C'^{-\frac{3}{2}}\sqrt{27}$. Then β is clearly less than 30°, and the three roots of the cubic are

$$2\sqrt{\frac{1}{3}C'}\cos(60° \pm \beta), \quad -2\sqrt{\frac{1}{3}C'}\cos\beta.$$

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[1] Amer. Journ. of Math. Vol. 1, pp. 5—29.

The third of these roots is essentially negative, and may be omitted as not corresponding to a geometrical solution. But the first two roots are positive and will give a real geometrical meaning to the solution provided that if $\rho > 1$,

$$r < \rho + 1$$
$$> \rho - 1;$$

and if $\rho < 1$,

$$r < \rho + 1$$
$$> 1 - \rho.$$

In some cases there are two solutions, in others one and in others none.

By the solution of a number of cubic equations I have found a number of values of r, ρ which satisfy $2\Omega = C$, and have thus traced the curves in Fig. 1, to the consideration of which I shall return below.

Some idea of the nature of the family of curves may be derived from general considerations; for when r and ρ are small the equation approximates to $\frac{2\nu}{r} + \frac{2}{\rho} = C$, and the curves are like the equipotentials due to two attractive particles of masses 2ν and 2.

Thus for large values of C they are closed ovals round S and J, the one round S being the larger. As C declines the ovals swell and coalesce into a figure-of-8, which then assumes the form of an hour-glass with a gradually thickening neck.

When on the other hand r and ρ are large the equation approximates to $\nu r^2 + \rho^2 = C$, and this represents an oval enclosing both S and J, which decreases in size as C decreases.

It is thus clear by general reasoning that for large values of C the curve consists of two closed branches round S and J respectively, and of a third closed branch round both S and J. The spaces within which the velocity of the planet is real are inside of either of the smaller ovals, and outside of the larger one. Since the larger oval shrinks and the hour-glass swells, as C declines, a stage will be reached when the two curves meet and coalesce. This first occurs at the end of the small bulb of the hour-glass which encloses J. The curve is then shaped like a horse-shoe, but is narrow at the toe and broad at the two points.

For still smaller values of C, the horse-shoe narrows to nothing at the toe, and breaks into two elongated pieces. These elongated pieces, one on each side of SJ, then shrink quickly in length and slowly in breadth, until they contract to two points when C reaches its minimum.

This sketch of the sequence of changes shows that there are four critical stages in the history of the curves,

(α) when the internal ovals coalesce to a figure-of-8;

(β) when the small end of the hour-glass coalesces with the external oval;

(γ) when the horse-shoe breaks;

(δ) when the halves of the broken shoe shrink to points.

The points of coalescence and rupture in (α), (β), (γ) are obviously on the line SJ (produced either way), and the points in (δ) are symmetrically situated on each side of SJ.

We must now consider the physical meaning of the critical points, and show how to determine their positions.

In the first three cases the condition which enables us to find the critical point is that a certain equation derived from $2\Omega = C$ shall have equal roots.

(α) The coalescence into a figure-of-8 must occur between S and J; hence $r = 1 - \rho$, and $2\Omega = C$ becomes

$$(6) \qquad \nu\left[(1-\rho)^2 + \frac{2}{1-\rho}\right] + \rho^2 + \frac{2}{\rho} = C.$$

This equation must have equal roots. Accordingly by differentiation we find that ρ must satisfy,

$$-\nu(1-\rho) + \frac{\nu}{(1-\rho)^2} + \rho - \frac{1}{\rho^2} = 0,$$

or

$$(\nu+1)\rho^5 - (3\nu+2)\rho^4 + (3\nu+1)\rho^3 - \rho^2 + 2\rho - 1 = 0,$$

a quintic equation from which ρ may be found.

This equation may be put in the form,

$$(3\nu+1)\rho^3 = 1 - \frac{\rho(1-\rho^3)\left(1-\frac{2}{3}\rho\right)}{1-\rho+\frac{1}{3}\rho^2}.$$

When the Sun is large compared with Jove ν is large, and ρ is obviously small, and we have approximately

$$(3\nu + 1)^{\frac{1}{3}}\rho = 1 - \frac{1}{3}\rho,$$

whence

$$\rho = \frac{1}{(3\nu + 1)^{\frac{1}{3}} + \frac{1}{3}}. \tag{7}$$

If this value of ρ be substituted in (6) we obtain the approximate result

$$C = 3\nu + \frac{2\nu}{3\nu + 1} + 3(3\nu + 1)^{\frac{1}{3}}. \tag{8}$$

In this paper the value adopted for ν is 10, and the approximate formulæ (7) and (8) give

$$\rho = \cdot 28779, \qquad r = \cdot 71221, \qquad C = 40 \cdot 0693.$$

The correct results derived from the quintic equation and from the full formula for C are

$$\rho = \cdot 28249, \qquad r = \cdot 71751, \qquad C = 40 \cdot 1821. \tag{9}$$

Thus for even so small a value of ν as 10, the approximation is near the truth, and for such cases as actually occur in the solar system it would be accurate enough for every purpose.

The formula from which ρ has been derived is equivalent to $\frac{\partial \Omega}{\partial x} = 0$, and since $y = 0$, we have also $\frac{\partial \Omega}{\partial y} = 0$. Hence the point is one of zero effective force at which the planet may revolve without motion relatively to the Sun and Jove.

This position of conjunction between the two larger bodies is obviously one of dynamical instability.

(β) The coalescence of the hour-glass with the external oval must occur at a point in SJ produced beyond J; hence $r = 1 + \rho$, and $2\Omega = C$ becomes

$$\nu\left[(1 + \rho)^2 + \frac{2}{1 + \rho}\right] + \rho^2 + \frac{2}{\rho} = C.$$

This equation must have equal roots, and ρ must satisfy

$$\nu(1+\rho)-\frac{\nu}{(1+\rho)^2}+\rho-\frac{1}{\rho^2}=0,$$

or

$$(\nu+1)\rho^5+(3\nu+2)\rho^4+(3\nu+1)\rho^3-\rho^2-2\rho-1=0.$$

This quintic equation may be written in the form

$$(3\nu+1)\rho^3=1+\frac{\rho(1-\rho^3)\left(1+\frac{2}{3}\rho\right)}{1+\rho+\frac{1}{3}\rho^2}.$$

With the same approximation as in (α)

(10) $$\rho=\frac{1}{(3\nu+1)^{\frac{1}{3}}-\frac{1}{3}},$$

(11) $$C=3\nu-\frac{2\nu}{3\nu+1}+3(3\nu+1)^{\frac{1}{3}}.$$

When ν is 10, the approximate formulæ (10), (11) give

$$\rho=\cdot 35612, \qquad r=1\cdot 35612, \qquad C=38\cdot 7790.$$

The correct results derived from the quintic equation are

(12) $$\rho=\cdot 34700, \qquad r=1\cdot 34700, \qquad C=38\cdot 8760.$$

The approximation is not so good as in (α), but in such cases as actually occur in the solar system the formulæ (10), (11) would lead to a high degree of accuracy.

This second critical point is another one at which the planet may revolve without motion relatively to the Sun and Jove, and such a motion is dynamically unstable.

(γ) The thinning of the toe of the horse-shoe to nothing must occur at a point in JS produced beyond S; hence $\rho=r+1$, and $2\Omega=C$ becomes

$$\nu\left(r^2+\frac{2}{r}\right)+(r+1)^2+\frac{2}{r+1}=C.$$

This equation must have equal roots, and r must satisfy

$$\nu\left(r - \frac{1}{r^2}\right) + (r + 1) - \frac{1}{(r + 1)^2} = 0,$$

or

$$(\nu + 1)r^5 + (2\nu + 3)r^4 + (\nu + 3)r^3 - \nu(r^2 + 2r + 1) = 0,$$

a quintic for finding r.

If we put $r = 1 - \xi$, the equation becomes

$$(\nu + 1)\xi^5 - (7\nu + 8)\xi^4 + (19\nu + 25)\xi^3 - (24\nu + 37)\xi^2 + (12\nu + 26)\xi - 7 = 0.$$

This equation may be solved by approximation, and the first approximation, which is all that I shall consider, gives

$$(13) \qquad \xi = 1 - r = \frac{7}{12\nu + 26}.$$

Thus the approximate solution is $r = 1 - \dfrac{7}{12\nu + 26}$.

We also find

$$(14) \quad C = \nu(1 - 2\xi + \xi^2 + 2 + 2\xi + 2\xi^2 \ldots) + 4 - 4\xi + \xi^2 + 1 + \frac{1}{2}\xi + \frac{1}{4}\xi^2$$

$$= 3\nu + 5 - \frac{7}{2}\xi + \left(3\nu + \frac{5}{4}\right)\xi^2.$$

If we take only the term in ξ in (14), and put $\nu = 10$ the approximate result is

$$r = \cdot 95205, \qquad \rho = 1 \cdot 95205, \qquad C = 34 \cdot 9012.$$

The exact solution derived from the quintic equation is

$$(15) \qquad r = \cdot 94693, \qquad \rho = 1 \cdot 94693, \qquad C = 34 \cdot 9054.$$

With large values of ν the first approximation would give nearly accurate results. This critical point is another one at which the three bodies may move round without relative motion, but as before the motion is dynamically unstable.

(δ) The fourth and last critical position occurs when C is a minimum. Now C is a minimum when $\frac{\partial C}{\partial r} = 0$, $\frac{\partial C}{\partial \rho} = 0$; whence $r = 1$, $\rho = 1$, and $C = 3\nu + 3$. We arrived above at this minimum value of C from another point of view.

If an equilateral triangle be drawn on SJ, its vertex is at this fourth critical point; and since this vertex may be on either the positive or negative side of SJ, there are two points of this kind.

It is well known that there is an exact solution of the problem of three bodies in which they stand at the corners of an equilateral triangle, which revolves with a uniform angular velocity. The motion is stable in this case.

Thus all the five critical points correspond with particular exact solutions of the problem, and of these solutions three are unstable and the symmetrical pair is stable.

Fig. 1 represents the critical curves of the family $2\Omega = C$, for the case $\nu = 10$. The points in the curves were determined, as explained above, by the solution of a number of cubic equations. I have only drawn the critical curves, because the addition of other members of the family would merely complicate the figure.

An important classification of orbits may be derived from this figure. When C is greater than $40 \cdot 1821$ the third body must be either a superior planet moving outside of the large oval, or an inferior planet moving inside of the larger internal oval, or a satellite moving inside the smaller internal oval; and it can never exchange one of these parts for either of the other two. The limiting case $C = 40 \cdot 1821$ gives superior limits to the radii vectores of inferior planets and of satellites, which cannot sever their connections with their primaries.

When C is less than $40 \cdot 1821$ but greater than $38 \cdot 8760$, the third body may be a superior planet, or an inferior planet or satellite, or a body which moves in an orbit which partakes of the two latter characteristics; but it can never pass from the first condition to any of the latter ones.

When C is less than $38 \cdot 8760$ and greater than $34 \cdot 9054$, the body may move anywhere save inside of a region shaped like a horse-shoe. The distinction between the two sorts of planetary motion and the motion as a satellite ceases to exist, and if the body is started in any one of these three ways it is possible for it to exchange the characteristics of its motion for either of the two other modes.

When C is less than $34\cdot9054$ and greater than 33, the forbidden region consists of two strangely shaped portions of space on each side of SJ.

Lastly when C is equal to 33, than which it cannot be less, the forbidden regions have shrunk to a pair of infinitely small closed curves enclosing the third angles of a pair of equilateral triangles erected on SJ as a base.

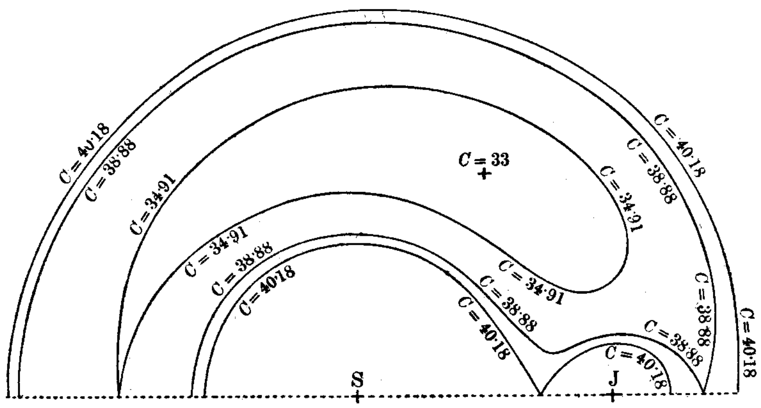

Fig.1

Curves of zero velocity, $10\left(r^2 + \frac{2}{r}\right) + \left(\rho^2 + \frac{2}{\rho}\right) = C.$

§ 4. *A certain partition of space according to the nature of the curvature of the orbit.*

It appears from (4) of § 2 that the curvature of an orbit is given by

$$\frac{V^2}{R} = P - 2nV, \quad \text{where} \quad P = -\frac{\partial\Omega}{\partial x}\cos\varphi - \frac{\partial\Omega}{\partial y}\sin\varphi.$$

Now if V_0 denotes any constant velocity, the equation $2\Omega = C + V_0^2$ defines a curve of constant velocity; it is one of the family of curves considered in § 3. We have seen that this family consists of a large oval enclosing two smaller ones, or of curves arising from the coalescence of ovals. In the mathematical sense of the term the »interior» of the curve of constant velocity consists of the space inside of either of the smaller ovals or outside of the large one, or of the corresponding spaces when there is coalescence of ovals. It is a convenient and ordinary convention that when the circuit of a closed curve is described in a positive direction, the »interior» of the curve is on the left-hand side. According to this convention the meaning of the »inward» normal of one of these curves of constant velocity is clear, for it is directed towards the »interior». Similarly the inward normal of an orbit is towards the left-hand side, as the body moves along its path.

It is clear then that P is the component of effective force estimated along the inward normal of the orbit. Also if T be the resultant effective force $T^2 = \left(\frac{\partial \Omega}{\partial x}\right)^2 + \left(\frac{\partial \Omega}{\partial y}\right)^2$; and if χ be the angle between T and the inward normal to the orbit, $P = T\cos\chi$.

Hence

$$\frac{V^2}{R} = T\cos\chi - 2nV.$$

If we consider curvature as a quantity which may range from infinite positive to infinite negative, it may be stated that of all the orbits passing through a given point the curvature is greatest for that orbit which is tangential to the curve of constant velocity, when the motion takes place in a positive direction along that curve.

If χ lies between $\pm\chi_0$, where $\cos\chi_0 = \frac{2nV}{T}$, the orbit has positive curvature; if $\chi = \pm\chi_0$, there is a point of contrary flexure in the orbit; and if χ lies outside of the limits $\pm\chi_0$, the curvature is negative.

If however T be less than $2nV$, there are no orbits, passing through the point under consideration, which have positive curvature. Hence the equation $T = 2nV$ defines a family of curves which separate the regions in which the curvature of orbits is necessarily negative, from those in which it may be positive.

Since

$$n^2 = \nu + 1, \qquad V^2 = \nu\left(r^2 + \frac{2}{r}\right) + \left(\rho^2 + \frac{2}{\rho}\right) - C, \qquad T^2 = \left(\frac{\partial\Omega}{\partial x}\right)^2 + \left(\frac{\partial\Omega}{\partial y}\right)^2,$$

the equation $T = 2nV$ becomes,

$$\nu^2\left(\frac{1}{r^2} - r\right)^2 + \left(\frac{1}{\rho^2} - \rho\right)^2 + 2\nu\left(\frac{1}{r^2} - r\right)\left(\frac{1}{\rho^2} - \rho\right)\cos(\theta - \phi)$$

$$= 4(\nu + 1)\left[\nu\left(r^2 + \frac{2}{r}\right) + \left(\rho^2 + \frac{2}{\rho}\right) - C\right].$$

Since $2r\rho\cos(\theta - \phi) = r^2 + \rho^2 - 1$, it may be written

$$(16) \qquad \nu^2\left(\frac{1}{r^4} - \frac{10}{r} - 3r^2\right) + \left(\frac{1}{\rho^4} - \frac{10}{\rho} - 3\rho^2\right)$$

$$+ \nu\left[\left(\frac{1}{r^3} - 1\right)\left(\frac{1}{\rho^3} - 1\right)(r^2 + \rho^2 - 1) - 4(r^2 + \rho^2) - 8\left(\frac{1}{r} + \frac{1}{\rho}\right)\right] + 4C(\nu + 1) = 0.$$

This equation is reducible to the sextic equation,

$$(16) \qquad \rho^6[3(\nu + 1)r^4 + \nu r]$$

$$+ \rho^4[3\nu(\nu + 1)r^6 - (4\nu C + 4C - \nu)r^4 + (10\nu^2 + 9\nu)r^3 - \nu r - \nu^2]$$

$$+ \rho^3[(9\nu + 10)r^4 - \nu r] + \rho\nu r(1 - r^2)(1 - r^3) - r^4 = 0.$$

It may also be written as a sextic in r, by interchanging r and ρ and by writing $\frac{1}{\nu}$ for ν and $\frac{C}{\nu}$ for C.

It would require a great deal of computation to trace the curves represented by (16), and for the present I have not thought it worth while to undertake the task.

When however we adopt Mr Hill's approximate value for the potential Ω, the equation becomes so much simpler that it may be worth while to consider it further.

If m, a, n be the mass, distance from Sun and orbital angular velocity of Jove, the expression for Ω reduces to

$$\Omega = \frac{m}{\rho} + \frac{3}{2}n^2(x-a)^2 + \frac{3}{2}n^2a^2.$$

The last term is constant, so that if C be replaced by C_0, where $C_0 = C - 3n^2a^2$, we may omit the last term in Ω and use C_0 in place of C.

Now taking units of length and time such that $m = 1$, $n = 1$; also writing $\xi = (x-a)$, $\eta = y$; we have

$$(17) \qquad \Omega = \frac{1}{\rho} + \frac{3}{2}\xi^2, \qquad V^2 = 2\Omega - C_0, \qquad \xi^2 + \eta^2 = \rho^2.$$

Then

$$T^2 = \left(\frac{\partial\Omega}{\partial\xi}\right)^2 + \left(\frac{\partial\Omega}{\partial\eta}\right)^2 = 3\left(3 - \frac{2}{\rho^3}\right)\xi^2 + \frac{1}{\rho^4}.$$

Hence the equation (16) becomes

$$3\left(3 - \frac{2}{\rho^3}\right)\xi^2 + \frac{1}{\rho^4} = 4\left(\frac{2}{\rho} + 3\xi^2 - C_0\right),$$

or

$$(18) \qquad \xi^2\left(1 + \frac{2}{\rho^3}\right) = \frac{4}{3}\left(C_0 - \frac{2}{\rho} + \frac{1}{4\rho^4}\right).$$

Since $\xi = \rho\cos\psi$, the polar equation to the curve is

$$(18) \qquad \cos^2\psi = \frac{4}{3}C_0\,\frac{\left(\rho - \frac{2}{C_0} + \frac{1}{4C_0\rho^3}\right)}{\rho^3 + 2}.$$

M^r^ Hill's curve $2\Omega = C_0$ gives

$$(19) \qquad \left\{\begin{array}{l} \xi^2 = \frac{1}{3}C_0\left(1 - \frac{2}{C_0\rho}\right), \\ \text{or} \\ \cos^2\psi = \frac{1}{3}\frac{C_0}{\rho^3}\left(\rho - \frac{2}{C_0}\right). \end{array}\right.$$

It is clear that the two curves present similar characteristics, but the former is the more complicated one.

The asymptotes of (18) are $\xi = \pm 2\sqrt{\left(\frac{1}{3}C_0\right)}$, whilst those of (19) are $\xi = \pm\sqrt{\left(\frac{1}{3}C_0\right)}$.

Again to find where the curves cut the positive half of the axis of η, we put $\xi = 0$, $\rho = \eta$ and find that (18) becomes

(20) $$\eta^4 - \frac{2}{C_0}\eta^3 + \frac{1}{4C_0} = 0,$$

whilst (19) becomes simply $\eta = \frac{2}{C_0}$.

The condition that (20) shall have equal roots is $4\eta = \frac{6}{C_0}$, or $\frac{1}{\eta} = \frac{2}{3}C_0$. But $C_0 = \frac{2}{\eta} - \frac{1}{4\eta^4}$, and therefore $C_0 = \frac{3}{2^{\frac{2}{3}}}$.

The quartic for η has two real roots if C_0 is less than $\frac{3}{2^{\frac{2}{3}}}$ or $1 \cdot 8899$, but no real roots if it is greater than this value.

It is easy to show that when the roots are real, one is greater than and the other less than $\frac{3}{2C_0}$.

It follows that if C_0 is greater than $1 \cdot 8899$ the curve does not cut the axis of η, but if less it does so twice.

To find the critical values of C_0 in the case of Mr Hill's curve (19), we put (as in § 3) $\eta = 0$ and therefore $\rho = \xi$, and we then find the condition that the equation shall have equal roots.

Now with $\rho = \xi$, (19) becomes

$$\xi^2 = \frac{1}{3}C_0 - \frac{2}{3\xi}.$$

This has equal roots when $\xi = \frac{1}{3^{\frac{1}{3}}}$. Hence $C_0 = 3\xi^2 + \frac{2}{\xi} = 3^{\frac{4}{3}} = 4 \cdot 3267$.

If C_0 be greater than $4 \cdot 3267$ the curve consists of an internal oval and of two asymptotic branches. With smaller values of C_0 the oval has coalesced with the two external branches.

Following the same procedure with our curve (18), we have to find when

$$\xi^2\left(1 + \frac{2}{\xi^3}\right) = \frac{4}{3}\left(C_0 - \frac{2}{\xi} + \frac{1}{4\xi^4}\right)$$

has equal roots.

The condition is that $3\xi^6 - 7\xi^3 + 2 = 0$, and the solutions are $\xi^3 = 2$, $\xi^3 = \frac{1}{3}$.

Now

$$C_0 = \frac{3}{4}\left(\xi^2 + \frac{2}{\xi}\right) + \frac{2}{\xi} - \frac{1}{4\xi^4}.$$

Hence when

$$\xi^3 = 2, \qquad C_0 = \frac{39}{8.2^{\frac{1}{3}}} = 3\cdot8693;$$

and when

$$\xi^3 = \frac{1}{3}, \qquad C_0 = 3^{\frac{4}{3}} = 4\cdot3267.$$

Thus there are three critical values of C_0, viz: $C_0 = 1\cdot8899$, which separates the curves which do from those which do not intersect the axis of η; $C_0 = 3\cdot8693$ when two branches coalesce; and $C_0 = 4\cdot3267$ when two branches again coalesce. The last is also a critical value of C_0 in the case of Mr Hill's curve.

It would seem then that if these curves were traced for the values $C_0 = 1\cdot5, 3, 4, 5$ a good idea might be obtained of their character, but I have not yet undertaken the task.

§ 5. *Formulæ of interpolation and quadrature.*

The object of this paper is to search for periodic orbits, but no general method has been as yet discovered by which they may be traced. I have therefore been compelled to employ a laborious method of tracing orbits by quadratures, and of finding the periodic orbits by trial. The formulæ of integration used in this process will now be exhibited.

According to the usual notation of the calculus of finite differences, u_x is to denote a function of x, and the operators E and Δ are defined by

$$Eu_x = u_{x+1}, \qquad \Delta u_x = u_{x+1} - u_x = (E - 1)u_x.$$

It is obvious that $E = e^{\frac{d}{dx}}$, where e is the base of Napierian logarithms, and that $E^x u_0 = u_x$.

In most of the work, as it presents itself in this investigation, the series of values $\ldots u_{n-2}, u_{n-1}, u_n$ are known, but $u_{n+1}, u_{n+2}, \ldots$ are as yet unknown.

Now

$$E = 1 + \Delta = (1 - \Delta E^{-1})^{-1},$$

and

$$u_x = E^x u_0 = (1 - \Delta E^{-1})^{-x} u_0,$$

so that

$$(21) \quad u_x = \left(1 + x\Delta E^{-1} + \frac{x(x+1)}{\underline{|2}} \Delta^2 E^{-2} + \frac{x(x+1)(x+2)}{\underline{|3}} \Delta^3 E^{-3} + \ldots\right) u_0.$$

In the course of the work occasion will arise for finding $u_{-\frac{1}{2}}$ by interpolation; putting then $x = -\frac{1}{2}$ in (21), we have

$$(22) \quad u_{-\frac{1}{2}} =$$

$$\left(1 - \frac{1}{2}\Delta E^{-1} - \frac{1}{8}\Delta^2 E^{-2} - \frac{1}{16}\Delta^3 E^{-3} - \frac{5}{128}\Delta^4 E^{-4} - \frac{7}{256}\Delta^5 E^{-5} \ldots\right) u_0.$$

In a subsequent section the two following well-known formulæ of interpolation will be of service,

$$(23) \quad u_x = \left\{1 + x \cdot \frac{1}{2}(\Delta + \Delta E^{-1}) + \frac{x^2}{\underline{|2}} \Delta^2 E^{-1} + \frac{x(x^2-1)}{\underline{|3}} \cdot \frac{1}{2}(\Delta^3 E^{-1} + \Delta^3 E^{-2}) + \frac{x^2(x^2-1)}{\underline{|4}} \Delta^4 E^{-2} \ldots \right\} u_0,$$

$$(23) \quad u_x = \left\{1 + x\Delta + \frac{x(x-1)}{\underline{|2}} \cdot \frac{1}{2}(\Delta^2 + \Delta^2 E^{-1}) + \frac{x(x-1)\left(x - \frac{1}{2}\right)}{\underline{|3}} \Delta^3 E^{-1} + \frac{x(x^2-1)(x-2)}{\underline{|4}} \cdot \frac{1}{2}(\Delta^4 E^{-1} + \Delta^4 E^{-2}) \ldots \right\} u_0.$$

Of these formulæ the first is the better when the interpolated value of u_x lies between $x = -\frac{1}{4}$ and $x = +\frac{1}{4}$; and the second is the better when it lies between $x = +\frac{1}{4}$ and $x = +\frac{3}{4}$.

In order to obtain a formula of integration we require to prove that

$$-\frac{1}{\log(1-a)} = \sum_{r=0}^{r=\infty} (-)^r a^{r-1} \int_0^1 \frac{v^{(r)}}{\underline{|r}} dv,$$

where $v^{(r)}$ denotes the factorial $v(v-1)\ldots(v-r+1)$.

This is easily proved as follows: —

$$\int_0^1 (1-a)^v dv = \left[\frac{e^{v\log(1-a)}}{\log(1-a)}\right]_0^1 = \frac{-a}{\log(1-a)}.$$

But

$$\int_0^1 (1-a)^v dv = \sum \int_0^1 (-)^r a^r \frac{v^{(r)}}{\underline{|r}} dv.$$

If the last two forms of this integral be equated to one another, we obtain the required formula.

Now

$$e^{\frac{d}{dx}} = (1-\Delta E^{-1})^{-1},$$

and therefore

$$\frac{d}{dx} = -\log(1-\Delta E^{-1}).$$

Hence

$$\int dx = \left(\frac{d}{dx}\right)^{-1} = -\frac{1}{\log(1-\Delta E^{-1})} = \sum (-)^r \Delta^{r-1} E^{-r+1} \int_0^1 \frac{v^{(r)}}{\underline{|r}} dv.$$

If the definite integrals on the right hand side be evaluated, we find

$$\int_0^n u_x dx = \Big(\Delta^{-1}E - \frac{1}{2} - \frac{1}{12}\Delta E^{-1} - \frac{1}{24}\Delta^2 E^{-2} - \frac{19}{720}\Delta^3 E^{-3}$$
$$- \frac{3}{160}\Delta^4 E^{-4} - \frac{863}{60480}\Delta^5 E^{-5}\ldots\Big)(u_n - u_0).$$

Since Δ^{-1} contains an arbitrary constant we may choose

(24) $$\Delta^{-1} u_1 = \frac{1}{2}u_0 + \frac{1}{12}\Delta u_{-1} + \frac{1}{24}\Delta^2 u_{-2} + \frac{19}{720}\Delta^3 u_{-3} + \ldots,$$

and we then have as our formula of integration,

$$(24)\qquad \int_0^n u_x dx = \Delta^{-1}u_{n+1} - \frac{1}{2}u_n - \frac{1}{12}\Delta u_{n-1} - \frac{1}{24}\Delta^2 u_{n-2}$$

$$- \frac{19}{720}\Delta^3 u_{n-3} - \frac{3}{160}\Delta^4 u_{n-4} - \frac{863}{60480}\Delta^5 u_{n-5}.$$

This is the most convenient formula of integration when only the integral from n to 0 is wanted, and the integrals from $n-1$ to 0, $n-2$ to 0, etc. are not also wanted. But in the greater part of the work the intermediate integrals are also required. Now on applying the operator Δ to (24), we have

$$(25)\qquad \int_n^{n+1} u_x dx = u_{n+1} - \frac{1}{2}\Delta u_n - \frac{1}{12}\Delta^2 u_{n-1} - \frac{1}{24}\Delta^3 u_{n-2} - \frac{19}{720}\Delta^4 u_{n-3}\ldots.$$

If this be added to the integral from n to 0 we have the integral from $n+1$ to 0.

I have found that a table of integration may be conveniently arranged as follows: —

Let us suppose that the integral from $n-1$ to 0 has been already found, and that the integral from n to 0 is required; write u_n and its differences Δu_{n-1}, $\Delta^2 u_{n-2}$, $\Delta^3 u_{n-3}$ in vertical column; below write $-\frac{1}{2}\Delta u_{n-1}$, $-\frac{1}{12}\Delta^2 u_{n-2}$, $-\frac{1}{24}\Delta^3 u_{n-3}$, and add them together; add u_n to the last; multiply the last sum by the common difference Δx, and the result is the integral from n to $n-1$; add to this the integral from $n-1$ to 0, and the result is the required integral from n to zero.

Thus each integration requires 13 lines of a vertical column, and the successive columns follow one another, headed by the value of the independent variable to which it applies.

A similar schedule would apply when the formula (24) is used; but when the initial value of Δ^{-1} has been so chosen as to insure the vanishing of the integral from 0 to 0, the final value of Δ^{-1} is to be found by adding to it the successive u's, so that the intermediate columns need not be written down.

When the successive values of u depend on their precursors, it is necessary at the first stage to take Δx small, because in the first integration it is only possible to take the first difference into account. At the second stage the second difference may be included and at the third the third difference.

But in almost every case I begin integration with such a value of the independent variable (say $x = 0$), that we either have u_x an even function of x, or an odd function of x; in the first case $u_x = u_{-x}$, in the second $u_x = -u_{-x}$. Both these cases present special advantages for the commencement of integration, for in the first integration we may take second differences into account. Thus when u_x is an even function, the second difference involved in the table of integration from 1 to 0 is $2\Delta u_0$; and when u_x is an odd function it is zero. In both cases third differences may be included in the second integration.

It is of course desirable to use the largest value of the increment of the independent variable consistent with adequate accuracy. If at any stage of the work it appears by the smallness of the second and third differences involved in the integrals, that longer steps may safely be employed, it is easy to double the value of Δx, by forming a new difference table with omission of alternate entries amongst the values already computed. Thus if the change is to be made at the stage where $x = n$, the new difference table will be formed from u_{n-4}, u_{n-2}, u_n; and thereafter Δx will have double its previous value.

When on the other hand it appears by the growth of the second and third differences that Δx is becoming too large, Δx can be halved, and the new difference table must be formed by interpolation. The formula (22) enables us to find $u_{n-\frac{1}{2}}$ from $u_n, u_{n-1}, u_{n-2}, \ldots$ with sufficient accuracy for the purpose of obtaining the differences of $u_{n-\frac{3}{2}}, u_{n-1}, u_{n-\frac{1}{2}}, u_n$. The process of halving the value of Δx is therefore similar to that of doubling it.

In some of the curves which I have to trace there are sharp bends or quasi-cusps, and in these cases the process is very tedious. It is sometimes necessary to repeatedly halve the increments of the independent variable, which is the arc s of the curve. Thus if (s) denotes the function of the arc to be integrated, and if s be the value of the arc at the point where the curvature begins to increase with great rapidity, and if δ be the previous increment of arc; then in integrating (s) from s to $s+\frac{1}{2}\delta$, the difference table is to be formed from $(s-\delta)$, $\left(s-\frac{1}{2}\delta\right)$, (s), the middle one of these three being an interpolated value. At the next step (s) has to be integrated from $s+\frac{1}{2}\delta$ to $s+\frac{3}{4}\delta$, and the difference table is formed from (s), $\left(s+\frac{1}{4}\delta\right)$, $\left(s+\frac{1}{2}\delta\right)$, the middle term being again an interpolation. This process may clearly be employed over and over again. In some of the curves traced the increment of arc has been 32 times less in one part than in another.

But the chief difficulty about these quasi-cusps arises when they are past, and when it is time to double the arc again. For the fact that the earlier values of the function to be used in the more open ranked difference tables are thrown back nearly to the cusp or even beyond it, makes the higher differences very large. Now the correctness of the formula of integration depends on the correctness of the hypothesis that an algebraic curve will give a good approximation to actuality. But in the neighbourhood of a quasi-cusp, and with increasing arcs this is far from correct. I have found then that in these cases of doubling the arc, a better result is obtained in the first and second integration by only including the second difference in the table of integration.

If we are tracing one member of a family of curves which are widely spaced throughout the greater part of their courses, but in one region are closely crowded into quasi-cusps, it is difficult to follow one member of the family through the crowded region, and on emerging from the region we shall probably find ourselves tracing a closely neighbouring member, and not the original one. I have applied the method to trace the curve drawn by a point attached to a circle at nine-tenths of its radius from the centre, as the circle rolls along a straight line. After the passage of the quasi-cusp I found that I was no longer exactly pursuing the correct line; nevertheless on a figure of the size of this page the difference between the two lines would be barely discernible. But the orbits which it is my object to trace do not quite resemble this case, since their cusps do not lie crowded together in one region of space. I believe therefore that these cases have been treated with substantial accuracy.

Another procedure has however been occasionally employed which I shall explain in § 7.

§ 6. *On the method of tracing a curve from its curvature.*

It will be supposed that the curve to be traced is symmetrical with respect to the x axis, and starts at right angles to it so that $x = x_0$, $y = 0$, $\varphi = 0$, $s = 0$. This is not a necessary condition for the use of the method, but it appears from § 5 that the start is thus rendered somewhat easier than would be the case otherwise. The curvature at each point of the curve is supposed to be a known function of the coordinates x, y of the point, and of the direction of the normal defined by the angle φ.

The first step is to compute the initial curvature $\frac{1}{R_0}$; it is then necessary to choose such a value for the increment of arc δs as will give the requisite degree of accuracy.

I have found that it is well to take, as a rule, δs of such a size that $\frac{\delta s}{R_0}$ shall not be greater than about 8°; but later, when all the differences in the tables of integration have come into use, I allow the increments of φ to increase to about 12°.

It is obvious that the curvature is even, when considered as a function of s. When nothing further is known of the nature of the curve, it is necessary to assume that the curvature is constant throughout the first arc δs, but it is often possible to make a conjecture that the curvature at the end of the arc δs will be say $\frac{1}{R_1}$. By the formula of integration with first and second differences we then compute $\varphi = \varphi_1$ at the end of the arc, by the first of equations (5) in § 2.

With this value of φ we find $\sin\varphi_1$, $\cos\varphi_1$, and observing that $\sin\varphi_0 = 0$, $\cos\varphi_0 = 1$, we compute x_1, y_1 by means of the second and third of (5), using first and second differences.

We next compute $\frac{1}{R_1}$ with these values of x, y, and if it agrees with the conjecture the work is done; and if not so, the work is repeated until there is agreement between the initial and final values of the curvature.

After the first arc, a second is computed, and higher differences are introduced into the tables of integration. We thus proceed by steps along the curve.

The approximation to the final result is usually so rapid, that in the recalculation it commonly suffices to note the changes in the last significant figure of the numbers involved in the original computation, without rewriting the whole.

The correction of the tables of integration is also very simple; for suppose that the first assumed value of the function to be integrated is u, and that the second approximation shows that it should have been $u + \delta u$; then all the differences in the column of the table have to be augmented by δu, and therefore the integral has to be augmented by

$$\left(1 - \frac{1}{2} - \frac{1}{12} - \frac{1}{24} - \ldots\right) \delta u \, \delta s.$$

If we stop with third differences, this gives the simple rule that the integral is to augmented by $\frac{3}{8} \delta u \, \delta s$.

It has been shown in § 5 how the chosen arc δs is to be increased or diminished according to the requirements of the case.

This method is the numerical counterpart of the graphical process described by Lord KELVIN in his Popular Lectures,[1] but it is very much more accurate, and when the formula for the curvature is complex it is hardly if at all more laborious. In the present investigation it would have been far more troublesome to use the graphical method, with such care as to attain the requisite accuracy, than to follow the numerical method.

In order to trace orbits I first computed auxiliary tables of $r^2 + \frac{2}{r}$, and of $\log\left(\frac{1}{r^2} - r\right)$ for $r < 1$, and of $\log\left(r - \frac{1}{r^2}\right)$ for $r > 1$; the tables

[1] Popular Lectures, vol. 1, 2nd ed. pp. 31—42; Phil. Mag. vol. 34, 1892, pp. 443—448.

extend from $r = 0$ to $1 \cdot 5$ at intervals of $\cdot 001$, but they will ultimately require further extension.

The following schedule shows the arrangement for the computation of the curvature at any point. The table has been arranged so as to be as compact as possible, and is not in strictly logical order; for the calculation of V^2 should follow that of r, ρ, but is entered at the foot of the first column. It will be observed that the calculation is in accordance with the formula (4) of § 2.

L denotes logarithm and C denotes cologarithm; ν the sun's mass is taken as 10, and $L\, 2n = \cdot 8217$, being $L\, 2\sqrt{11}$, a constant. The brackets indicate that the numbers so marked are to be added together.

Schedule for computation of curvature.

s

φ	$x - 1$
x	y
Ly	Ly
Cx	$C(x - 1)$
$L \tan \theta$	$L \tan \psi$
θ	ψ
$\varphi - \theta$	$\varphi - \psi$
$L \sec \theta$	$L \sec \psi$
Lx	$L(x - 1)$
Lr	$L\rho$
r	ρ
$L\left(\frac{1}{r^2} - r\right)$	$L\left(\frac{1}{\rho^2} - \rho\right)$
$L\nu \cos(\varphi - \theta)$	$L \cos(\varphi - \psi)$
$C V^2$	$C V^2$
La	Lb
a	$C V$
b	$L 2n$
$\nu\left(r^2 + \frac{2}{r}\right)$	$L \frac{2n}{V}$
$\rho^2 + \frac{2}{\rho}$	$-\frac{2n}{V}$
$V^2 + C$	$a + b$
V^2	$\frac{1}{R}$

The formulæ $r = y \operatorname{cosec} \theta$, $\rho = y \operatorname{cosec} \psi$ are used, when the values of θ or ψ show that these are the better forms.

The tables of integration are kept on separate sheets in the forms indicated in § 5.

As the computation proceeds I keep tables of differences of x, y, φ, r, ρ, V^2, and this check has been of immense advantage in detecting errors.

The auxiliary tables of logarithms are computed to 5 figures, but the last figure is not always correct to unity, and the fifth figure is principally of use in order to make correct interpolation possible.

The conversion of φ from circular measure to degrees and the values of $\sin \varphi$ and $\cos \varphi$ are obtained from Bottomley's four-figured table.

Most of the work has been done with these tables, but as it appears that the principal source of error lies in the determination of r and ρ, five-figured logarithms have generally been used in this part of the work, and the values of θ and ψ are written down to $0'{\cdot}1$.

In those parts of an orbit in which V^2 becomes small I have often ceased to use the auxiliary table for $\nu\left(r^2 + \frac{2}{r}\right)$; for since the auxiliary table of this function only contains four decimal places and since ν is 10, it follows that only three places are obtainable from the table, and of course there may be an error of unity or even of 2 in the last significant figure of V^2.

In order to test the method, I computed an unperturbed elliptic orbit by means of the curvature. The formulæ were $V^2 = \frac{2}{r} - \frac{1}{10}$, $\frac{1}{R} = \frac{P}{V^2}$, where $P = \frac{1}{r^2}\cos(\varphi - \theta)$, and the initial values were $x_0 = 5$, $y_0 = 0$, $\varphi_0 = 0$, $s_0 = 0$.

The curve described should be the ellipse of semiaxes 10 and $5\sqrt{3}$, and x, y ought to satisfy the equation

$$\left(\frac{x+5}{10}\right)^2 + \left(\frac{y}{5\sqrt{3}}\right)^2 = 1.$$

I take the square root of the left hand side of this equation, with computed x, y, as one measure of the error of position in the ellipse.

Again if $\tan\chi = \dfrac{\frac{4}{3}y}{x+5}$, χ ought to be identical with φ; hence $\chi - \varphi$ measures the error of the direction of motion.

Lastly the area conserved h is $5\sqrt{\frac{3}{10}}$ or $2{\cdot}7386$; but it is also $Vr\cos(\varphi-\theta)$, if the computation gives perfect results. Hence $h - Vr\cos(\varphi-\theta)$ measures the error in the equable description of areas. The semi-period should be $\pi\sqrt{1000}$ or $99{\cdot}346$.

The computations were made partly with five-figured and partly with four-figured logarithms, and the process followed the lines of my other work very closely.

The following table exhibits the results together with the errors. It will be observed that when $s = 24$ there is a sudden increase in the second column of errors, but I have not been able to detect the arithmetical mistake which is probably responsible for it. The accordance still remains so close, that it appeared to be a waste of time to work any longer at this example.

Computed positions in an ellipse described under the action of a central force.

s	x	y	φ	$\chi-\varphi$	$\left[\left(\frac{x+5}{10}\right)^2+\left(\frac{y}{5\sqrt{3}}\right)^2\right]^{\frac{1}{2}}-1$	$h-Vr\cos(\varphi-\theta)$
0	5·0000	·0000	0° 0′	0′·0	+·00000	·0000
1	4·9337	·9971	7° 37′	+0·3	+·00002	·0000
2	4·7364	1·9768	15° 8′	+0·8	+·00005	—·0001
3	4·4137	2·9227	22° 29′	+0·3	+·00004	—·0001
4	3·9749	3·8205	29° 35′	—0·3	+·00004	—·0002
5	3·4304	4·6586	36° 23′	0·0	+·00004	—·0001
6	2·7925	5·4281	42° 53′	+0·1	+·00004	—·0001
8	1·2843	6·7363	55° 1′	+0·2	—·00002	+·0001
10	— ·4567	7·7147	66° 9′	+1·0	—·00001	+·0002
12	— 2·3497	8·3507	76° 36′	+0·6	·00000	+·0003
14	— 4·3259	8·6407	86° 39′	+0·1	—·00001	·0000
16	— 6·3225	8·5845	96° 35′	+0·4	+·00003	·0000
18	— 8·2787	8.1823	106° 43′	+0·6	+·00010	+·0003
20	—10·1305	7·4349	117° 21′	+0·8	+·00012	+·0003
22	—11·8051	6·3481	128° 47′	+1·0	+·00001	+·0004
24	—13·2181	4·9385	141° 17′	+0·8	+·00028	+·0004
25	—13·7968	4·1237	148° 0′	—0·4	+·00027	+·0003
26	—14·2740	3·2456	155° 0′	—0·8	+·00027	+·0001
27	—14·6385	2·3151	162° 15′	—0·5	+·00023	+·0003
28	—14·8808	1·3456	169° 43′	—0·5	+·00021	+·0003
29	—14·9938	·3526	177° 19′	—0·6	+·00020	+·0002
30	—14·9740	— ·6465	184° 57′	—0·6	+·00019	+·0004
29·3546	—15·0020	·0000	180° 1′	+1·0		

The last line in the above table was found by interpolation.

The computed values of the semiaxes of the ellipse (both involving interpolations) were found to be 10·0010 and ·86604; their correct values are 10·0000 and ·866026. The computed semiperiod (requiring another integration and interpolation) was found to be 99·346, agreeing with the correct value to the last place of decimals.

Considering that a considerable part of the computation was done with four-figured tables, the accuracy shown in this table is surprising.

This calculation is exactly comparable with the best of my calculations of orbits, but there has been from time to time a good deal of variety in my procedure. My object has been throughout to cover a wide field with adequate accuracy rather than a far smaller one with scrupulous exactness, for economy of labour is of the greatest importance in so heavy a piece of work. I shall in the appendix generally indicate which are the more exact and which the less exact computations. I do not think it would in any case have been possible in the figures to show the difference between an exactly computed and a roughly computed curve, because the lines would be almost or quite indistinguishable on the scale of the plates of figures. This however might not be quite true of the orbits which have very sharp bends in them.

§ 7. *Development in powers of the time; the form of cusps.*

In a few cases the quasi-cusps of orbits have been computed by means of series; the mode of development will therefore now be considered.

If for brevity we write

$$2n = m, \qquad \frac{dx}{dt} = u, \qquad \frac{dy}{dt} = v,$$

the equations of motion (1) become

$$\frac{du}{dt} = mv + \frac{\partial \Omega}{\partial x}, \qquad \frac{dv}{dt} = -mu + \frac{\partial \Omega}{\partial y}. \tag{26}$$

Now let

$$D_i = \frac{d^i u}{dt^i}\frac{\partial}{\partial x} + \frac{d^i v}{dt^i}\frac{\partial}{\partial y}, \quad \text{where } i \text{ is } 0, 1, 2, 3 \ldots.$$

Then total differentiation of a function of x, y, t or of x, y, u, v is expressed in terms of partial differentials as follows:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + D_0.$$

It is obvious that $\frac{\partial}{\partial t} D_i = D_{i+1}$, and $\frac{d}{dt}$ performed on a function of x, y, but not of u, v, is simply D_0.

If we differentiate (26) repeatedly with respect to the time, we have

$$(27) \qquad \frac{d^{i+1}u}{dt^{i+1}} = m\frac{d^i v}{dt^i} + \left(\frac{d}{dt}\right)^i \frac{\partial \Omega}{\partial x}, \qquad \frac{d^{i+1}v}{dt^{i+1}} = -m\frac{d^i u}{dt^i} + \left(\frac{d}{dt}\right)^i \frac{\partial \Omega}{\partial y}.$$

Now $\frac{\partial \Omega}{\partial x}$ and $\frac{\partial \Omega}{\partial y}$ are functions of x, y only, and not also of u, v; therefore in the last terms of these equations,

$$(27) \qquad \left\{ \begin{array}{ll} \text{when } i = 1, & \frac{d}{dt} = D_0, \\ \text{when } i = 2, & \left(\frac{d}{dt}\right)^2 = D_1 + D_0^2, \\ \text{when } i = 3, & \left(\frac{d}{dt}\right)^3 = D_2 + 3D_0 D_1 + D_0^3, \\ \text{when } i = 4, & \left(\frac{d}{dt}\right)^4 = D_3 + 4D_0 D_2 + 3D_1^2 + 6D_0 D_1 + D_0^4, \\ & \qquad\qquad \text{and so forth.} \end{array} \right.$$

The function Ω consists of two parts, one being a function of r, the other of ρ; if in the latter part we write $\xi = (x - 1)$, $\eta = y$,

$$\Omega = \frac{1}{2}\nu(x^2 + y^2) + \frac{1}{2}(\xi^2 + \eta^2) + \frac{\nu}{r} + \frac{1}{\rho}.$$

The partial differentials of Ω with respect to x, y may be regarded also as consisting of two parts viz. of the partial differentials with respect to

x, y of $\frac{1}{2}\nu(x^2+y^2)+\frac{\nu}{r}$, and of the partial differentials with respect to ξ, η of $\frac{1}{2}(\xi^2+\eta^2)+\frac{1}{\rho}$. These two parts may be considered separately, since, except as regard the factor ν, the one is the exact counterpart of the other.

The partial differentials of $\frac{1}{2}\nu(x^2+y^2)$ disappear after the first two orders, and those of $\frac{\nu}{r}$ are exactly those functions which occur in the theory of spherical harmonic analysis.

Thus

$$\frac{\partial}{\partial x}\frac{1}{r} = -\frac{1}{r^2}\cos\theta, \qquad \frac{\partial}{\partial y}\frac{1}{r} = -\frac{1}{r^2}\sin\theta;$$

$$\frac{\partial^2}{\partial x^2}\frac{1}{r} = \frac{1}{r^3}(3\cos^2\theta - 1), \qquad \frac{\partial^2}{\partial x\partial y}\frac{1}{r} = \frac{3}{r^3}\sin\theta\cos\theta,$$

$$\frac{\partial^2}{\partial y^2}\frac{1}{r} = \frac{1}{r^3}(3\sin^2\theta - 1);$$

$$\frac{\partial^3}{\partial x^3}\frac{1}{r} = \frac{3}{r^5}(3\cos\theta - 5\cos^3\theta), \qquad \frac{\partial^3}{\partial x^2\partial y}\frac{1}{r} = \frac{3}{r^5}(\sin\theta - 5\sin\theta\cos^2\theta),$$

$$\frac{\partial^3}{\partial x\partial y^2}\frac{1}{r} = \frac{3}{r^5}(\cos\theta - 5\cos\theta\sin^2\theta), \qquad \frac{\partial^3}{\partial y^3}\frac{1}{r} = \frac{3}{r^5}(3\sin\theta - 5\sin^3\theta);$$

and so forth.

It thus appears that the calculation of the successive differentials of u, v with regard to the time is easy, although laborious. These differentials, when appropriately divided by the factorials of 1, 2, 3, 4 etc., are the successive coefficients of the powers of the time in the developments of x, y. If the series for x, y be differentiated, we obtain those for u, v.

The Jacobian integral is useful as a control to the applicability of the series; for the square of the velocity corresponding to any position computed from the series for x and y should agree with the value of u^2+v^2 as computed from the series for u und v.

The computation of an orbit by series is however so tedious, that I have made very little use of this method.

I have also obtained a less extended development for x, y in terms of powers of the arc of the orbit, but the formulæ are so cumbrous as to be of little service.

The development in powers of the time becomes much less laborious if we start from a point in the curve of zero velocity, and in this case the symbols D_i may be replaced by their full expressions in terms of the partial differentials of Ω. But is does not seem worth while to give these special forms, except as regard the first two terms.

If we have initially $x = x_0$, $y = y_0$, $u = 0$, $v = 0$, D_0 and all its powers vanish, and

$$\frac{du}{dt} = \frac{\partial \Omega}{\partial x}, \qquad \frac{dv}{dt} = \frac{\partial \Omega}{\partial y},$$

$$\frac{d^2u}{dt^2} = m\frac{\partial \Omega}{\partial y}, \qquad \frac{d^2v}{dt^2} = -m\frac{\partial \Omega}{\partial x}.$$

Hence as far as the cube of the time,

$$x - x_0 = \frac{1}{2}t^2\frac{\partial \Omega}{\partial x} + \frac{1}{6}t^3 m\frac{\partial \Omega}{\partial y},$$

$$y - y_0 = \frac{1}{2}t^2\frac{\partial \Omega}{\partial y} - \frac{1}{6}t^3 m\frac{\partial \Omega}{\partial x}.$$

These may be written

$$(x - x_0)\frac{\partial \Omega}{\partial y} - (y - y_0)\frac{\partial \Omega}{\partial x} = \frac{1}{6}t^3 . mT^2,$$

$$(x - x_0)\frac{\partial \Omega}{\partial x} + (y - y_0)\frac{\partial \Omega}{\partial y} = \frac{1}{2}t^2 . T^2,$$

where $T^2 = \left(\frac{\partial \Omega}{\partial x}\right)^2 + \left(\frac{\partial \Omega}{\partial y}\right)^2$.

By elimination of t, and substitution of $2n$ for m, we obtain the equation to the cusp,

$$8n^2\left[(x - x_0)\frac{\partial \Omega}{\partial x} + (y - y_0)\frac{\partial \Omega}{\partial y}\right]^3 = 9T^2\left[(x - x_0)\frac{\partial \Omega}{\partial y} - (y - y_0)\frac{\partial \Omega}{\partial x}\right]^2.$$

The cusp is therefore a semicubical parabola, with the tangent at the cusp normal to the curve $2\Omega = C$.

§ 8. *Variation of orbit.*

The object of this paper is not only to discover periodic orbits but also to consider their stability.

Now the stability of a periodic orbit is determinable by discovering whether the motion is oscillatory or not, when the path varies by infinitely little from that of the periodic orbit. The variation of an orbit may be of two kinds, for the constant of relative energy may be varied, or the planet may be displaced from the periodic orbit.

Suppose that the constant C undergoes a small variation and becomes $C + \delta C$; then there must be a periodic orbit, corresponding to $C + \delta C$, which differs by very little from that corresponding to C.

Now if a planet be moving in a periodic orbit, and if C suddenly becomes $C + \delta C$, we may henceforth refer the motion to the varied periodic orbit, and may consider the constant of relative energy as $C + \delta C$ and invariable. The periodic orbit of reference then varies *per saltum,* but the instantaneous position of the planet is unvaried, and therefore the planet is now displaced from its orbit of reference. Hence the result of a variation of C will virtually be determined by regarding C as constant, and by supposing the planet to be displaced from the periodic orbit. This subject is considered in the present section.

The whole of the following investigation is founded on the work of M[r] Hill,[1] but it is presented in a different form.

If the Jacobian integral (2) be differentiated with respect to the time, and if the equations $\frac{dx}{dt} = - V \sin\varphi$, $\frac{dy}{dt} = V \cos\varphi$ be used in the result, we obtain

$$\frac{dV}{dt} = - \sin\varphi \frac{\partial \Omega}{\partial x} + \cos\varphi \frac{\partial \Omega}{\partial y}. \tag{28}$$

Again if the first of the equations of motion (1) be multiplied by

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[1] *On the part of the motion of the moon's perigee* etc. Acta Mathem. Vol. 8, pp. 1—36.

$-\cos\varphi$, and the second by $-\sin\varphi$, and if the two be added together, the result may be written

$$\cos\varphi\frac{d}{dt}(V\sin\varphi)-\sin\varphi\frac{d}{dt}(V\cos\varphi)+2nV=-\cos\varphi\frac{\partial\Omega}{\partial x}-\sin\varphi\frac{\partial\Omega}{\partial y}.$$

Completing the differentiations on the left-hand side, we have

$$(29)\qquad V\left(\frac{d\varphi}{dt}+2n\right)=-\cos\varphi\frac{\partial\Omega}{\partial x}-\sin\varphi\frac{\partial\Omega}{\partial y}.$$

Let s be the arc of the orbit, and p the arc of an orthogonal trajectory of the orbit, estimated in the direction of the outward normal of the orbit; then

$$(30)\qquad \left\{\begin{aligned}\frac{\partial}{\partial s}&=-\sin\varphi\frac{\partial}{\partial x}+\cos\varphi\frac{\partial}{\partial y},\\ \frac{\partial}{\partial p}&=\quad\cos\varphi\frac{\partial}{\partial x}+\sin\varphi\frac{\partial}{\partial y}.\end{aligned}\right.$$

Accordingly (28), (29) and the Jacobian integral become

$$(30)\qquad \left\{\begin{aligned}\frac{dV}{dt}&=\frac{\partial\Omega}{\partial s},\\ V\left(\frac{d\varphi}{dt}+2n\right)&=-\frac{\partial\Omega}{\partial p},\\ V^2&=2\Omega-C.\end{aligned}\right.$$

The equations (30) are equivalent to (1) and (2).

Now suppose that x, y are the coordinates of a point on an orbit, and that $x+\delta x$, $y+\delta y$ are the coordinates of a point on an adjacent orbit. Then if we put

$$\delta p=\quad\delta x\cos\varphi+\delta y\sin\varphi,$$

$$\delta s=-\delta x\sin\varphi+\delta y\cos\varphi,$$

δp, δs are the distances, measured along the outward normal and along the arc of the unvaried orbit, from the original point x, y to the adjacent point $x+\delta x$, $y+\delta y$.

If, with x, y as origin, rectangular axes be drawn along the outward normal and along the arc of the unvaried orbit, we may regard δp, δs

as the coordinates of the new point relatively to the old one. The new axes rotate with angular velocity $\frac{d\varphi}{dt} + n$, the first term representing the angular velocity of the normal and the second that of our original axes of x and y.

The well-known formulæ for the component accelerations of a point along two directions, which instantaneously coincide with a pair of rotating rectangular axes by reference to which the position of the point is determined, give the accelerations

$$(31)\quad \begin{cases} \dfrac{d^2}{dt^2}\delta p - \delta p\left(\dfrac{d\varphi}{dt} + n\right)^2 - 2\dfrac{d\delta s}{dt}\left(\dfrac{d\varphi}{dt} + n\right) - \delta s\dfrac{d^2\varphi}{dt^2}; \text{ along the normal} \\[2ex] \dfrac{d^2}{dt^2}\delta s - \delta s\left(\dfrac{d\varphi}{dt} + n\right)^2 + 2\dfrac{d\delta p}{dt}\left(\dfrac{d\varphi}{dt} + n\right) + \delta p\dfrac{d^2\varphi}{dt^2}, \text{ along the tangent.} \end{cases}$$

These are the accelerations of the new point relatively to the old, estimated along lines fixed in space which coincide instantaneously with the normal and tangent of the unvaried orbit.

The function Ω includes the potential of the rotation n of the original axes of x and y. Hence $\Omega - \frac{1}{2}n^2r^2$ is the true potential of the forces under which the body moves in the unvaried orbit, and

$$\frac{\partial}{\partial p}\left(\Omega - \frac{1}{2}n^2r^2\right), \qquad \frac{\partial}{\partial s}\left(\Omega - \frac{1}{2}n^2r^2\right)$$

are the components of force in the unvaried orbit along the normal and along the arc.

Therefore the excess of the forces in the varied orbit above those in the unvaried orbit are

$$\left(\delta p\frac{\partial^2}{\partial p^2} + \delta s\frac{\partial^2}{\partial p\,\partial s}\right)\left(\Omega - \frac{1}{2}n^2r^2\right) \text{ and } \left(\delta p\frac{\partial^2}{\partial p\,\partial s} + \delta s\frac{\partial^2}{\partial s^2}\right)\left(\Omega - \frac{1}{2}n^2r^2\right).$$

Now by considering the meaning (30) of the operations $\frac{\partial}{\partial p}$, $\frac{\partial}{\partial s}$, it is easy to prove that

$$\frac{1}{2}\frac{\partial^2 r^2}{\partial p^2} = \frac{1}{2}\frac{\partial^2 r^2}{\partial s^2} = 1, \qquad \frac{1}{2}\frac{\partial^2}{\partial p\,\partial s}r^2 = 0.$$

Hence the excess of the forces in the varied orbit above those in the unvaried orbit are

$$\delta p \frac{\partial^2 \Omega}{\partial p^2} + \delta s \frac{\partial^2 \Omega}{\partial p \partial s} - n^2 \delta p, \quad \text{and} \quad \delta p \frac{\partial^2 \Omega}{\partial p \partial s} + \delta s \frac{\partial^2 \Omega}{\partial s^2} - n^2 \delta s,$$

along the normal and along the arc of the unvaried orbit.

But these are necessarily equal to the accelerations (31) of which they are the cause. Then transferring $-n^2 \delta p$, $-n^2 \delta s$ to the left hand sides of the equations, we have

$$(32) \quad \begin{cases} \dfrac{d^2}{dt^2}\delta p + \delta p\left[n^2 - \left(\dfrac{d\varphi}{dt} + n\right)^2\right] - 2\dfrac{d\delta s}{dt}\left(\dfrac{d\varphi}{dt} + n\right) - \delta s \dfrac{d^2\varphi}{dt^2} \\ \qquad\qquad = \delta p \dfrac{\partial^2 \Omega}{\partial p^2} + \delta s \dfrac{\partial^2 \Omega}{\partial p \partial s}, \\ \dfrac{d^2}{dt^2}\delta s + \delta s\left[n^2 - \left(\dfrac{d\varphi}{dt} + n\right)^2\right] + 2\dfrac{d\delta p}{dt}\left(\dfrac{d\varphi}{dt} + n\right) + \delta p \dfrac{d^2\varphi}{dt^2} \\ \qquad\qquad = \delta p \dfrac{\partial^2 \Omega}{\partial p \partial s} + \delta s \dfrac{\partial^2 \Omega}{\partial s^2}. \end{cases}$$

These are the equations of motion in the varied orbit.

The variation of the last of (30), the Jacobian integral, gives

$$(33) \qquad V\delta V = \delta p \frac{\partial \Omega}{\partial p} + \delta s \frac{\partial \Omega}{\partial s}.$$

Now δV is the tangential velocity of the point $x + \delta x$, $y + \delta y$ in the varied orbit, relatively to the original point x, y. But as we only want to consider a velocity relatively to the axes of x and y, which themselves rotate with angular velocity n, our p, s axes must be regarded as rotating with angular velocity $\frac{d\varphi}{dt}$, instead of $\frac{d\varphi}{dt} + n$.

Accordingly

$$(34) \qquad \delta V = \frac{d}{dt}\delta s + \delta p \frac{d\varphi}{dt}.$$

This may also be proved by putting $V\delta V = \frac{dx}{dt}\frac{d\delta x}{dt} + \frac{dy}{dt}\frac{d\delta y}{dt}$, and by substituting for the differentials in terms of δp, δs, V, φ.

The formula (34) enables us to get rid of δV in (33) but we may also get rid of $\frac{\partial\Omega}{\partial p}$ and $\frac{\partial\Omega}{\partial s}$ by means of the equations of motion (30). Thus the variation of the Jacobian integral leads to

$$V\left(\frac{d}{dt}\delta s + \delta p\frac{d\varphi}{dt}\right) = -V\left(\frac{d\varphi}{dt} + 2n\right)\delta p + \frac{dV}{dt}\delta s.$$

Therefore

$$(35)\qquad \begin{cases} \dfrac{d}{dt}\delta s + 2\delta p\left(\dfrac{d\varphi}{dt} + n\right) - \dfrac{1}{V}\dfrac{dV}{dt}\delta s = 0, \\ \text{or} \\ V\dfrac{d}{dt}\left(\dfrac{\delta s}{V}\right) + 2\delta p\left(\dfrac{d\varphi}{dt} + n\right) = 0. \end{cases}$$

The equations (35) are two forms of the varied Jacobian integral.

A great simplication of the equations of motion (32) is possible by reference to the unvaried motion.

Let us suppose then that $\delta p, \delta s$ are no longer displacements to a varied orbit, but are the actual displacements occurring in time δt in the unvaried orbit. Thus $\delta p = 0$, $\delta s = V\delta t$.

The equations (32) then give

$$(36)\qquad \begin{cases} -2\dfrac{dV}{dt}\left(\dfrac{d\varphi}{dt} + n\right) - V\dfrac{d^2\varphi}{dt^2} = V\dfrac{\partial^2\Omega}{\partial p\partial s}, \\ \dfrac{d^2V}{dt^2} + V\left[n^2 - \left(\dfrac{d\varphi}{dt} + n\right)^2\right] = V\dfrac{\partial^2\Omega}{\partial s^2}. \end{cases}$$

The first of (36) may be written

$$\frac{d^2\varphi}{dt^2} + \frac{\partial^2\Omega}{\partial p\partial s} = -\frac{2}{V}\frac{dV}{dt}\left(\frac{d\varphi}{dt} + n\right).$$

These two terms, multiplied by δs, occur in the first of (32), which may therefore be written

$$\frac{d^2\delta p}{dt^2} + \delta p\left[n^2 - \left(\frac{d\varphi}{dt} + n\right)^2\right] - 2\frac{d\delta s}{dt}\left(\frac{d\varphi}{dt} + n\right) + \frac{2\delta s}{V}\frac{dV}{dt}\left(\frac{d\varphi}{dt} + n\right) - \delta p\frac{\partial^2\Omega}{\partial p^2} = 0.$$

The terms in this which involve δs may now be eliminated by the first of (35), and we have

$$\frac{d^2\delta p}{dt^2} + \delta p\left[n^2 - \left(\frac{d\varphi}{dt} + n\right)^2 + 4\left(\frac{d\varphi}{dt} + n\right)^2 - \frac{\partial^2 \Omega}{\partial p^2}\right] = 0.$$

If then we put

$$(37) \qquad \theta = n^2 + 3\left(\frac{d\varphi}{dt} + n\right)^2 - \frac{\partial^2 \Omega}{\partial p^2},$$

we have

$$(37) \qquad \left\{\begin{aligned} &\frac{d^2\delta p}{dt^2} + \theta\delta p = 0, \\ &\frac{d}{dt}\left(\frac{\delta s}{V}\right) + 2\frac{\delta p}{V}\left(\frac{d\varphi}{dt} + n\right) = 0. \end{aligned}\right.$$

The differential equation for δp is Mr Hill's well-known result.

We have now to consider the form of the function θ.

Let us write $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial p^2} + \frac{\partial^2}{\partial s^2}$; then adding $V\frac{\partial^2 \Omega}{\partial p^2}$ to each side of the second of (36), we have

$$\frac{1}{V}\frac{d^2 V}{dt^2} + n^2 - \left(\frac{d\varphi}{dt} + n\right)^2 + \frac{\partial^2 \Omega}{\partial p^2} = \nabla^2 \Omega,$$

so that

$$n^2 - \frac{\partial^2 \Omega}{\partial p^2} = \frac{d}{dt}\left(\frac{dV}{Vdt}\right) + \left(\frac{dV}{Vdt}\right)^2 - \left(\frac{d\varphi}{dt} + n\right)^2 + 2n^2 - \nabla^2 \Omega.$$

Substituting in (37),

$$\theta = 2n^2 - \nabla^2 \Omega + 2\left(\frac{d\varphi}{dt} + n\right)^2 + \frac{d}{dt}\left(\frac{dV}{Vdt}\right) + \left(\frac{dV}{Vdt}\right)^2.$$

If we put $u = x + y\iota$, $s = x - y\iota$, $\frac{d}{dt} = \iota D$, where $\iota = \sqrt{-1}$, it is easy to show that $Du = Ve^{\varphi\iota}$, $Ds = -Ve^{-\varphi\iota}$, and

$$2\frac{d\varphi}{dt} = \frac{D^2 u}{Du} - \frac{D^2 s}{Ds}, \qquad 2\frac{dV}{Vdt} = \iota\left(\frac{D^2 u}{Du} + \frac{D^2 s}{Ds}\right).$$

Mr Hill's form for the function θ follows as once from these transformations.

Another form for θ, deducible directly from (37), is

$$\theta = n^2 - \frac{1}{2}\nabla^2\Omega - \frac{1}{2}\left(\frac{\partial^2\Omega}{\partial x^2} - \frac{\partial^2\Omega}{\partial y^2}\right)\cos 2\varphi - \frac{1}{2}\frac{\partial^2\Omega}{\partial x\partial y}\sin 2\varphi + 3\left(\frac{d\varphi}{dt} + n\right)^2,$$

whence

$$\theta = \frac{\nu}{r^3} + \frac{1}{\rho^3} - \frac{3\nu}{r^3}\cos^2(\varphi - \theta) - \frac{3}{\rho^3}\cos^2(\varphi - \psi) + 3V^2\left(\frac{1}{R} + \frac{n}{V}\right)^2.$$

§ 9. *Change of independent variable from time to arc of orbit.*

For the purpose of future developments it is now necessary to change the independent variable from the time t to the arc s.

Let

$$\delta q = \delta p V^{\frac{1}{2}}. \tag{38}$$

Then

$$\frac{d^2\delta p}{dt^2} = V\frac{d}{ds}\left(V\frac{d}{ds}\left(\frac{\delta q}{V^{\frac{1}{2}}}\right)\right) = V\frac{d}{ds}\left(V^{\frac{1}{2}}\frac{d\delta q}{ds} - \frac{1}{2V^{\frac{1}{2}}}\delta q\frac{dV}{ds}\right),$$

$$= V^{\frac{3}{2}}\frac{d^2\delta q}{ds^2} - \frac{1}{2}\delta q V\frac{d}{ds}\left(\frac{1}{V^{\frac{1}{2}}}\frac{dV}{ds}\right).$$

But

$$V\frac{d}{ds}\left(\frac{1}{V^{\frac{1}{2}}}\frac{dV}{ds}\right) = \frac{d}{dt}\left(\frac{1}{V^{\frac{3}{2}}}\frac{dV}{dt}\right) = -\frac{3}{2V^{\frac{5}{2}}}\left(\frac{dV}{dt}\right)^2 + \frac{1}{V^{\frac{3}{2}}}\frac{d^2V}{dt^2},$$

$$= -\frac{3}{2V^{\frac{1}{2}}}\left(\frac{dV}{ds}\right)^2 + \frac{1}{V^{\frac{3}{2}}}\frac{d^2V}{dt^2}.$$

Hence

$$\frac{d^2\delta p}{dt^2} = V^{\frac{3}{2}}\frac{d^2\delta q}{ds^2} + \frac{3}{4V^{\frac{1}{2}}}\left(\frac{dV}{ds}\right)^2\delta q - \frac{\delta q}{2V^{\frac{3}{2}}}\frac{d^2V}{dt^2}.$$

Also

$$\theta\delta p = \frac{\theta\delta q}{V^{\frac{1}{2}}}.$$

If these two be added together, and divided by $V^{\frac{3}{2}}$, we obtain

$$(39)\quad \begin{cases} \dfrac{d^2\delta q}{ds^2} + \Psi\delta q = 0, \\ \text{where} \\ \Psi = \dfrac{\theta}{V^2} + \dfrac{3}{4}\left(\dfrac{dV}{Vds}\right)^2 - \dfrac{1}{2V^3}\dfrac{d^2V}{dt^2}. \end{cases}$$

It remains to obtain the expression for the function Ψ

Since

$$\frac{d\varphi}{ds} = \frac{1}{R}, \quad \text{and} \quad n^2 = \nu + 1,$$

$$\theta = \nu + 1 + 3\left(\frac{V}{R} + n\right)^2 - \frac{\partial^2\Omega}{\partial p^2}.$$

Now from the first of (30) and the second of (36),

$$V\frac{dV}{ds} = \frac{\partial\Omega}{\partial s},$$

$$\frac{1}{V}\frac{d^2V}{dt^2} = \left(\frac{V}{R} + n\right)^2 + \frac{\partial^2\Omega}{\partial s^2} - \nu - 1.$$

Then by substitution in the second of (39),

$$\Psi V^2 = \frac{3}{2}(\nu + 1) + \frac{5}{2}\left(\frac{V}{R} + n\right)^2 + \frac{3}{4}\left(\frac{dV}{ds}\right)^2 - \frac{\partial^2\Omega}{\partial p^2} - \frac{1}{2}\frac{\partial^2\Omega}{\partial s^2}.$$

Also

$$\frac{\partial^2\Omega}{\partial p^2} + \frac{1}{2}\frac{\partial^2\Omega}{\partial s^2} = \frac{1}{2}\nabla^2\Omega + \frac{1}{2}\frac{\partial^2\Omega}{\partial p^2}.$$

Now $2\Omega = \nu\left(r^2 + \frac{2}{r}\right) + \left(\rho^2 + \frac{2}{\rho}\right)$, and

$$\frac{\partial^2\Omega}{\partial x^2} = \nu + 1 - \frac{\nu}{r^3} - \frac{1}{\rho^3} + \frac{3\nu}{r^3}\cos^2\theta + \frac{3}{\rho^3}\cos^2\psi,$$

$$\frac{\partial^2\Omega}{\partial x\,\partial y} = \frac{3\nu}{r^3}\sin\theta\cos\theta + \frac{3}{\rho^3}\sin\psi\cos\psi,$$

$$\frac{\partial^2\Omega}{\partial y^2} = \nu + 1 - \frac{\nu}{r^3} - \frac{1}{\rho^3} + \frac{3\nu}{r^3}\sin^2\theta + \frac{3}{\rho^3}\sin^2\psi.$$

Hence

$$\nabla^2 \Omega = 2(\nu + 1) + \frac{\nu}{r^3} + \frac{1}{\rho^3},$$

and

$$\frac{\partial^2 \Omega}{\partial p^2} = \cos^2\varphi \frac{\partial^2 \Omega}{\partial x^2} + 2 \sin\varphi \cos\varphi \frac{\partial^2 \Omega}{\partial x \partial y} + \sin^2\varphi \frac{\partial^2 \Omega}{\partial y^2},$$

$$= \nu + 1 - \frac{\nu}{r^3} - \frac{1}{\rho^3} + \frac{3\nu}{r^3}\cos^2(\varphi - \theta) + \frac{3}{\rho^3}\cos^2(\varphi - \psi).$$

Therefore

$$(40) \quad \Psi = \frac{5}{2}\left(\frac{1}{R} + \frac{n}{V}\right)^2 - \frac{3}{2V^2}\left[\frac{\nu}{r^3}\cos^2(\varphi - \theta) + \frac{1}{\rho^3}\cos^2(\varphi - \psi)\right] + \frac{3}{4}\left(\frac{dV}{Vds}\right)^2.$$

Also since

$$V\frac{dV}{ds} = \frac{\partial \Omega}{\partial s} = -\sin\varphi \frac{\partial \Omega}{\partial x} + \cos\varphi \frac{\partial \Omega}{\partial y},$$

$$(40) \qquad \frac{dV}{Vds} = \frac{\nu}{V^2}\left(\frac{1}{r^2} - r\right)\sin(\varphi - \theta) + \frac{1}{V^2}\left(\frac{1}{\rho^2} - \rho\right)\sin(\varphi - \psi).$$

This completes the formula for Ψ in terms of the coordinates, the velocity, the curvature and of φ.

It may be useful to obtain the expressions for δs and $\delta\varphi$ in terms of the new independent variable s.

The second of (37) may be written down at once, namely

$$(41) \qquad \frac{d}{ds}\left(\frac{\delta s}{V}\right) = -\frac{2\delta q}{V^{\frac{3}{2}}}\left(\frac{1}{R} + \frac{n}{V}\right).$$

Also it is clear from geometrical considerations that

$$\delta\varphi = -\frac{d}{ds}\delta p + \frac{\delta s}{R},$$

whence

$$(42) \qquad \delta\varphi = -\frac{1}{V^{\frac{1}{2}}}\left[\frac{d\delta q}{ds} - \frac{1}{2}\delta q\left(\frac{dV}{Vds}\right)\right] + \frac{\delta s}{R}.$$

§ 10. *The solution of the differential equation for δq.*

The function Ψ has a definite value at each point of a periodic orbit whose complete arc is S. Therefore Ψ is a function of the arc s of the orbit, measured from any point therein, and when s has increased from zero to S, Ψ has returned to its initial value. Also since a periodic orbit is symmetrical with respect to the x-axis, Ψ is an even function of the arc s, when s is measured from an orthogonal intersection of the orbit with the x-axis. If the periodic orbit only goes once round S or J, or round both, all the intersections with the x-axis are necessarily orthogonal. I call such an orbit simply periodic, but the term must have its meaning extended so as to embrace the possibility of loops. But when there are loops all the intersections with the x-axis are not necessarily orthogonal, and if the orbit is only periodic after several revolutions some of the intersections cannot be orthogonal.

With the understanding that s is measured from an orthogonal intersection with the x-axis, Ψ is an even function of s and is expressible by the Fourier series

$$\Psi = \Psi_0 + 2\Psi_1 \cos\frac{2\pi s}{S} + 2\Psi_2 \cos\frac{4\pi s}{S} + \dots.$$

Now multiply the differential equation (39) for δq by $\frac{S^2}{\pi^2}$, write σ for $\frac{\pi s}{S}$, and put $\Phi = \frac{S^2}{\pi^2}\Psi$, and we have

$$\frac{d^2}{d\sigma^2}\delta q + \Phi\delta q = 0. \tag{43}$$

Also if $\Phi_j = \frac{S^2}{\pi^2}\Psi_j$,

$$\Phi = \Phi_0 + 2\Phi_1 \cos 2\sigma + 2\Phi_2 \cos 4\sigma + \dots.$$

If then we write $\zeta = e^{\sigma\sqrt{-1}}$,

$$\zeta\frac{d}{d\zeta} = \frac{1}{\sqrt{-1}}\frac{d}{d\sigma},$$

and the equation (43) becomes

$$\left(\zeta\frac{d}{d\zeta}\right)^2\delta q = \Phi\delta q, \tag{44}$$

where $\Phi = \Sigma_j \Phi_j \zeta^{2j}$, the summation being taken from $j = +\infty$ to $j = -\infty$, and Φ_{-j} being equal to Φ_j.

Let us assume as the solution of (44)

$$\begin{aligned}\delta q &= \Sigma_j[(b_j + e_{-j}) \cos(c + 2j)\sigma + (b_j - e_{-j})\sqrt{-1} \sin(c + 2j)\sigma],\\ &= \Sigma_j[b_j \zeta^{c+2j} + e_j \zeta^{-c+2j}].\end{aligned}$$

The equation (44) must be separately satisfied for the terms involving b and for those involving e; hence we need only regard one series of terms.

On substituting in (44) the assumed expression for δq, and equating to zero the coefficients of the several powers of ζ, we have

$$b_j(c + 2j)^2 = \Sigma_i b_{j-i} \Phi_i,\ [1]$$

written *in extenso* this is

$$\ldots - b_{j-2}\Phi_2 - b_{j-1}\Phi_1 + b_j[(c + 2j)^2 - \Phi_0] - b_{j+1}\Phi_1 - b_{j+2}\Phi_2 - \ldots = 0.$$

There are an infinite number of equations like the above, but the infinity must be regarded as an odd number.

If from these equations the b's be eliminated, we have an infinite determinantal equation for determining c. If we write

$$(c + 2j)^2 - \Phi_0 = \{j\},$$

the equation is

$$\begin{vmatrix} \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \{-1\}, & -\Phi_1, & -\Phi_2 & \ldots \\ \ldots & -\Phi_1, & \{0\}, & -\Phi_1 & \ldots \\ \ldots & -\Phi_2, & -\Phi_1, & \{1\} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix} = 0.$$

This is the same in form as Mr Hill's determinantal equation.

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[1] The equation of condition for the e's is easily shown to be

$$e_{-j}(c + 2j)^2 = \Sigma_i e_{i-j} \Phi_{-i};$$

and since $\Phi_i = \Phi_{-i}$, this is exactly the same as that for the b's save that e_{-j} corresponds with b_j.

As much has been written on the subject, it is unnecessary to reproduce the arguments by which it may be shown that if

$$[j] = \Phi_0 - 4j^2,$$

and

$$(45) \qquad \Delta = \begin{vmatrix} \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots \; 1 \;, \; \frac{\Phi_1}{[1]} \;, \; \frac{\Phi_2}{[1]} \; \cdots \\ \cdots \; \frac{\Phi_1}{[0]} \;, \; 1 \;, \; \frac{\Phi_1}{[0]} \; \cdots \\ \cdots \; \frac{\Phi_2}{[1]} \;, \; \frac{\Phi_1}{[1]} \;, \; 1 \; \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{vmatrix},$$

the solution of the determinantal equation is given by

$$(45) \qquad \sin^2\frac{1}{2}\pi c = \Delta \sin^2\frac{1}{2}\pi\sqrt{\Phi_0}.$$

§ 11. *On the stability or instability of an orbit.*

When c is real, δq is expressible by a series of sines and cosines of multiples of the arc. Since V is an even function of the arc, it is expressible by a series of cosines of the same form as that for Φ; hence δp, which is equal to $V^{\frac{1}{2}}\delta q$, is expressible in a series, similar in form to that for δq.

But δp denotes normal displacement from the periodic orbit, and therefore the motion in the varied orbit is oscillatory with reference to the periodic orbit. In other words the periodic orbit is stable.

If c_0 be any one value of c, all its infinite values are comprised in the formula $\pm c_0 \pm 2i$, where i is an integer. It is however convenient to choose one value of c as fundamental. When the choice has been made we may refer to the terms in the series for δq of which the argument is c_0 as the principal terms, although it does not appear to be necessary that these terms should have the largest coefficients. In fact since two arbitrary constants are involved in the specification of a definite variation of orbit, it is probable that the terms, which are numerically the most important in one variation, will not be so in another.

If the body be considered as moving in an elliptic orbit, it will be at its pericentre or apocentre, when δp is a negative or positive maximum, respectively. The principal terms of δq, and therefore also of δp, have the argument $c\sigma$ or $\frac{c\pi s}{S}$; hence if we may assume that the principal term is also the most important, the body has passed through a complete anomalistic circuit when s has increased from zero to $2\frac{S}{c}$. Since S is the synodic arc in the relative orbit, $\frac{1}{2}c$ is the ratio of the anomalistic to the synodic arc, both arcs being measured on the orbit as drawn with reference to the moving axes.

Now I propose to adopt as a convention that the fundamental value of c shall be that value which lies nearest to $\sqrt{\Phi_0}$, where Φ_0 denotes the mean value of Φ. This convention certainly attributes to $\frac{1}{2}c$ a physical meaning, which is correct in all those cases which have any resemblance to the motion of an actual satellite in the solar system. I shall accordingly use the value of c which lies nearest to $\sqrt{\Phi_0}$ as fundamental.

We have just arrived at a physical meaning for c by considering the principal term in the series; now in so doing we were in effect considering only the mean motion of the body with reference to the moving axes; therefore $\frac{1}{2}c$ is also the ratio of the synodic to the anomalistic period.[1]

If T denotes the synodic period, the mean motion of the body referred to axes fixed in space is $\frac{2\pi}{T} + n$; and if $\frac{d\omega}{dt}$ denotes the mean angular velocity of the pericentre with reference to axes fixed in

[1] It may be observed that when V is constant (as is the case when we only consider mean motion) $V^2\Psi = \Theta$, and Mr Hill's equation for δp becomes identical with the present one for δq. It is well to remark that what I denote by c is $2c$ of Mr Hill's notation.

space, the mean motion of the body with reference to the pericentre is $\frac{2\pi}{T} + n - \frac{d\omega}{dt}$. Then, since angular velocities vary inversely as periods,

$$\frac{1}{2}c = \frac{\frac{2\pi}{T} + n - \frac{d\omega}{dt}}{\frac{2\pi}{T}}, \quad \text{where} \quad n^2 = \nu + 1.$$

Therefore

$$(46) \qquad \begin{cases} \dfrac{d\omega}{dt} = n - \dfrac{2\pi}{T}\left(\dfrac{1}{2}c - 1\right), \\ \text{or} \\ T\left(n - \dfrac{d\omega}{dt}\right) = 2\pi\left(\dfrac{1}{2}c - 1\right). \end{cases}$$

Mr Hill's c is equal to one half of my c, and accordingly the first of (46) is identical with the formula from which Mr Hill derives »a part of the motion of the lunar perigee».[1]

The angular velocity of regression of the pericentre being $n - \frac{d\omega}{dt}$, it follows from (46) that $2\pi\left(\frac{1}{2}c - 1\right)$ is the amount of that regression with respect to the moving axes in the synodic period.

Whilst the pericentre regredes with reference to the moving axes, it advances with reference to fixed axes; the advance in the synodic period is $nT - 2\pi\left(\frac{1}{2}c - 1\right)$, and in the sidereal period the advance is $2\pi\left[1 - \dfrac{\frac{1}{2}c}{1 + \frac{nT}{2\pi}}\right]$.

In the numerical treatment of stable periodic orbits I tabulate the apparent regression $2\pi\left(\frac{1}{2}c - 1\right)$, and the actual advance $nT - 2\pi\left(\frac{1}{2}c - 1\right)$ in the synodic period; also $2\pi\left[1 - \dfrac{\frac{1}{2}c}{1 + \frac{nT}{2\pi}}\right]$ the advance in the sidereal period.

[1] Acta Mathem. vol. 8.

Let us now consider the case where c is imaginary, so that the motion is no longer oscillatory with respect to the periodic orbit, and the periodic orbit is unstable.

The form of (45) shows that c becomes imaginary either when $\Delta \sin^2 \frac{1}{2}\pi\sqrt{\Phi_0}$ is negative, or when it is greater than unity; this function will therefore be described below as the criterion of stability.

If Φ_0 were negative it would indicate that the mean force of restitution towards the periodic orbit was negative. Hence it seems obvious that the body would then depart from the periodic orbit, which would therefore be unstable. If however Δ were negative as well as Φ_0, it would seem as if it were possible to have a real value for c; but it is not easy to see how this condition could lead to a stable orbit.

I have not yet come on any case where Φ_0 is negative and accordingly that condition is left out of consideration for the present. We are left then with the two conditions, Δ negative or $\Delta \sin^2 \frac{1}{2}\pi\sqrt{\Phi_0}$ greater than unity; these lead to two kinds of instability.

In instability of the first kind Δ is negative; for reasons which will appear below, I shall call this »even instability».

In this case let us put

$$\Delta \sin^2 \frac{1}{2}\pi\sqrt{\Phi_0} = -D^2,$$

so that (45) becomes $\sin \frac{1}{2}\pi c = \pm D\sqrt{-1}$.

The sine in this case is hyperbolic, and if we write $c = 2i + k\sqrt{-1}$, where i is an integer, the equation for k becomes $\sinh \frac{1}{2}\pi k = \pm D$.

Since the values of c occur in pairs, equal in magnitude and opposite in sign, it is only necessary to consider the upper sign and the result may be written

$$(47) \qquad \left\{ \begin{aligned} & e^{\frac{1}{2}\pi k} = \sqrt{(D^2 + 1)} + D, \\ \text{or} & \\ & k = \frac{2}{\pi}\log_e [\sqrt{(D^2 + 1)} + D]. \end{aligned} \right.$$

I shall return in § 12 to the form of solution adapted to the case of »even instability».

Turning to the instability of the second kind, which I shall call »uneven instability», we have

$$\sin^2\frac{1}{2}\pi c = \Delta \sin^2\frac{1}{2}\pi\sqrt{\Phi_0} = D^2,$$

where D^2 is greater than unity, so that c is imaginary.

The sine in this case also becomes a hyperbolic function, and if we write $c = 2i + 1 + k\sqrt{-1}$, where i is an integer, we have

$$\sin\frac{1}{2}\pi c = (-)^i \cosh\frac{1}{2}\pi k,$$

a hyperbolic cosine.

Hence

$$\cosh\frac{1}{2}\pi k = \pm D.$$

Taking only the upper sign as before, this may be written

$$(48)\qquad \left\{\begin{array}{l} e^{\frac{1}{2}\pi k} = \sqrt{(D^2 - 1)} + D, \\ \text{or} \\ k = \frac{2}{\pi}\log_e[\sqrt{(D^2 - 1)} + D]. \end{array}\right.$$

I shall return in § 12 to the form of solution adapted to the case of »uneven instability», but I wish now to consider the nature of the transitions from instability to stability.

Suppose that we are considering a family of periodic orbits, the members of which are determined by the continuous increase or decrease of the constant C of relative energy; and let us suppose that $\Delta\sin^2\frac{1}{2}\pi\sqrt{\Phi_0}$, being at first negative, increases and reaches the value zero. At the moment of the transition of this function from negative to positive, there is transition from even instability to stability. If on the other hand this function were positive and less than unity, and were to increase up to and beyond unity there would be a transition from stability to uneven instability.

In all the cases of stability which I have investigated, except one,[1] the fundamental value of c lies between 2 and 3, and the apparent

[1] The orbit in question is $C = 40\cdot0$, $x_0 = 1\cdot0334$; see Appendix.

regression of pericentre in the synodic period, namely $2\pi\left(\frac{1}{2}c - 1\right)$, lies between 0 and 180°, these extreme values corresponding with transitional stages.

It will now conduce to brevity to regard c as lying between 2 and 3, instead of regarding it as a multiple-valued quantity.

If we refer back to the form of solution assumed for the equation (44), we see that when $c = 2$, the solution is

$$\delta q = (b_{-1} + e_1) + (b_0 + e_0 + b_{-2} + e_2) \cos\frac{2\pi s}{S} \cdots$$
$$+ (b_0 - e_0 - b_{-2} + e_2)\sqrt{-1}\sin\frac{2\pi s}{S} \cdots,$$

and that when $c = 3$, it is

$$\delta q = (b_1 + e_{-1} + b_{-2} + e_2)\cos\frac{\pi s}{S} + (b_0 + e_0 + b_{-3} + e_3)\cos\frac{3\pi s}{S} \cdots$$
$$+ (b_1 - e_{-1} - b_{-2} + e_2)\sqrt{-1}\sin\frac{\pi s}{S} + (b_0 - e_0 - b_{-3} + e_3)\sqrt{-1}\sin\frac{3\pi s}{S} \cdots.$$

In the first case it is clear that when $s = S$, δq has gone through a complete period and has returned to its initial value; but in the second case whilst δq is equal in value, it is opposite in sign to what it was at first.

Consider then the first case where $c = 2$, and suppose that the body is displaced from the periodic orbit along the normal, at a conjunction. Then the body starts moving at right angles to the line of syzygies, and when $s = S$ it has again returned to the same point, and is again moving at right angles to the line of syzygies.

Hence it follows that we have found a new periodic orbit differing by infinitely little from the original one. Thus the original orbit is a double solution of the problem, and the interpretation to be put on the result $c = 2$ is, that we have found a periodic orbit which is a member of two distinct families.

The $\Delta \sin^2\frac{1}{2}\pi\sqrt{\Phi_0}$ corresponding to our family of orbits has been supposed to be increasing from a negative to a positive value; at the instant of transition the same function for the other family must also be passing through the value zero.

If C be the value of the constant of relative energy for the critical orbit which gives $c = 2$, there must be *two* orbits, infinitely near to one another, for which the constant is $C - \partial C$.

If the orbits were classified according to values of the parameter $\Delta \sin^2 \frac{1}{2}\pi\sqrt{\Phi_0}$, instead of according to values of C, these two families would have to be regarded as a single family, and the critical stage would be that in which C reached a maximum or minimum value.

But when the classification is according to values of C, we say that there are two families which coalesce at the critical value of C; it is also clear that, as the orbit we were following was unstable up to this critical value, the other must have been stable.

An interesting example of this will be found below, where the families of orbits B and C spring from a single orbit.

Now reverting again to the question of the transition from instability to stability, let us suppose that as the constant C varies, $\Delta \sin^2 \frac{1}{2}\pi\sqrt{\Phi_0}$, being at first greater than unity, diminishes, passes through the value unity and continues diminishing. Then the orbit was at first unstable with uneven instability and c of the form $3 + k\sqrt{-1}$; it becomes stable at the critical stage with c less than 3. But there is now no real double solution at the moment of transition and no coalescence of families.[1] It is probable that there is coalescence with another family of imaginary orbits at this crisis, but I do not discuss this, since I am not looking at the subject from the point of view of the theory of differential equations. Accordingly in our figures of orbits there will be nothing to mark the transition from uneven instability to stability, and it will only be by the consideration of the function $\Delta \sin^2 \frac{1}{2}\pi\sqrt{\Phi_0}$ that we shall be aware of the change.

The conclusions arrived at in this section seem to accord with those of M. Poincaré in his Mécanique Céleste, who remarks that periodic orbits will disappear in pairs.

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[1] When I explained the results at which I have arrived to M. Poincaré, he suggested that there may be coalescence between a doubly periodic orbit and a singly periodic one, when the two circuits of the former become identical with one another and with the latter.

It is clear from this discussion that uneven instability can never graduate directly into even instability, but the transition must take place through a range of stability.

But this last conclusion must not be held to be contradictory of a very remarkable method of transition, of which we shall find an example below.

Suppose we have two independent orbits in either of which the body may move, and that as the constant of energy varies these two orbits approach until they have a common tangent. Then when the constant of energy varies still further, we shall find only a single orbit replacing the two independent ones. Now we shall see reason to suppose that two independent orbits one of which is evenly unstable, and the other unevenly unstable may fuse together so as to form an evenly unstable orbit. In this case we have, in some sense, a direct transition from uneven instability to even instability, without the interposition of stability. An example of this will be noted in § 18, where we shall find the satellite A fusing its orbit with the oscillatory orbit a and forming a figure-of-8 orbit.

§ 12. *Modulus of instability, and form of solution.*

The cases of instability will now be considered.

When the instability is of the first or even kind, we have $c = 2i + k\sqrt{-1}$, and

$$(49) \qquad \begin{cases} e^{\frac{1}{2}\pi k} = \sqrt{(D^2 + 1)} + D, \\ e^{-\frac{1}{2}\pi k} = \sqrt{(D^2 + 1)} - D, \end{cases}$$

where $D^2 = -\Delta \sin^2 \frac{1}{2} \pi \sqrt{\Phi_0}$.

The solution of (44) was

$$\delta q = \Sigma_j [(b_j + e_{-j}) \cos (c + 2j)\sigma + (b_j - e_{-j}) \sqrt{-1} \sin (c + 2j)\sigma].$$

Now if we take the integer i involved in the expression for c as zero,

$$\cos (c + 2j)\sigma = \cosh k\sigma \cos 2j\sigma - \sqrt{-1} \sinh k\sigma \sin 2j\sigma,$$

$$\sqrt{-1} \sin (c + 2j)\sigma = -\sinh k\sigma \cos 2j\sigma + \sqrt{-1} \cosh k\sigma \sin 2j\sigma.$$

Therefore when the sign of summation only runs from ∞ to 0, instead of to $-\infty$, and when b_0 and e_0 are supposed to be the halves of their values when the summation ran from $+\infty$ to $-\infty$, the solution may be written

$$\delta q = \sum_0^\infty \{\cosh k\sigma[(b_j + e_{-j} + b_{-j} + e_j)\cos 2j\sigma + (b_j - e_{-j} - b_{-j} + e_j)\sqrt{-1}\sin 2j\sigma]$$
$$+ \sinh k\sigma[-\sqrt{-1}(b_j + e_{-j} - b_{-j} - e_j)\sin 2j\sigma - (b_j - e_{-j} + b_{-j} - e_j)\cos 2j\sigma]\}.$$

Putting

$$b_j + b_{-j} = B_j, \qquad e_{-j} + e_j = E_j,$$
$$b_j - b_{-j} = \beta_j\sqrt{-1}, \qquad e_{-j} - e_j = \varepsilon_j\sqrt{-1},$$

and writing the hyperbolic functions as exponentials, we have

$$(50) \qquad \delta q = \sum_0^\infty \{e^{k\sigma}(E_j \cos 2j\sigma + \varepsilon_j \sin 2j\sigma) + e^{-k\sigma}(B_j \cos 2j\sigma - \beta_j \sin 2j\sigma)\}.$$

By means of (49) this may be written

$$(50) \qquad \delta q = \sum_0^\infty \Big\{(\sqrt{(D^2+1)} + D)^{\frac{2\sigma}{\pi}}(E_j \cos 2j\sigma + \varepsilon_j \sin 2j\sigma)$$
$$+ (\sqrt{(D^2+1)} - D)^{\frac{2\sigma}{\pi}}(B_j \cos 2j\sigma - \beta_j \sin 2j\sigma)\Big\}.$$

In (50) it is not safe to assume that the most important term is that for which $j = 0$; indeed this will usually not be the case. All that we know is that the series contains sines and cosines of even multiples of σ, that one set of terms increases without limit and that the other set diminishes.

In the numerical treatment of unstable periodic orbits it will be well to have a modulus of the degree of instability; and these considerations afford a convenient means of obtaining such a modulus.

This modulus may be taken to be the number of synodic revolutions in which the augmenting factor doubles its initial value; that is to say we are to put

$$e^{k\sigma} = [\sqrt{(D^2+1)} + D]^{\frac{2\sigma}{\pi}} = 2.$$

Therefore

$$(51) \qquad \frac{s}{S} = \frac{\sigma}{\pi} = \frac{\log\sqrt{2}}{\log[\sqrt{(D^2+1)} + D]}.$$

This is the modulus of instability, when it is of the even kind.

A consideration of the form of the series for δq shows that it increases without limit, and that the planet or satellite crosses and recrosses the periodic orbit an even number of times in a single circuit; it is on this account that I have called this »even instability».

When the instability is of the second or uneven kind, we have $c = 2i + 1 + k\sqrt{-1}$, or if we take i as zero, $c = 1 + k\sqrt{-1}$; also

$$(52)\qquad \begin{cases} e^{\frac{1}{2}\pi k} = D + \sqrt{(D^2 - 1)}, \\ e^{-\frac{1}{2}\pi k} = D - \sqrt{(D^2 - 1)}, \end{cases}$$

where $D^2 = \Delta \sin^2 \frac{1}{2}\pi\sqrt{\Phi_0}$.

Then

$$\cos(c + 2j)\sigma = \cos(2j + 1)\sigma \cosh k\sigma - \sqrt{-1}\sin(2j + 1)\sigma \sinh k\sigma,$$

$$\sqrt{-1}\,.\sin(c + 2j)\sigma = -\cos(2j + 1)\sigma \sinh k\sigma + \sqrt{-1}\sin(2j + 1)\sigma \cosh k\sigma.$$

And the solution, expressed with singly infinite summation and with the proper change in the meanings of b_0 and e_0, is

$$\begin{aligned}\delta q = \sum_0^\infty \Big\{ &\cosh k\sigma[(b_j + b_{-j-1} + e_{-j} + e_{j+1})\cos(2j + 1)\sigma \\ &\qquad + (b_j - b_{-j-1} - e_{-j} + e_{j+1})\sqrt{-1}\sin(2j + 1)\sigma] \\ &+ \sinh k\sigma[-\sqrt{-1}\,(b_j - b_{-j-1} + e_{-j} - e_{j+1})\sin(2j + 1)\sigma \\ &\qquad - (b_j + b_{-j-1} - e_{-j} - e_{j+1})\cos(2j + 1)\sigma]\Big\}.\end{aligned}$$

Putting

$$b_j + b_{-j-1} = B_j, \qquad e_{-j} + e_{j+1} = E_j,$$
$$b_j - b_{-j-1} = \beta_j\sqrt{-1}, \qquad e_{-j} - e_{j+1} = \varepsilon_j\sqrt{-1},$$

and writing the hyperbolic functions as exponentials, we have

$$(52)\qquad \delta q = \sum_0^\infty \Big\{ e^{k\sigma}(E_j \cos(2j + 1)\sigma + \varepsilon_j \sin(2j + 1)\sigma) + e^{-k\sigma}(B_j \cos(2j + 1)\sigma - \beta_j \sin(2j + 1)\sigma)\Big\}.$$

By means of (52) this may be written

$$(53)\qquad \delta q = \sum_0^\infty \Big\{ (D + \sqrt{(D^2 - 1)})^{\frac{2\sigma}{\pi}} (E_j \cos(2j+1)\sigma + \varepsilon_j \sin(2j+1)\sigma)$$
$$+ (D - \sqrt{(D^2 - 1)})^{\frac{2\sigma}{\pi}} (B_j \cos(2j+1)\sigma - \beta_j \sin(2j+1)\sigma) \Big\}.$$

In this case again the terms for which $j = 0$ are not usually the most important ones, but we see that the series contains sines and cosines of odd multiples of σ; and that one set of terms increases without limit and that the other diminishes. As in the first sort of instability, a convenient modulus is the number of synodic revolutions in which the amplitude of the increasing oscillation doubles its initial value; that is to say we put

$$e^{k\sigma} = (D + \sqrt{(D^2 - 1)})^{\frac{2\sigma}{\pi}} = 2.$$

Therefore

$$(54)\qquad \frac{s}{S} = \frac{\sigma}{\pi} = \frac{\log \sqrt{2}}{\log [D + \sqrt{(D^2 - 1)}]},$$

where

$$D^2 = \Delta \sin^2 \frac{1}{2}\pi\sqrt{\Phi_0}.$$

This is the modulus of instability, when it is of the uneven kind. A consideration of the principal term has shown us that there is an oscillation, whose amplitude increases without limit. The planet or satellite crosses and recrosses the periodic orbit an odd number of times in a single circuit, making ever increasing excursions on each side; it is on this account that I have called this »uneven instability».

It is interesting to consider the form which the equations of condition assume in the two sorts of instability.

In the case of even instability we have $c = k\sqrt{-1}$, and the equations for the determination of the b's are given by

$$(55)\qquad b_j(c + 2j)^2 = \Sigma_i b_{j-i} \Phi_i,$$
$$= b_0 \Phi_j + \sum_1^\infty {}_i b_i \Phi_{j-i} + \sum_1^\infty {}_i b_{-i} \Phi_{j+i}.$$

We now have

$$2b_j = B_j + \beta_j\sqrt{-1}, \qquad 2b_{-j} = B_j - \beta_j\sqrt{-1},$$

$$2b_j(c+2j)^2 = (4j^2-k^2)B_j - 4jk\beta_j + \sqrt{-1}\,[4jkB_j + (4j^2-k^2)\beta_j].$$

Then noting that β_0 is necessarily zero, and equating to zero the real and imaginary parts of the equation of condition (55), we have

$$(56) \qquad \begin{cases} (4j^2-k^2)B_j - 4jk\beta_j = B_0\Phi_j + \sum_1^\infty{}_i B_i(\Phi_{j-i}+\Phi_{j+i}), \\ 4jkB_j + (4j^2-k^2)\beta_j = \qquad + \sum_1^\infty{}_i B_i(\Phi_{j-i}-\Phi_{j+i}). \end{cases}$$

In the case of $j=0$, the second equation is identically true, and the first becomes

$$(55) \qquad -k^2B_0 = B_0\Phi_0 + 2\sum_1^\infty{}_i B_i\Phi_i.$$

It is easy to show that if we take j as negative, we are led to the same equations; thus it is only necessary to consider the case of j positive.

These equations suffice to determine all the B's and β's in terms of one of them, say B_0, which is an arbitrary constant of the solution.

We have already seen that the equations of condition for e_{-j} are exactly the same as those for b_j. Hence bearing in mind the definitions of E_j and ε_j, we see that the equations of condition for E_j, ε_j are the same as those for B_j, β_j. Then since $\varepsilon_0=0$, E_j, ε_j are the same multiples of E_0 as B_j, β_j are of B_0. Thus E_0 is the second arbitrary constant of the solution.

Suppose that we put $B_0=1$, and solve the equations finding $B_j=\Lambda_j$, $\beta_j=\lambda_j$, then the general solution is

$$(57) \quad \delta q = \sum_0^\infty{}_j\,[E_0e^{k\sigma}(\Lambda_j\cos 2j\sigma + \lambda_j\sin 2j\sigma) + B_0e^{-k\sigma}(\Lambda_j\cos 2j\sigma - \lambda_j\sin 2j\sigma)].$$

Now turn to the case of uneven instability where $c = 1 + k\sqrt{-1}$; the equation of condition may be written

$$(58) \qquad b_j(c+2j)^2 = \sum_0^\infty{}_i b_i\Phi_{j-i} + \sum_0^\infty{}_i b_{-i-1}\Phi_{j+i+1},$$

where

$$2b_j = B_j + \beta_j\sqrt{-1}, \qquad 2b_{-j-1} = B_j - \beta_j\sqrt{-1},$$

$$2b_j(c+2j)^2 = [(2j+1)^2 - k^2]B_j - 2(2j+1)k\beta_j$$
$$+\sqrt{-1}\{2(2j+1)kB_j + [(2j+1)^2 - k^2]\beta_j\}.$$

Then equating to zero the real and imaginary parts of the equation of condition (58),

$$(59) \quad \begin{cases} [(2j+1)^2 - k^2]B_j - 2(2j+1)k\beta_j = \sum\limits_0^\infty{}_i B_i(\Phi_{j-i} + \Phi_{j+i+1}), \\ 2(2j+1)kB_j + [(2j+1)^2 - k^2]\beta_j = \sum\limits_0^\infty{}_i \beta_i(\Phi_{j-i} - \Phi_{j+i+1}). \end{cases}$$

It is easy to show that it is only necessary to consider the positive values of j.

These equations suffice to determine all the B's and β's in terms of B_0, which is one of the arbitrary constants of the solution.

From the definitions of E_j, ε_j it is easy to see that the equations of condition are the same as (59), and that E_j, ε_j are the same multiples of E_0, (the second arbitrary constant) that B_j, β_j are of B_0.

Suppose that (59) are solved with $B_0 = 1$, and that we find $B_j = A_j$, $\beta_j = \lambda_j$; then the general solution is

$$(60) \quad \delta q = \sum_0^\infty{}_j [E_0 e^{k\sigma}(A_j \cos(2j+1)\sigma + \lambda_j \sin(2j+1)\sigma)$$
$$+ B_0 e^{-k\sigma}(A_j \cos(2j+1)\sigma - \lambda_j \sin(2j+1)\sigma)].$$

It follows therefore that when k has been found from the infinite determinant the solutions for the varied orbit are expressible by means of two arbitrary constants in both kinds of instability. Such solutions would of course only express the true motion for a short time.

I have actually applied this method to one of the unstable periodic orbits which was computed, but as the work leads to no useful conclusion I shall not give the details of it.

§ 13. *Numerical determination of stability.*

When a periodic orbit has been found by quadratures, it is not obvious by mere inspection whether it is stable or not, and we must consider the numerical processes requisite to obtain an answer to the question.

The points which are determined by quadratures in a periodic orbit do not divide the arc S into a number of equal parts. The distance along the arc from the first orthogonal crossing of the x axis to the second orthogonal crossing is $\frac{1}{2}S$; this may be determined by interpolation, for we may find what value of s makes y vanish.

In general there are two orbits computed, which differ from exact periodicity in opposite directions by small amounts. The arc $\frac{1}{2}S$, measured from the first orthogonal crossing to the second, which is not exactly orthogonal, is determined in each of these cases. The subsequent proceedings are then carried out in duplicate, and the final step is an interpolation between the two results to obtain the result for the exactly periodic orbit. In many cases however the computed orbit differs from a truly periodic one by an amount which is so small, that it may be attributed to the errors inherent to the method of calculation. In such cases the duplicate computation is unnecessary, and since the operations on the approximately periodic orbits are exactly like those on the truly periodic ones, we may henceforth speak as if the true orbit had been found.

The next step is the computation of Φ corresponding to each computed point of the orbit. In order to take advantage of the work already carried out in the quadratures, I arrange the computation of Φ in the following form:

Computation of Φ.

$\varphi - \theta$	$\varphi - \psi$
$\mathrm{L}r$	$\mathrm{L}\rho$
$\mathrm{L}r^3$	$\mathrm{L}\rho^3$
$\mathrm{L}\left(\frac{1}{r^2} - r\right)$	$\mathrm{L}\left(\frac{1}{\rho^2} - \rho\right)$
$\mathrm{L}\nu \sin(\varphi - \theta)$	$\mathrm{L}\sin(\varphi - \psi)$
$\mathrm{C}V^2$	$\mathrm{C}V^2$
La	Lb
a	$\frac{1}{R}$
b	$\frac{n}{V}$
$\frac{dV}{Vds}$	c
	c^2
$\left(\frac{dV}{Vds}\right)^2$	$\frac{10}{6}\mathrm{c}^2$
	$\frac{1}{2}\left(\frac{dV}{Vds}\right)^2$
	A
$\mathrm{L}\nu \cos^2(\varphi - \theta)$	$\mathrm{L}\cos^2(\varphi - \psi)$
$\mathrm{C}r^3$	$\mathrm{C}\rho^3$
$\mathrm{C}V^2$	$\mathrm{C}V^2$
Ld	Le
d	A
e	$-\mathrm{B}$
B	$\mathrm{A} - \mathrm{B}$
$\mathrm{L}\Psi$	$\frac{1}{2}(\mathrm{A} - \mathrm{B})$
$\mathrm{L}\frac{S^2}{\pi^2}$	Ψ
$\mathrm{L}\Phi$	Φ

As before L, C stand for logarithm and cologarithm, and the brackets indicate additions.

It would be tedious to find the Fourier's series for Φ from its computed values, and it is best to find interpolated values of Φ at exact submultiples of the arc S. I therefore interpolate Φ at the points for which the arc is $\frac{1}{24}S, \frac{2}{24}S \dots \frac{12}{24}S$, 13 values in all. These interpolations are made by one of the formulæ (23).

The next step is the harmonic analysis of these 13 values of Φ, which is an even function of the arc.

The analysis may be conveniently arranged in a schedule of the following form.

Harmonic analysis of an even function of which 24 values $a_0, a_1 \dots a_{11}, a_{12}, a_{11} \dots a_1$ are given.

i	ii	iii		iv		v		vi
		i − ii	M	M × iii	M	M × iii	M	M × iii
a_0	a_{12}	$a_0 - a_{12}$ (α)	1	α	1	α	1	α
a_1	a_{11}	$a_1 - a_{11}$ (β)	σ_5	$\sigma_5\beta$	σ_1	$\sigma_1\beta$	$-\sigma_1$	$-\sigma_1\beta$
a_2	a_{10}	$a_2 - a_{10}$ (γ)	σ_4	$\sigma_4\gamma$	$-\sigma_4$	$-\sigma_4\gamma$	$-\sigma_4$	$-\sigma_4\gamma$
a_3	a_9	$a_3 - a_9$ (δ)	σ_3	$\sigma_3\delta$	$-\sigma_3$	$-\sigma_3\delta$	σ_3	$\sigma_3\delta$
a_4	a_8	$a_4 - a_8$ (ε)	1	ε	1	ε	1	ε
a_5	a_7	$a_5 - a_7$ (ζ)	σ_1	$\sigma_1\zeta$	σ_5	$\sigma_5\zeta$	$-\sigma_5$	$-\sigma_5\zeta$
a_6	a_6	0	0	0	0	0	0	0
Sum 0 to 6	Sum 7 to 12			24 | Sum		24 | Sum		24 | Sum
	Sum 0 to 6			Φ_1		Φ_5		Φ_7
	Sum 0 to 12							
	2 × Sum 0 to 12							
	$-(a_0 + a_{12})$							
	24 | Sum							
	Φ_0							

$$\Phi_3 = \frac{1}{24}[\alpha - 2\varepsilon + \sigma_3(\beta - \delta - \zeta)]$$

(see iii)

$\sigma_1 = 2\sin 15^\circ = \cdot 5176$
$\sigma_3 = 2\sin 45^\circ = 1\cdot 4142$
$\sigma_4 = 2\sin 60^\circ = 1\cdot 7321$
$\sigma_5 = 2\sin 75^\circ = 1\cdot 9319$

vii	viii	ix	x
i + ii	Last 4 of vii reversed	vii − viii	vii + viii
$a_0 + a_{12}$	$a_6 + a_6$	$(a_0 + a_{12}) - (a_6 + a_6)$ (η)	$(a_0 + a_{12}) + (a_6 + a_6)$ (λ)
$a_1 + a_{11}$	$a_5 + a_7$	$(a_1 + a_{11}) - (a_5 + a_7)$ (θ)	$(a_1 + a_{11}) + (a_5 + a_7)$ (μ)
$a_2 + a_{10}$	$a_4 + a_8$	$(a_2 + a_{10}) - (a_4 + a_8)$ $(\varkappa)$	$(a_2 + a_{10}) + (a_4 + a_8)$ (ν)
$a_3 + a_9$	$a_3 + a_9$	0	$(a_3 + a_9) + (a_3 + a_9)$ (ρ)
$a_4 + a_8$			
$a_5 + a_7$			
$a_6 + a_6$			

$$\Phi_2 = \frac{1}{24}[\eta + \varkappa + \sigma_4\theta], \qquad \Phi_4 = \frac{1}{24}[(\lambda + \mu) - (\nu + \rho)].$$

$$\Phi_6 = \frac{1}{24}[\eta - 2\varkappa], \qquad \Phi_8 = \frac{1}{24}[(\lambda - \mu) - (\nu - \rho)].$$

(see ix) (see x)

If we write $\theta = 2\sigma = \frac{2\pi s}{S}$, the function Φ is equal to

$$\Phi_0 + 2\,\Phi_1 \cos\theta + 2\,\Phi_2 \cos 2\theta + \ldots + 2\,\Phi_8 \cos 8\theta.$$

In order to test the accuracy of the work and the convergency of the series, it is well to compute the values of several of the a's directly from the harmonic expansion. For this purpose we have

$$\begin{cases} a_0 \\ a_{12} \end{cases} = \Phi_0 + 2(\Phi_2 + \Phi_4 + \Phi_6 + \Phi_8) \pm 2(\Phi_1 + \Phi_3 + \Phi_5 + \Phi_7),$$

$$\begin{cases} a_2 \\ a_{10} \end{cases} = \Phi_0 + \Phi_2 - \Phi_4 - 2\,\Phi_6 - \Phi_8 \pm \sigma_4(\Phi_1 - \Phi_5 - \Phi_7),$$

$$\begin{cases} a_3 \\ a_9 \end{cases} = \Phi_0 - 2\,\Phi_4 + 2\,\Phi_8 \pm \sigma_3(\Phi_1 - \Phi_3 - \Phi_5 + \Phi_7),$$

$$\begin{cases} a_4 \\ a_8 \end{cases} = \Phi_0 - \Phi_2 - \Phi_4 + 2\,\Phi_6 - \Phi_8 \pm (\Phi_1 - 2\,\Phi_3 + \Phi_5 + \Phi_7),$$

$$a_6 = \Phi_0 + 2(\Phi_4 + \Phi_8) - 2(\Phi_2 + \Phi_6).$$

It may be remarked that if the harmonic expansion of Φ is convergent, the determinant from which the stability is determinable is also convergent.

But if the representation of Φ by the harmonic expansion up to the 8th harmonic is very imperfect, it is necessary to give up the attempt to determine the stability numerically. In such cases however it is nearly always possible to see that the orbit is unstable, although it may not sometimes be so easy to perceive whether the instability is even or uneven.

We next have to calculate the several members of the determinant Δ by the formula

$$\frac{\Phi_i}{\Phi_0 - 4j^2}.$$

This is the entry for the j^{th} row above or below the centre of the determinant, and it is the i^{th} member to the right and to the left of the leading diagonal, all the members on the diagonal being unity. The values of Φ_i computed by the preceding analysis suffice to enable us to write down 17 columns and rows of Δ. The method of computing Δ will be considered in the next section.

§ 14. *The calculation of a determinant of many columns and rows.*

The following transformation contains the principle by which the number of columns and rows of a determinant may be diminished by unity

$$\Delta = \begin{vmatrix} a_1, & a_2, & a_3, & \dots \\ b_1, & b_2, & b_3, & \dots \\ c_1, & c_2, & c_3, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = a_1 \begin{vmatrix} 1, & \frac{a_2}{a_1}, & \frac{a_3}{a_1}, & \dots \\ 0, & b_2 - b_1\frac{a_2}{a_1}, & b_3 - b_1\frac{a_3}{a_1}, & \dots \\ 0, & c_2 - c_1\frac{a_2}{a_1}, & c_3 - c_1\frac{a_3}{a_1}, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 - b_1\frac{a_2}{a_1}, & b_3 - b_1\frac{a_3}{a_1}, & \dots \\ c_2 - c_1\frac{a_2}{a_1}, & c_3 - c_1\frac{a_3}{a_1}, & \dots \\ \dots & \dots & \dots \end{vmatrix}.$$

Now if we write $b_2' = b_2 - b_1\frac{a_2}{a_1}$, and so on, and then extract the factor b_2', another column and row may be removed, and the process may be repeated until the determinant is reduced to a single member, say z_n; then

$$\Delta = a_1 b_2' c_3'' \dots z_n.$$

If the determinant is convergent and if the rows and columns be removed in proper succession, the factors tend to unity.

By interchanges of columns and rows any member of a determinant may be brought to stand at a corner, but if the number of interchanges is odd the sign of the determinant is changed.

It is not therefore necessary to work from a corner, as in the above example, but any column and any row may be chosen for elimination.

The member which stands at the intersection of the chosen column and row may be called the centre of elimination. Then if the centre of elimination be at an odd or even number of moves from a corner, the sign of the whole is or is not changed.

In the determinants which arise in this investigation the centre of elimination is always taken on the diagonal, and thus no change of sign is introduced.

Let us suppose that the determinant to be evaluated is a symmetrical one, and that the columns and rows are numbered, as in the following example:

$$\begin{array}{r|ccccc|}
\multicolumn{1}{r}{} & -2 & -1 & 0 & 1 & \multicolumn{1}{c}{2} \\
-2 & C, & c_1, & c_2, & c_3, & c_4 \\
-1 & b_1, & B, & b_1, & b_2, & b_3 \\
0 & a_2, & a_1, & A, & a_1, & a_2 \\
1 & b_3, & b_2, & b_1, & B, & b_1 \\
2 & c_4, & c_3, & c_2, & c_1, & C
\end{array}$$

Let $(-1, -1)$ be the first centre of elimination, and $(1, 1)$ the second; then if the double elimination be carried out and algebraic reductions effected, it will be found that the result is

$$B^2\left(1-\frac{b_2^2}{B^2}\right)\begin{array}{|ccc|l}
\multicolumn{1}{c}{-2} & 0 & \multicolumn{1}{c}{2} & \\
B', & b_1', & b_2' & -2 \\
a_1', & A', & a_1' & 0 \\
b_2', & b_1', & B' & 2
\end{array}$$

Where

$$B' = C - \frac{b_3c_1 + b_1c_3}{B + b_2} - \frac{(b_1 - b_3)(c_1 - c_3)}{B - \frac{b_2^2}{B}}, \qquad b_1' = c_2 - \frac{b_1(c_1 + c_3)}{B + b_2},$$

$$b_2' = c_4 - \frac{b_1c_1 + b_3c_3}{B + b_2} + \frac{(b_1 - b_3)(c_1 - c_3)}{B - \frac{b_2^2}{B}}, \qquad a_1' = a_2 - \frac{a_1(b_1 + b_3)}{B + b_2},$$

$$A' = A - \frac{2a_1b_1}{B + b_2}.$$

If the determinant is convergent, with an odd number of columns and rows, (0 , 0) is the heart of the determinant; if the elimination proceeds away from the heart, at any stage of the process the approximation consists of the product of all the factors extracted, multiplied by (0 , 0), the heart of the remaining determinant.

Thus in the above example after one double elimination the approximation is

$$B^2\left(1-\frac{b_2^2}{B^2}\right)\left(A-\frac{2a_1b_1}{B+b_2}\right).$$

This is in fact the full expression for the determinant

$$\begin{vmatrix} B, & b_1, & b_2 \\ a_1, & A, & a_1 \\ b_2, & b_1, & B \end{vmatrix}$$

I have found it most convenient in practice first to extract a squared factor, such as B^2 (thus reducing (—1 , —1) and (1 , 1) to unity), and afterwards to extract a single factor, such as $1-\frac{b_2^2}{B^2}$.

This process cannot of course be applied with advantage, when the work is algebraical, but some process of the kind seems to be practically necessary, when the approximate numerical value is to be found of a determinant of a large number of columns and rows.

It will be noticed that after each pair of eliminations the primitive symmetry is restored; but the work might equally well be arranged otherwise. For we might first eliminate from the centre (0, 0), which would not affect the symmetry, and we might then take the pair (—1, —1) and (1, 1). This variation of procedure would afford a valuable check on the arithmetic.

Where the outer fringe of the determinant obviously has but little influence on the final result, and where we are in any case going to use all the members in the original determinant, I have found it best to begin from the outside. In such a case four or five columns and rows may, as it were, be shelled off the outside, with scarcely any alteration of the central entries.

The actual numerical work of evaluating a determinant may be arranged as follows:

The number of decimal places to be retained is first fixed on. A paper is then marked with a gridiron of columns and rows, numbered from zero at the centre upwards and downwards. Each square should be large enough to contain four or five rows of figures. The original determinant is then written in the squares, the numbers being put as near the top of each square as possible. I have found it convenient to omit decimal points, and to express the numbers in units of the last decimal place retained. In most of my work, where only a rough result was required, I have adopted three places of decimals; thus the unit in which the entries are expressed is ·001, and the diagonal members are all written as 1000.

The pair of symmetrical diagonal members, which is to form the first pair of centres, is then chosen. As stated above, I have in my later work usually worked from the outside. In the first pair of eliminations these diagonals are already unity, but this is not so subsequently, and we first reduce them to unity by dividing the rows on which they stand by their values, and by extracting a squared factor.

It will be found convenient to run a red line through the column and row to be removed. If the red lines be regarded as coordinate axes, the row being x and the column y, any member of the determinant may be specified by its x and y. If the member of the determinant whose coordinates are x, y be a; and if the member whose coordinates are $x, 0$ be b; and if the member whose coordinates are $0, y$ be c; then the number which has to be substituted for a is $a - bc$.

In other words each number on the horizontal red line has to be multiplied by each number on the vertical red line, and the products have to be subtracted from the numbers which stand at the remote corners of the rectangles.

In effecting this process I form a separate table of the subtrahends, and write down the differences immediately under the numbers which they displace.

After the first elimination, which has rendered the determinant unsymmetrical, a single factor corresponding to the other chosen diagonal member is extracted, its row is correspondingly altered, red lines are drawn to mark the column and row to be removed, and the similar process is repeated. The symmetry of the determinant should now be restored, but any pair of numbers which should agree are arrived at by different numerical processes.

The restoration of symmetry affords a very valuable check on arithmetical processes which I have found it singularly difficult to work correctly.

As only a limited number of decimal places are employed there is often a discrepancy of unity in the last significant figure between two numbers which ought to agree. It is sometimes possible to determine by inspection which of the two numbers is arrived at by the less risky series of operations, and I then adopt that number to represent both entries. But where there is no obvious reason for choosing one result more than the other, I choose one or other at hasard, and restore the perfect symmetry.

The process of elimination is continued until the determinant is reduced to $(0, 0)$, but in the last two or three stages it is well to increase the number of decimals retained.

If at any stage the factor to be extracted becomes small, the whole row to which it belongs becomes large, and the symmetry may perhaps be seriously affected. In this case it is well not to choose this pair of centres of elimination, but to take another pair, leaving this pair to a later stage in the calculation.

If the determinant is negative, a negative factor will be extracted at some stage. In all the cases which have been worked out it is easy to see that no other negative factor will ever arise, and thus the determinant will remain clearly negative. Most of the determinants have been written with 17 columns and rows; then beginning with $(-8, -8)$ and $(8, 8)$ I find that it is often possible to erase 8 columns and 8 rows on a single sheet of paper, with scarcely any modification of the central part of the determinant. Thus the determinant which at first had 289 spaces (although many only contain zeros) is reduced to 81 spaces, with but little labour.

The multiplications have been done with Crelle's table, but a specially computed auxiliary table of products, from $\cdot 000 \times \cdot 000$ up to

$\cdot 040 \times \cdot 040$ to three places of decimals, has rendered the work much more rapid.

I believe that the values obtained by this process are correct to within about one per cent. For the same determinant when reduced with different order of elimination agrees with its previous determination within less than that amount of discrepancy.

PART II.

§ 15. *Periodic Orbits.*

An orbit in which the third body can continually revolve, so as always to present the same character relatively to the two other bodies, is said to be periodic. If the motion is referred to a plane which is carried round with Jove and revolves about the Sun as a centre, any re-entrant orbit of the third body is periodic. Periodic orbits may consist of any number of revolutions round either of the primaries, or round other points in space. Periodic orbits, which are only re-entrant after several circuits, are much more difficult to discover than those which only make a single one; as hardly anything is known up to the present time about this subject, I determined to confine my attention to »simple periodic orbits», which are re-entrant after a single circuit. This definition of a simply periodic orbit must not preclude the consideration of orbits with loops, for the inclusion of such loops is necessary to the comprehension of the subject.

It appears from the differential equations of motion that periodic orbits must in general be symmetrical with respect to the line of syzygy; or if any periodic orbit consists of a closed circuit round a point which does not lie on this line, there must be a similar closed circuit round a symmetrical point on the other side of it.

Periodic orbits are critical cases which separate the orbits of one class from those of another, and the chief difficulty in tracing them consists in the fact that it is necessary to trace the gradual change of an orbit, as its parameters change, and to discover its form at the instant of its transformation into an orbit of a different character.

The partition of space derived from the Jacobian integral (§ 3) shows that the constant of relative energy C is of primary importance in the classification of orbits. The work of this investigation being numerical, I was compelled to assume a definite ratio for the mass of the Sun in terms of that of Jove; this ratio is taken as 10. The mass of the actual Sun in terms of that of the actual planet Jupiter is about 1000, and accordingly all the phenomena of perturbation are greatly exaggerated in our figures as compared with the real solar system. This exaggeration appeared to me advantageous for the purpose of giving a clear view of the phenomena.

The mass of the Sun being 10, that of Jove being unity and the distance between them being unity, we found in (9) that when C is greater than 40·1821 the third body must be either a superior planet, or an inferior planet, or a satellite, but cannot change from one of these conditions to another.

These larger values of C then bring us to those cases which are treated in the Planetary and Lunar Theories; I therefore cease my consideration of the problem for all values of C which are greater than 40·5. On the other hand C can never be less than 33. Hence the whole field to be treated is covered by the values of C between 33 and 40·5, and the problem is to obtain a complete synopsis of simply periodic orbits and of their stabilities between these limits.

The field of investigation is however so large that in the present paper I am compelled to make further restrictions. In the first place, the case of superior planets has not been touched at all; although, at the point at which I have now arrived, they must soon be taken into account.

Secondly all the orbits considered are direct; the retrograde orbits would afford an interesting field of research.

Lastly the present paper only covers the field from C equal to 38 to 40·5; and even this has occupied me for three years.

The slowness with which results are attained by arithmetical processes has been very tantalising, but the interest of the work has been sustained by the fact that the results have presented a succession of surprises. I have, over and over again, been deceived when I imagined I could foresee the shape which would be assumed by the next orbit to be treated, and thus the subject was continually presenting itself under a new light. Nevertheless a point has, I think, been now reached at which some forecasts are possible, and I shall venture to say something hereafter in § 19 on this head, with the full knowledge however that the conjectures may prove erroneous.

Being ignorant of the nature of the orbits of which I was in search, I determined to begin by a thorough examination of one case. It seemed likely that the most instructive results would be obtained from cases in which it should be possible for an inferior planet and satellite to interchange their parts. Now when C is greater than 38·8760 but less than 40·1821, the two interior ovals of the curve of zero velocity coalesce into the shape of an hour-glass, and thus interchange of parts is possible. I therefore began by the consideration of the case where C is 39, and traced a large number of orbits which start at right angles to SJ, and in some cases I also traced the orbit with reference to axes fixed in space.

The two curves, which represent the orbit in space and with reference to the moving plane, contain a complete solution of the problem.

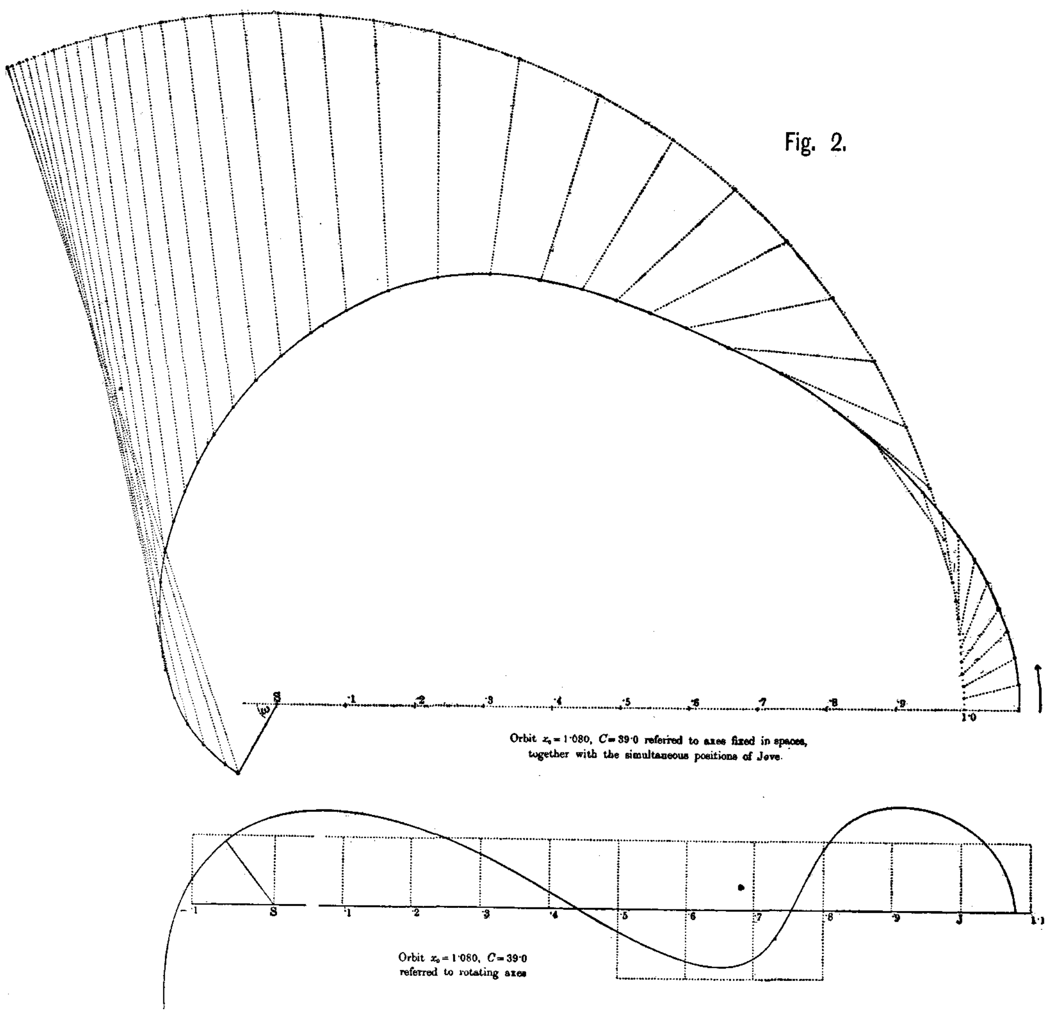
For if the curve on the moving plane be drawn as a transparency, and if the Sun in the two figures be made to coincide, and if the transparent figure be made to revolve uniformly about the sun, the intersection of the two curves will give the position of the body both in time and place.

In order to exhibit this I show in fig. 2 a certain orbit with reference to axes fixed in space and also the same orbit referred to rotating axes. In the former figure the simultaneous positions of the planet and of Jove are joined by dotted lines. It is interesting to observe how the body hangs in the balance between the two centres, before the elliptic form of the orbit asserts itself, as the body approaches the Sun.

This figure, and others of the same sort, are instructive as illustrating the usual sequence of events in orbits of this class.

If a planet be started to move about the Sun in an orbit of a certain degree of eccentricity, it will at first move with more or less exactness in an ellipse with advancing perihelion. But as the aphelion approaches conjunction with Jove the perturbations will augment at each passage of the aphelion. At length the perturbation becomes so extreme that the elliptic form of the orbit is entirely lost for a time, and the body will either revert to the Sun, or it will be drawn off and begin

Fig. 2.



a circuit round Jove. In either case after the approximate concurrence of aphelion with conjunction, the orbit will have lost all resemblance to its previous form.

The figure 2 exhibits the special case in which the body only makes a single circuit round Jove, and where the heliocentric elliptic orbit before and after the crisis has the same form; the perihelion has however advanced through twice the angle marked ω on the figure. In general the body would, after parting from the Sun, move several times round Jove until a concurrence of apojove with conjunction produced a severance of the connection, but in the figure this concurrence happens after the first circuit. If the neck of the hour-glass defining the curve of zero velocity be narrow, the body may move hundreds of times round one of the centres before its removal to the other.

It seems likely that a body of this kind would in course of time find itself in every part of the space within which its motion is confined. Sooner or later it must pass indefinitely near either to the Sun or to Jove, and as in an actual planetary system those bodies must have finite dimensions, the wanderer would at last collide with one of them and be absorbed. We thus gain some idea of the process by which stray bodies are gradually swept up by the Sun and planets.

It might be supposed that all possible orbits for any value of C will pass through a similar series of changes and that the bodies which move in them will be thus finally absorbed. Lord KELVIN is of opinion that this must be the case, and that all orbits are essentially unstable.[1] This may be so when sufficient time is allowed to elapse, but we shall see later that, even when the hour-glass has an open neck, there are still stable orbits, as far as our approximation goes. The only approximation permitted in this investigation is the neglect of the perturbation of Jove by the planet. For a very small planet the instability must accordingly be a very slow process, and I cannot but believe that the whole history of a planetary system may be comprised in the interval required for the instability to render itself manifest. Henceforward then I shall speak as though the stability of stable orbits were absolute, instead of being, as it probably is, only approximate.

§ 16. *Non-periodic orbits;* $C = 39{\cdot}0$.

(*a*) *Orbits round Jove.* Fig. 3.

The Sun S is outside of the figure towards the left. A small portion of the curve $2\Omega = 39$ is shown to the right of J, and another portion at the narrowing of the neck of the hour-glass. The two points of zero force given by $\frac{\partial\Omega}{\partial x} = 0$, $\frac{\partial\Omega}{\partial y} = 0$ (see § 3) are also marked.

The complete circuits are shown in order to obtain a better idea of the nature of the orbits, although this is unnecessary for the search for periodic orbits.

[1] Sir WILLIAM THOMSON, *On the Instability of Periodic Motion,* Philosophical Magazine, vol. 32, 1891, p. 555. M. POINCARÉ also considers that orbits may have a temporary, but not a secular stability. Acta Mathem. T. 13, 1890, p. 101.

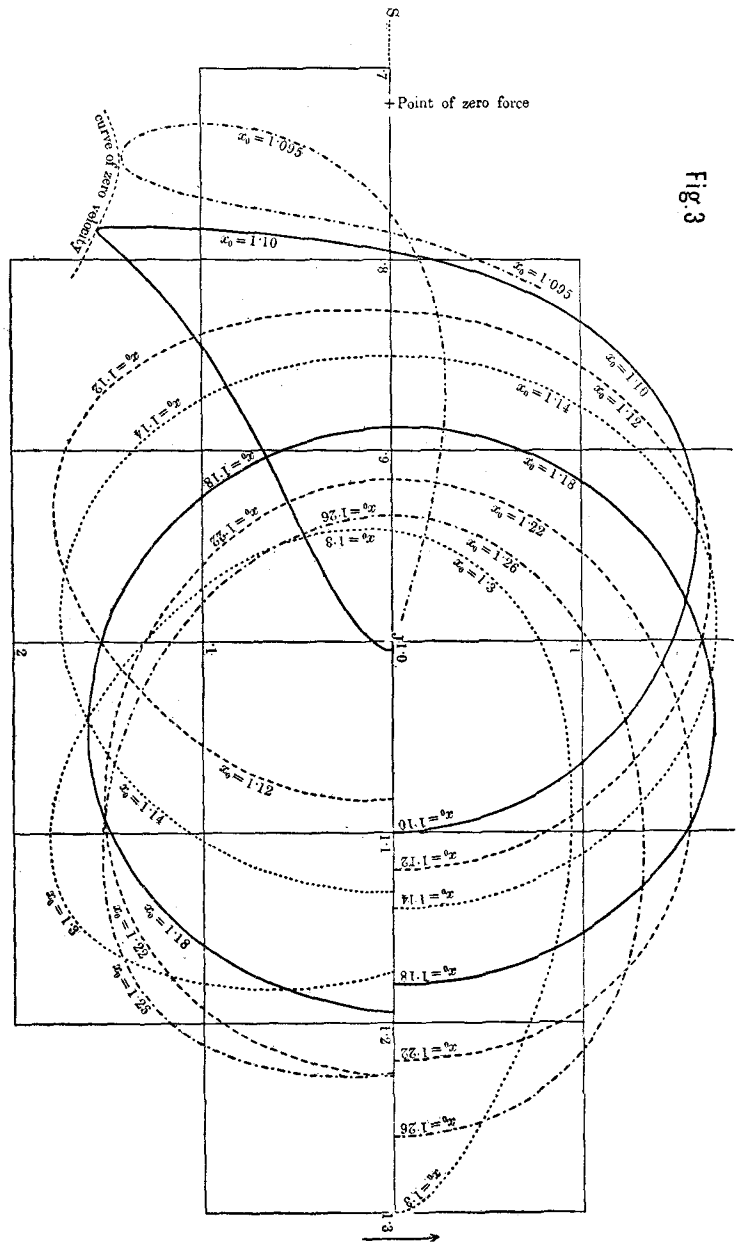
The satellite is supposed to be started at right angles to SJ at the conjunction remote from the sun, and enough of the orbits are shown to obtain a synopsis of the class. Here and elsewhere I define the orbits by the initial value of x, which is denoted by x_0; in this case the final value of x after the complete circuit may be called x_1.

The first on the right (dotted-line) starts with $x_0 = 1{\cdot}3$, and x_1 is much less than x_0. The second (chain-dotted) has $x_0 = 1{\cdot}26$, and x_1 has considerably increased so as to approach x_0. The third (broken-line) has $x_0 = 1{\cdot}22$, and x_1 has now become greater than x_0; therefore we have passed an orbit for which x_1 was equal to x_0, and such an orbit is periodic.

In the fourth (full-line) with $x_0 = 1{\cdot}18$, x_1 exceeds x_0 by more than it did in the third orbit. But in the fifth (dotted) with $x_0 = 1{\cdot}14$, x_1 has again become less than x_0; therefore we have passed another periodic orbit.

In the sixth orbit, (broken-line) $x_0 = 1{\cdot}12$, x_1 has decreased very much, and in the seventh (full-line) $x_0 = 1{\cdot}10$, x_1 has become quite small. This last has very nearly a cusp. It is not so accurately computed as the preceding ones, having been the first difficult orbit undertaken, and my methods at that time were not quite so satisfactory as they became subsequently. In this seventh orbit at the final intersection φ has just passed through the value zero, and I think it is probable that there is an orbit of very nearly this form, with the final φ exactly zero. Such an orbit would be periodic, but as it would not be simply periodic, it falls outside the scope of this paper.

The first part of the eighth orbit (chain-dot) was derived by interpolation between $x_0 = 1{\cdot}1$ and $x_0 = 1{\cdot}09$ (shown in a future figure); the beginning of this orbit, which I call $x_0 = 1{\cdot}095$, is not shown. It is a very remarkable curve, for after the loop, the body recrosses SJ, and going directly towards J, passes so close to it that it is impossible without more accurate computation to say what would happen subsequently. This orbit was so unexpected that I have thought it well to show in Fig. 4 its form with respect to axes fixed in space; in this figure (which does not claim close accuracy) the interpolated portion has been inserted. I do not think that any one could have conjectured how the body should have been projected so as to fall into Jove.



Non-periodic orbits round Jove, $C = 29{\cdot}0$

curve of zero velocity

+Point of zero force

For smaller values of x_0 the bodies are no longer simple satellites, as they part company with Jove and pass away towards the Sun.

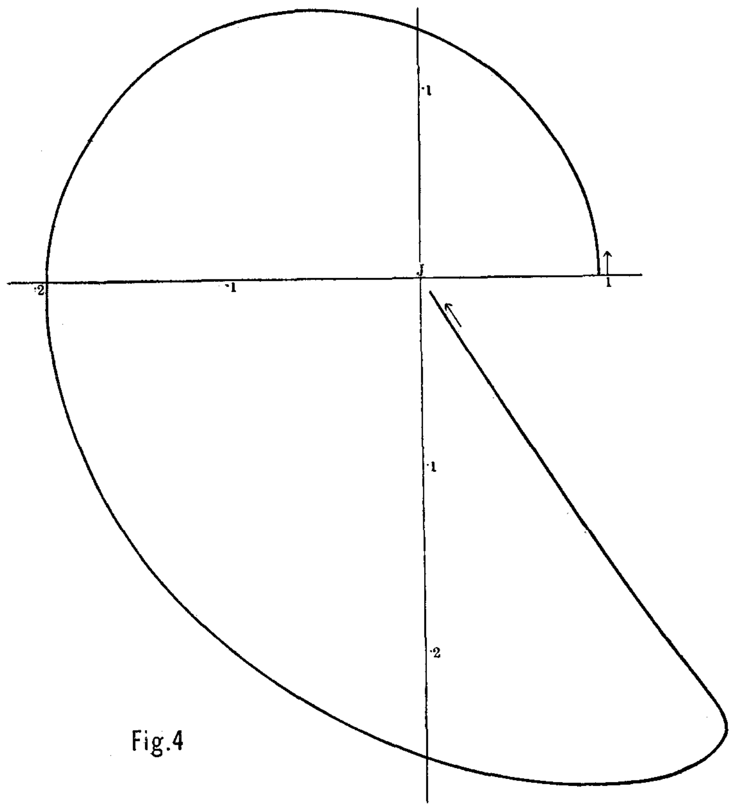


Fig. 4

Orbit round Jove referred to axes fixed in space ($x_0 = 1{\cdot}095$, $C = 39{\cdot}0$)

(β) *Orbits passing from Jove to Sun.* Fig. 5.

The curve of zero velocity for $C = 39$ having been computed, it is shown in this figure, although it is not necessary.

The starting points are again from conjunction remote from the Sun. The first orbit (broken-line) is the one with which we ended in Figure 3, viz. $x_0 = 1{\cdot}095$; the interpolated portion is however now drawn, as well as the computed portion. The body in this case does not pass away to the Sun.

We next come to an orbit (dotted) of which the first part was found by interpolation and which I call $x_0 = 1{\cdot}09375$; the earlier portion of the curve is not drawn.

Where the orbit $x_0 = 1{\cdot}095$ crosses SJ for the third time, φ is clearly negative, but where the orbit $x_0 = 1{\cdot}09375$ crosses for the third time φ is positive. There must therefore be an intermediate case for which φ vanishes, and this will give us a third periodic orbit round J. The orbit $x_0 = 1{\cdot}09375$ passes away to the sun; and we then come to four more orbits $x_0 = 1{\cdot}09, 1{\cdot}08, 1{\cdot}06, 1{\cdot}04$ which follow a similar course, but with diminishing depression towards the negative side of SJ. The next orbit is $x_0 = 1{\cdot}02$, in which the depression has disappeared. This curve has a slight hump in the place of the depression; it is the sort of feature which would present itself in a computed curve, when there has been an arithmetical error in the calculation, but we shall soon see that this hump is not explicable in this way.

The next curve which is traced (although others have been computed)

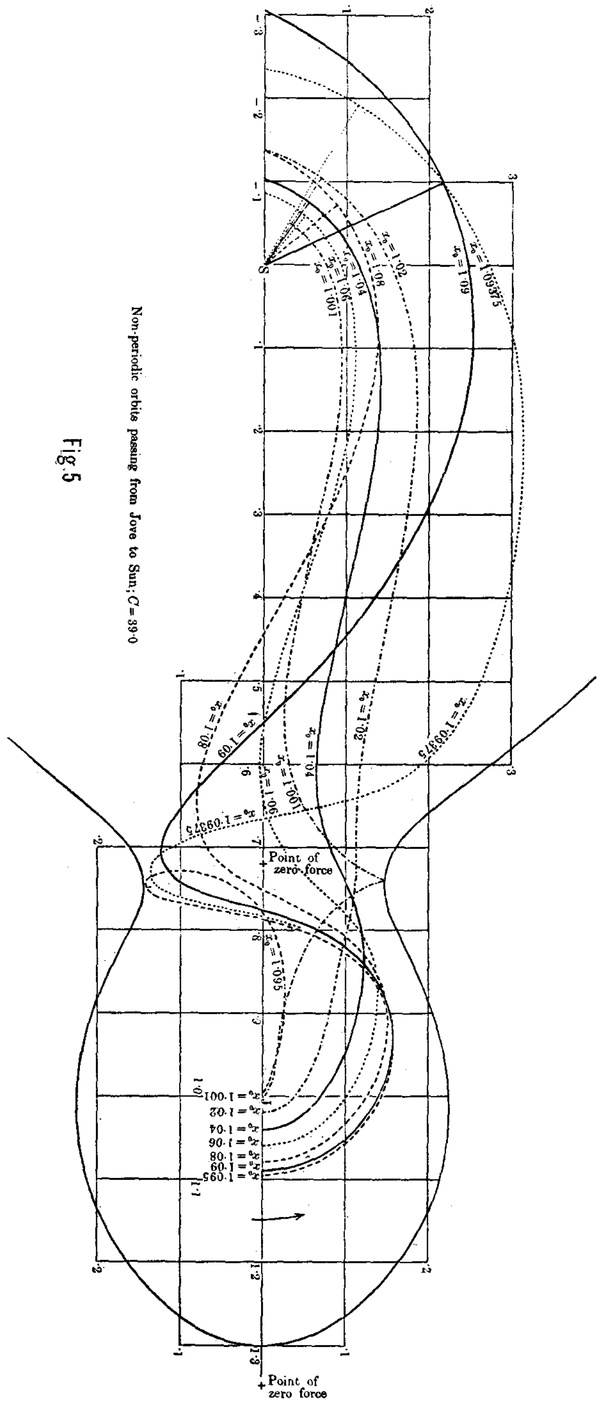


Fig. 5

Non-periodic orbits passing from Jove to Sun; $C = 39{\cdot}0$

starts with $x_0 = 1{\cdot}001$ (chain-dot); in a figure of this scale, it apparently starts actually from J. It will be observed that we now have a remarkable cusp, and it becomes obvious that the hump referred to above was an incipient elevation towards the cusp.

Passing now to the other end of the figure where the body passes round the Sun, we see from the incidence of the perihelia (which are indicated by radii from the Sun) that there can be no periodic orbit which is partly the path of a satellite and partly that of a planet; for such an orbit must have the longitude of the perihelion 180°.

The positions of the perihelia and the perihelion-distances seem to be almost chaotic in the figure, but I believe that the calculations are substantially correct, and a consideration of the numbers representing the positions of the perihelia shows that the chaos is rather apparent than real.

The following table gives the results:

Name of orbit.	Longitude of Perihelion.	Perihelion Distance.
$x_0 = 1{\cdot}001$	$\pi - 32°45'$	·058
$= 1{\cdot}02$	$\pi - 34°$	·125
$= 1{\cdot}04$	$\pi - 35°45'$	·093
$= 1{\cdot}06$	$\pi - 39°15'$	·078
$= 1{\cdot}08$	$\pi - 52°15'$	·115
$= 1.09$	$\pi - 64°15'$	·240
$= 1{\cdot}09375$	$\pi - 30°45'$	·222

Now if we were to plot out the defect of the longitudes from 180°, taking x_0 as abscissa and the defects of longitude as ordinates, we should obtain a sweeping curve starting from a minimum of 33°, rising to a maximum of 64°, and falling abruptly to 31°. If the perihelion-distances be treated similarly, we find a somewhat less satisfactory curve, for there is a small maximum, then a minimum and then a large maximum, followed by a fall in value. As I have said above, I believe that these results are substantially correct; but as each one of these curves represents three or four weeks hard work, I have not thought it good economy of labour to pursue the inquiry further in this respect.

(γ) *orbits round the Sun;* $C = 39{\cdot}0$. Fig. 6 (see p. 181).

These curves are drawn with less accuracy than the others, being computed with three-figured logarithms. I thought that sufficient accuracy would be attainable with this degree of approximation, but when I found that the saving of labour was not considerable, whilst the loss of accuracy was very great, I returned to the use of four-figured tables. It did not however seem necessary to recompute these curves.

The complete circuit is drawn for four of the curves, but the rest are only carried half way round.

The orbits start to the left of the Sun at the conjunction remote from Jove. The first orbit is $x_0 = -{\cdot}6$ (full-line), and at the second crossing of the line of conjunction the angle φ is negative. The second orbit $x_0 = -{\cdot}4$ (dotted) has φ positive, but small, at the second crossing; hence there is a periodic orbit for a value of x_0 a little less than $-{\cdot}4$.

All the succeeding orbits viz. $x_0 = -{\cdot}337$, $-{\cdot}3$, $-{\cdot}2$, $-{\cdot}1$, $-{\cdot}04$, $-{\cdot}001$ have φ positive and successively increasing at the second crossing; and thus there is no other periodic orbit. The last two of these orbits have loops.

The orbit $x_0 = -{\cdot}337$ was found in part by interpolation. It has been inserted because the third crossing of the line SJ appears to be orthogonal, and therefore the orbit is periodic, but not *simply* periodic. No search was being made for this sort of orbit, and the discovery was accidental.

§ 17. *Periodic Orbits classified according to values of C.*

Plates I, II, III.

Plate I, fig. 1. $C = 40{\cdot}0$.

When C is greater than $40{\cdot}18$, the inner branches of the curve of zero velocity, $2\Omega = C$, consist of two ovals, as seen in fig. 1; the periodic orbits then consist of two approximately circular orbits round S and J respectively. These cases may be treated by the methods of the Planetary and Lunar Theories, and fall outside the scope of this paper.

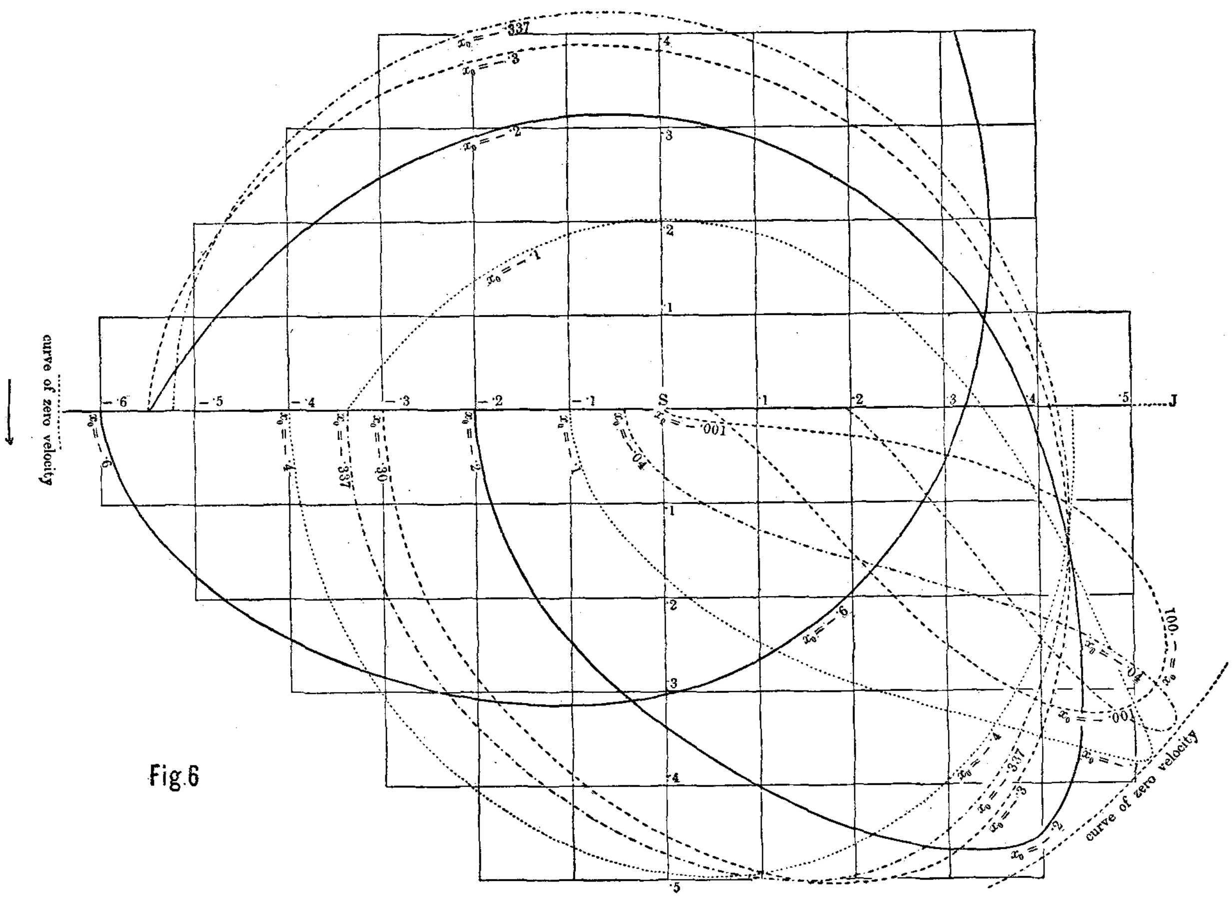


Fig. 6

Non-periodic orbits round the Sun; $C=39\cdot0$

When $C = 40\cdot18$ there is a third periodic orbit consisting of the point $x = \cdot7175$, $y = 0$. At this point a body is in unstable equilibrium, and this point is the beginning of a family of orbits; for, whilst in general periodic orbits begin in pairs, a single orbit may begin at a point.

In discussing these figures I shall denote the initial value of any function by the suffix 0; the suffix 1 will denote the value after the completion of a half circuit, and the suffix 2 the value on the completion of the whole circuit.

The planet A starts from $x_0 = -\cdot 414$, $\varphi_0 = \pi$, and φ increases.

When $x_0 \lessgtr -\cdot 414$, $\varphi_1 \lessgtr 0$, $x_2 \gtrless x_0$ ($x_0 < -\cdot 414$ of course denotes a starting point more remote from S, with x_0 numerically greater than $\cdot 414$).

This orbit is stable with $c = 2\cdot 81$.

The satellite A starts from $x_0 = 1\cdot 03341$, $\varphi_0 = 0$, and φ increases.

This orbit changes its shape rapidly with changes of C, as will appear below in the classification by families. Great care was bestowed on this case, and it was very troublesome to compute, since a considerable variation of x_0 corresponded with a small variation of φ_1.

When $x_0 \gtrless 1\cdot 03341$, $\varphi_1 - \pi \lessgtr 0$, $x_2 \lessgtr x_0$.

The orbit is stable, but borders closely on instability, with $c = 3\cdot 7$.

The third orbit is the oscillating satellite a, moving slowly with a retrograde revolution round the point of zero force $x = \cdot 7175$, $y = 0$, which was described above as the commencement of a series of orbits.

The orbit a starts from $x_0 = \cdot 705$, $\varphi_0 = 0$, and φ diminishes.

When $x_0 \gtrless \cdot 705$, $\varphi_1 - \pi \gtrless 0$. That is to say if the body starts too near to Jove the change of direction at the sharp turn is not quite sufficient for periodicity; and if it starts too near the Sun the converse is true. In the first case after one or more circuits the body passes away towards J, and in the second case towards S.

This orbit is very unstable, and the instability is almost certainly of the even type.

Plate I, fig. 2. $C = 39 \cdot 5$ and $39 \cdot 3$.

The planetary orbit A ($C = 39 \cdot 5$) differs little from the preceding case.

It starts from $x_0 = - \cdot 424$, $\varphi_0 = \pi$, and φ increases.

When $x_0 \lesseqgtr - \cdot 424$, $\varphi_1 \lesseqgtr 0$, $x_2 \gtrless x_0$.

The orbit is stable with $c = 2 \cdot 90$; but it is less stable than when $C = 40$.

The classification by families below shows that as C falls below $40 \cdot 0$, the orbit of the satellite A stretches out rapidly towards S, and at the same time the oval a expands. When C is very little greater than $39 \cdot 5$ (perhaps about $39 \cdot 6$), these two curves touch one another.

At this stage the body may either move entirely on A or entirely on a, or it may move alternately on A and on a, thus describing a figure-of-8.

When C has diminished to $39 \cdot 5$ there is no alternative; for the orbit A is necessarily a figure-of-8, whilst the orbit a remains a closed oval.

The satellite A starts from $x_0 = 1 \cdot 0650$, $\varphi_0 = 0$, and φ begins increasing. When the body has passed half round J so that y vanishes, φ is equal to $\pi - 15° 37'$; shortly after this φ diminishes and continues doing so until when y again vanishes $\varphi_1 = 0$.

We have $x_0 \gtrless 1 \cdot 0650$, $\varphi_1 \lesseqgtr 0$. When the body starts too far from J, it will move in some orbit round J, and when it starts too near J it will pass away to S.

This orbit is very unstable with even instability.

The oscillating orbit a was not computed for $C = 39{\cdot}5$;[1] during one part of its course it would be indistinguishable from part of A, and the rest is shown conjecturally by a dotted line.

This orbit is very unstable, with even instability.

It has already been remarked that after the first half circuit of satellite A φ was $\pi - 15° 37'$, or as we may now write it $\pi - \varphi_1 = 15° 37'$. Now when x_0 is made to increase from $1{\cdot}0650$ until it reaches the curve $2\Omega = C$, $\varphi_1 - \pi$ will always be negative, or $\pi - \varphi_1$ positive. It appears however that $\pi - \varphi_1$ has a minimum value, which very nearly reaches zero. In fact when $x_0 = 1{\cdot}140$, $\varphi_1 = \pi - 0° 20'$.

Since $\pi - \varphi_1$ is large when x_0 approaches $2\Omega = C$, and is $15° 37'$ when $x_0 = 1{\cdot}0650$, it follows that if it vanishes at all, it must vanish twice. That is to say if there is another periodic orbit, there must be two.

As C diminishes the minimum value of $\pi - \varphi_1$ falls, and I found that when $C = 39{\cdot}4$ the minimum is reached when x_0 is about $1{\cdot}15$; for this value of x_0, $\pi - \varphi_1$ is $0° 9'$, and there is still no value of x_0 for which $\pi - \varphi_1$ vanishes.

But when $C = 39{\cdot}3$ I computed the four orbits $x_0 = 1{\cdot}18$, $1{\cdot}17$, $1{\cdot}16$, $1{\cdot}15$ and found that for the two middle ones $\pi - \varphi_1$ was negative. By interpolation the pair of periodic orbits B and C were found.

The orbit B is given by

$$x_0 = 1{\cdot}1575, \qquad \varphi_0 = 0;$$

and the orbit C by

$$x_0 = 1{\cdot}1751, \qquad \varphi_0 = 0.$$

In both cases φ increases.

The relationship to the neighbouring orbits is given by the inequalities

$$x_0 > 1{\cdot}1751\,,\ \varphi_1 - \pi < 0\,,\ x_2 < x_0.$$

$$x_0 \begin{matrix} < 1{\cdot}1751 \\ > 1{\cdot}1575 \end{matrix}\,,\ \varphi_1 - \pi > 0\,,\ x_2 > x_0.$$

$$x_0 < 1{\cdot}1575\,,\ \varphi_1 - \pi < 0\,,\ x_2 < x_0.$$

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[1] At least the computation was not completed, for it was found to be so troublesome, that it appeared that the work could be better bestowed elsewhere.

The orbit B is slightly unstable, with even instability, and $c = \cdot 156\sqrt{-1}$; the orbit C is stable, but approaches instability, and $c = 2\cdot 163$.

Plate II, fig. 1. $C = 39\cdot 0$.

These are the periodic orbits which belong to the families of non-periodic orbits shown in figs. 3, 4, 6 above.

The planetary orbit A starts from $x_0 = -\cdot 434$, $\varphi_0 = \pi$, and φ increases. The incidence amongst the neighbouring orbits is shown by the inequalities

$$x_0 \lesseqgtr -\cdot 434 \ , \ \varphi_1 \lesseqgtr 0 \ , \ x_2 \gtreqless x_0.$$

This orbit is unstable with slight uneven instability and $c = 1 + \cdot 10\sqrt{-1}$. It thus appears that for some value of C between $39\cdot 5$ and $39\cdot 0$ we should find the passage of the planetary orbit A from stability to instability. It is certainly surprising to find that the instability of the planet sets in when the planet is a little less than half way to Jove at conjunction.

The satellite A starts from $x_0 = 1\cdot 0941$, $\varphi_0 = 0$, and φ increases until when y vanishes it is equal to about $\pi - 13° 30'$; it then diminishes to zero.

Its incidence among neighbouring orbits (figs. 3 and 4) is given by the inequalities

$$x_0 \gtreqless 1\cdot 0941 \ , \ \varphi_1 \lesseqgtr 0.$$

When it starts too far from Jove it will move in some orbit round J, and when it starts too near Jove it will pass away towards S.

This orbit is very unstable, with even instability and $c = \cdot 46\sqrt{-1}$.

The orbit of the oscillating satellite a is indistinguishable from A throughout part of its course, but falls more remote from J on the side towards S. It starts from $x_0 = \cdot 687$, $\varphi_0 = 0$, and φ diminishes.

When $x_0 \gtreqless \cdot 687$, $\varphi_1 - \pi \gtreqless 0$; thus if the body starts too near Jove the total change of direction is insufficient for periodicity; and if it starts too near the Sun the converse in true. In the first case it passes away towards Jove, and in the second towards the Sun.

This orbit is very unstable with even instability, and c is about $2\sqrt{-1}$.

The satellite B starts with $x_0 = 1 \cdot 1500$, $\varphi_0 = 0$, and φ increases.

When $x_0 \gtrless 1 \cdot 1500$, $\varphi_1 - \pi \gtrless 0$, $x_2 \gtrless x_0$.

This orbit is unstable, with even instability, and $c = \cdot 38 \sqrt{-1}$.

The satellite C starts with $x_0 = 1 \cdot 2338$, $\varphi_0 = 0$, and φ increases.

When $x_0 \gtrless 1 \cdot 2338$, $\varphi_1 - \pi \lesseqgtr 0$, $x_2 \lesseqgtr x_0$.

This orbit is stable, with $c = 2 \cdot 46$.

Plate II, fig. 2. $C = 38 \cdot 5$.

The planet A starts from $x_0 = - \cdot 444$, $\varphi_0 = \pi$, and φ increases.

When $x_0 \lesseqgtr - \cdot 444$, $\varphi_1 \lesseqgtr 0$, $x_2 \gtrless x_0$.

The orbit is unstable, with uneven instability and $c = 1 + \cdot 18 \sqrt{-1}$.

The satellite A starts from $x_0 = 1 \cdot 1164$, $\varphi_0 = 0$, and φ increases until when y vanishes it is equal to about $\pi - 12°$; it then diminishes to zero. It will be observed that at the first vanishing of y, the curve cuts the axis more nearly at right angles than was the case when $C = 39 \cdot 0$ and $39 \cdot 5$. When $x_0 \gtrless 1 \cdot 1164$, $\varphi_1 \lesseqgtr 0$. When it starts too far from Jove it will move in some orbit round J, and when its starts too near Jove it will pass away to the Sun. The orbit is very unstable, with even instability.

The oscillating satellite a starts with $x_0 = \cdot 6814$, $\varphi_0 = 0$, and φ diminishes. When $x_0 \gtrless \cdot 6814$, $\varphi_1 - \pi \gtrless 0$. In the first case it passes away towards Jove, in the second towards the Sun. The orbit is very unstable with even instability.

The satellite B starts with $x_0 = 1 \cdot 1497$, $\varphi_0 = 0$ and φ increases.

When $x_0 \gtrless 1 \cdot 1497$, $\varphi_1 - \pi \gtrless 0$, $x_2 \gtrless x_0$.

The orbit is unstable with even instability, and $c = \cdot 70 \sqrt{-1}$.

The satellite C starts with $x_0 = 1 \cdot 2760$, $\varphi_0 = 0$, and φ increases.

When $x_0 \gtrless 1 \cdot 2760$, $\varphi_1 - \pi \lesseqgtr 0$, $x_2 \lesseqgtr x_0$.

This orbit is very unstable, and as will appear below the instability is uneven. There has in fact been a passage from stability to uneven instability for some value of C between $39 \cdot 0$ and $38 \cdot 75$.

This orbit is interesting because it corresponds almost exactly to the cusped orbit described by Mr Hill as the moon of greatest lunation. It would seem however that this description is incorrect, for the satellite C moves with a still longer period when the cusp is replaced by a loop. Mr Hill's orbit was, on the account of his approximation, necessarily a symmetrical one with reference to the line of quadratures, but it will be observed that when the solar parallax is taken into account the orbit is very unsymmetrical.

When $C = 38{\cdot}88$ à new periodic orbit arises in the point $x_0 = 1{\cdot}3470$, $y = 0$ marked in the figure. This is the beginning of a second family of oscillating satellites, referred to here as b.

When $C = 38{\cdot}5$ this orbit begins with $x_0 = 1{\cdot}2919$, $\varphi_0 = 0$, and φ diminishes.

When $x_0 \gtrless 1{\cdot}2919$, $\varphi_1 - \pi \gtrless 0$. That is to say if the body starts too far from Jove for periodicity, it will pass away in an orbit as a superior planet; if on the other hand it starts too near Jove for periodicity, it will pass to some orbit about Jove. This orbit is very unstable.

Plate III, fig. 1. $C = 38{\cdot}0$.

The planet A starts from $x_0 = -{\cdot}455$, $\varphi_0 = \pi$, and φ increases.

When $x_0 \lesseqgtr -{\cdot}455$, $\varphi_1 \lesseqgtr 0$, $x_2 \gtrless x_0$.

The orbit is unstable, with uneven instability, and $c = 1 + {\cdot}193\sqrt{-1}$.

The satellite A starts from $x_0 = 1{\cdot}1305$, $\varphi_0 = 0$, and φ increases.

When $x_0 \gtrless 1{\cdot}1305$, $\varphi_1 \lesseqgtr 0$. The remarks concerning this orbit in previous cases apply again here.

At the point where the orbit crosses the axis of x for the second time $\pi - \varphi$ is less than it was in the preceding case.

The oscillating satellite a starts from $x_0 = {\cdot}6760$, $\varphi_0 = 0$ and φ decreases. When $x_0 \gtrless {\cdot}6760$, $\varphi_1 - \pi \gtrless 0$. It is very unstable, with even instability.

The satellite B starts from $x_0 = 1{\cdot}1470$, $\varphi_0 = 0$, and φ increases. When $x_0 \gtrless 1{\cdot}1470$, $\varphi_1 - \pi \gtrless 0$, $x_2 \gtrless x_0$. The orbit is very unstable with even instability, and $c = {\cdot}96\sqrt{-1}$.

The orbit B is on the point of coalescing with part of the orbit A, for the crossing point of the figure-of-8 in A is tending to become perpendicular to SJ, and the two curves nearly coincide.

The satellite C starts from $x_0 = 1{\cdot}2480$, $\varphi_0 = 0$, and φ increases.

When $x_0 \gtrless 1{\cdot}2480$, $\varphi_1 - \pi \lesseqgtr 0$, $x_2 \lesseqgtr x_0$.

This orbit was very troublesome, and is not computed with a high degree of accuracy. A very small variation of C would make a large change in the size of the loops in the curve.

The orbit is very unstable with uneven instability.

The oscillating satellite b starts with $x_0 = 1·2595$, $\varphi_0 = 0$, and φ decreases.

When $x_0 \gtrless 1·2595$, $\varphi_1 - \pi \gtrless 0$. The remarks made concerning this curve for $C = 38·5$ apply again here.

This orbit is very unstable.

The orbit C seems to be about to coalesce, in part of its course, with the loop b.

§ 18. *Classification of orbits by families.*

Several orbits are given in this classification which were not included in § 17.

Table of results.

Constant of Energy C	Coord. of starting point x_0	Synodic Period nT	Criterion of Stability $\Delta \sin^2 \frac{1}{2}\pi\sqrt{\Phi_0}$	Apparent advance of pericentre in synodic period $2\pi\left(\frac{1}{2}c-1\right)$	Regression of pericentre in sid. period $2\pi\left(1-\frac{\frac{1}{2}c}{1+\frac{nT}{2\pi}}\right)$	Description of instability	Modulus of instability $\frac{\log\sqrt{2}}{\log[D+\sqrt{D^2\pm 1}]}$	Remarks
				Satellite A, Plate IV, fig. 1.				
40·5	1·1135	61° 20′	+ ·112	39° 0′	22° 20′	....	....	
40·25	1·1150	65° 40′	+ ·063	29° 0′	31° 0′	....	....	minimum of criterion
40·2	1·1090	66° 50′	+ ·064	29° 10′	31° 40′	....	....	maximum of x_0
40·0	1·0334	98° 0′	+ ·226	303°	— 161°	....	....	minimum of x_0
39·5	1·0650	229°	— ?	....	....	even	?	Figure-of-8 begins
39·0	1·0941	240°	— 1·06	....	....	even	0·5	
38·5	1·1164	258°	— ?	....	....	even	?	
38·0	1·1305	299°	— ?	....	....	even	?	
				Satellite B, Plate IV, fig. 2.				
39·3	1·1575	87° 40′	— ·061	....	....	even	1·42	
39·0	1·1500	97° 0′	— ·402	....	....	even	0·58	
38·5	1·1497	113° 20′	— 1·82	....	....	even	0·31	
38·0	1·1470	131° 50′	— 4·5	....	....	even	0·23	
				Satellite C, Plate IV, fig. 3.				
39·3	1·1751	89° 20′	+ ·064	81° 0′	24° 30′	....	....	
39·0	1·2338	114° 0′	+ ·435	82° 40′	23° 30′	....	....	
38·75	1·2873	179° 30′	+ 1·95	....	....	uneven	0·4	maximum of x_0
38·5	1·2760	210° 50′	> + 1	....	....	uneven	?	
38·0	1·2480	235° 20′	> + 1	....	....	uneven	?	

Constant of Energy C	Coord. of starting point x	Synodic Period nT	Criterion of Stability $\Delta \sin^2 \frac{1}{2}\pi\sqrt{\Phi_0}$	Apparent advance of pericentre *in* synodic period $2\pi\left(\frac{1}{2}c-1\right)$	Regression of pericentre in sid. period $2\pi\left(1-\frac{\frac{1}{2}c}{1+\frac{nT}{2\pi}}\right)$	Description of instability	Modulus of *instability* $\frac{\log\sqrt{2}}{\log[D+\sqrt{D^2\pm1}]}$	Remarks
				Oscillating Satellite *a*.				
40·18	·7175	....	— ?	....	....	even		a point on *SJ*
40·0	·705	138°	— ?	....	....	even	?	
39·5	·693	?	— ?	....	....	even	?	
39·0	·687	146°	— 148	....	....	even	0·1	
38·5	·681	150°	— ?	....	....	even	?	
38·0	·676		— ?	....	....	even	?	
				Oscillating Satellite *b*.				
38·88	1·3470	....	?			?	?	a point on *SJ*
38·5	1·2919	214° ?	?	....	....	?	?	
38·0	1·2595	208° ?	?	....	....	?	?	
				Planet *A*, Plate IV, fig. 4.				
40·0	— ·414	154°	+ ·91	145°	6° 30′	....	....	
39·5	— ·424	165°	+ ·98	162°	2°	....	....	
39·0	— ·434	177°	+ 1·03	....	....	uneven	2·1	
38·5	— ·444	191°	+ 1·08	....	....	uneven	1·25	
38·0	— ·455	207°	+ 1·09	....	....	uneven	1·14	

Although the above table gives most of the facts, it will be well to draw attention to a few important points.

The passage of the family *A* of satellites into the figure-of-8 form is interesting. When *C* lies between 40·18 and some value a little less than 40·0, the oval orbit *A* and the oval *a* must be considered, in an algebraical sense, as a single orbit. But I think that we must imagine *a* to be described twice, so that when one of the two *a* orbits fuses with *A* to form the 8, the other may maintain a separate existence. The doctrine of the double nature of *a* receives confirmation from what is pointed out below in § 19 as to the manner in which the *C* orbit fuses with the oval *b*.

I think it is almost certain that a more complex sort of figure-of-8 also exists, for we may imagine a body which describes two, three or more circuits round the point of zero force in an oval like a, before passing off into the branch round Jove.

We have seen that the confluence of a circuit round a alone with a circuit round a and round A leads to a figure-of-8 and a circuit round a. It seems likely then that a pair of complex figures-of-8, one with a double circuit round a and the other with a triple circuit may spring from a single orbit. However these orbits can hardly be described as simply periodic, and I have not considered them in detail.

It appears from our table that the satellite orbit A is stable, but with only a very small margin of stability when $C = 40$. It is worthy of note that the criterion of stability after passing a minimum value of ·063, is rapidly increasing, so that the orbit is tending towards uneven instability. I do not know whether or not that instability has set in before the fusion with the oval a and the formation of the figure-of-8 orbit A; but the figure-of-8 is evenly unstable, and we thus have the fusion of a stable, or unevenly unstable, orbit with an evenly unstable orbit, and the resultant is evenly unstable.

This throws light on the fusion of the planetary orbit A with the oval a, which must occur for a value of C less than 38. In the case of the planet we have seen that $\Delta \sin^2 \frac{1}{2}\pi\sqrt{\Phi_0}$ has increased until it is greater than unity and there has as yet been no fusion with a. Hence amongst the planetary orbits we shall have the fusion of an unevenly unstable orbit with an evenly unstable one, and the resultant will be evenly unstable.

M^r^ Hill has drawn an interesting family of orbits of satellites, beginning with the orbit of the moon and ending with a cusped orbit. Now our moon undoubtedly belongs to the family A, whilst the cusped and looped orbits belong to the family C. He neglects the solar parallax, and this approximation has in fact led to the absorption of two families into one another. It appears now that it is not possible to comprehend the part played by this class of orbit without the inclusion of the solar parallax, for the asymmetry of the family C with regard to the line of quadratures is an essential feature in it. This will appear still more clearly in the next section.

M^r^ Hill draws attention to the minimum of distance at syzygies in the orbits of satellites, and this is observable in our family C, but we also find a maximum of distance in the family A at the superior syzygy.

The periods of some of the satellites is extraordinarily long, that of the last figure-of-8 A being $\frac{299}{360}$ or $\frac{5}{6}$ of that of Jove, and that of the last looped orbit C being $\frac{235}{360}$ or nearly $\frac{2}{3}$ of that of Jove.

§ 19. *On the probable forms of periodic orbits for values of C less than* 38.

It is obvious from Plate III, fig. 1 that a portion of the figure-of-8 orbit A and the orbit B will coalesce for some value of C a little less than $38{\cdot}0$. The oval a will however continue to exist and to expand.

The planetary orbit A will continue to expand, but the heliocentric distance at the conjunction remote from J will shortly reach a maximum and will then diminish, whilst the heliocentric distance at the other conjunction will increase rapidly. This will continue until the planetary orbit A touches the oval a; a new series of figure-of-8 planetary orbits will then arise, and the heliocentric distance at the remote conjunction will then increase.

At some stage a pair of new planetary orbits B and C will arise from a single orbit; of these B will be evenly unstable and C stable.

The orbit B will expand, coalesce with a portion of A, and then both will disappear.

Reverting now to the satellite C, we are able to foresee its future course. The fig. 2, Plate II and fig. 1, Plate III or fig. 3, Plate IV, show the growth of the two loops from two cusps. In order to throw light on the future development of these curves I have drawn Plate III, fig. 2, which shows a non-periodic orbit for $C = 38{\cdot}5$;[1] in it we see that the upper loop has descended below the line of conjunction, and the lower loop has risen above. For some value of C a little less than

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[1] It would have been better to have drawn the similar curve for $C = 38{\cdot}0$, but this one suffices for the present purpose.

38 there must be a periodic orbit of this general form. We shall thus have a periodic orbit with five full moons in the month. In this sort of orbit the crossing point P will be at first a point of contact; the distance JP will then diminish to a minimum and afterwards increase. When P has moved outwards and Q has moved inwards, so as to meet, the upper loop will have spread so as to coincide with the lower, and the lower with the upper, and both will coincide with the oval b. I think that after this stage the orbit C will disappear, but the oval b will continue to exist.

This conclusion is interesting when taken in connection with the looped orbit to which M. POINCARÉ [1] drew attention, and which has been traced by Lord KELVIN.[2] They both neglected the solar parallax, and with the degree of approximation adopted by them, the central space might be made as small and the loops as large as we like. But the inclusion of the solar parallax now appears to be essential to the proper consideration of these orbits.

It appears from fig. 1 that when $C = 34{\cdot}91$, there is a new periodic orbit consisting of the point $x = -{\cdot}9469$, $y = 0$. This point is the origin of a new family of oscillating planets, say c, which describe ovals with retrograde revolution round the point of zero force, for values of C less than 34·91

Turning now to our conjectural planetary orbit C, we see that whilst initially it will be nearly circular, it will ultimately produce two excrescences near the ends of the oval c. These excrescencces will become cusps, and then loops; the loops will cross one another, become identical with one another and with the oval c, and the orbit C will probably then disappear.

The case of the superior planets has not yet been considered, and there is not much concerning them of which I feel confident.[3] It is obvious however that they are described with an apparently retrograde revolution, and that they contract as C falls in value. The orbits will be nearly circular, but will bulge inwards in the neighbourhood of Jove. At

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[1] Méc. Cél., p. 109.

[2] Phil. Mag., Nov. 1892.

[3] I have now (July 1897) traced some of them.

some stage the inward depression of the orbit will meet the oval b in contact. This stage will be the commencement of a new family of orbits. having the form of a sort of inverted figure-of-8. If the old figure-of-8 be likened to two circles touching one another externally, the new figure may be compared with a small circle touching a large one internally. A similar series of changes must ultimately take place with the oval c, and probably we may have an orbit with loops at both ends of the line of conjunctions.

I will not hasard detailed conjectures as to the future of the three ovals a, b, c. I think however that it is probable that they will stretch out towards the vertices of the two equilateral triangles which may be erected on SJ as base. These vertices must be themselves the origins of a pair of similar ovals, and perhaps the extremities of a, b, c will stretch out to contact with this fourth system of ovals.

§ 20. *Classification of stable orbits of satellites.*

We have seen that amongst satellites there are two classes of stable orbits, namely those of the A and C families. Plate III, fig. 3 exhibits the limits of the orbits which have been shown to be stable. The exact orbits which possess limiting stability would of course differ slightly from those drawn in this figure.

When C is large the stable orbits of the A family are approximate circles of small radius. As C decreases the orbits swell, but when C reaches $40{\cdot}25$ the radius vector at superior syzygy reaches a maximum. Hence the orbit $x_0 = 1{\cdot}1150$, $C = 40{\cdot}25$ gives one limit of the stable orbits of this family. The orbit $x_0 = 1{\cdot}0334$, $C = 40{\cdot}0$ gives approximately another limit as regards the inferior syzygy. The shaded space between these two orbits is filled with stable orbits.

The stable orbits of the C family begin when C is a little greater than $39{\cdot}3$, and the first one traced is that for which $x_0 = 1{\cdot}1751$ and $C = 39{\cdot}3$. The stability of these orbits still subsists when $C = 39{\cdot}0$, but this orbit is already very unstable when C has fallen to $38{\cdot}75$. Accordingly I take for the other limit of orbits of this kind $x_0 = 1{\cdot}2338$, $C = 39{\cdot}0$. The shaded space between these two is filled with stable orbits.

It will be observed that there remains an unshaded tract within which no stable orbit can exist. I think moreover that it is probable that with a smaller mass for Jove we should have found a complete annulus within which stability is impossible.

This conclusion is interesting when viewed in connection with the distribution of the satellites and planets of our system, and it appears to me to be the first exact result, which throws any light on Bode's empirical law as to the mean distances of planets and satellites from their primaries.

It is as yet too soon to make a similar classification of stable planetary orbits, but this will follow in due course.

We have seen in an earlier section that unstable orbits are such as ultimately to lead to the absorption of bodies moving in them into one or other of the perturbing centres. If there were a large number of perturbing centres, as in our planetary system, the problem would become incomparably more difficult, but I think that the present investigation affords evidence that if we were to have a system consisting of a large planet moving round the sun, and of a cloud of infinitesimal bodies circling about them, a system would ultimately be evolved where there would be inferior and superior planets and a pair of satellites moving in certain zones indicated by our figures.

Postscript.

It is stated in § 3, p. 112 that if the third body be placed at the vertex of the equilateral triangle drawn on SJ, it is stable. I have to thank Mr S. S. Hough for pointing out to me that this is not universally true, but that if Jove is greater in mass than one twenty-fifth of the Sun, such a body is unstable.

This may be proved as follows:

The coordinates of the point for which $r = \rho = 1$ are $x = \frac{1}{2}$, $y = \frac{1}{2}\sqrt{3}$; also $\frac{\partial \Omega}{\partial x} = \frac{\partial \Omega}{\partial y} = 0$, but $\frac{\partial^2 \Omega}{\partial x^2} = \frac{3}{4}(\nu + 1)$, $\frac{\partial^2 \Omega}{\partial x \partial y} = \frac{3}{4}\sqrt{3}(\nu - 1)$,

$\frac{\partial^2 \Omega}{\partial y^2} = \frac{9}{4}(\nu + 1)$. Hence at a point whose coordinates are $x = \frac{1}{2} + \xi$, $y = \frac{1}{2}\sqrt{3} + \eta$,

$$2\Omega = 3(\nu + 1) + \frac{3}{4}(\nu + 1)\xi^2 + \frac{3}{2}\sqrt{3}(\nu - 1)\xi\eta + \frac{9}{4}(\nu + 1)\eta^2 + \dots,$$

and the equations of motion are

$$\frac{d^2\xi}{dt^2} - 2n\frac{d\eta}{dt} = \frac{3}{4}(\nu + 1)\xi + \frac{3}{4}\sqrt{3}(\nu - 1)\eta,$$

$$\frac{d^2\eta}{dt^2} + 2n\frac{d\xi}{dt} = \frac{3}{4}\sqrt{3}(\nu - 1)\xi + \frac{9}{4}(\nu + 1)\eta.$$

Noting that $n^2 = \nu + 1$, and assuming $\xi = ae^{\lambda t}$, $\eta = be^{\lambda t}$, we easily find

$$\lambda^4 + (\nu + 1)\lambda^2 + \frac{27}{4}\nu = 0.$$

It is clear that if $(\nu + 1)^2 > 27\nu$, λ^2 is negative, and the motion is oscillatory; but if $(\nu + 1)^2 < 27\nu$, λ is semi-imaginary and the solution will represent an oscillation with increasing amplitude.

The limiting value of ν consistent with stability is therefore given by $(\nu + 1)^2 = 27\nu$, the solution of which is $\nu = 24{\cdot}9599$. The second solution is of course the reciprocal of the first.

In the numerical work in this paper I have taken $\nu = 10$, and there will accordingly be no stable orbits encircling the point $r = \rho = 1$.

APPENDIX.

Computations of Periodic Orbits, and of their Stability.

Explanation.

The orbits are given in families, arranged according to descending values of C, the constant of relative energy. The families are distinguished by the initials A, B, C, a, b. The initial A is attached to one of the families of satellites and also to the family of planets, because the satellite A appears to bear the same relationship to Jove and the Sun that the planet A bears to the Sun and Jove.

The data for the orbits are given as follows: — The first column is the arc of the relative orbit measured from conjunction; the second and third are the rectangular coordinates $x - 1$, y for satellites, or x, y for planets; the fourth gives φ the inclination of the outward normal to the line SJ; the fifth and sixth are the coordinates ρ, ψ for satellites, or r, θ for planets; the last column contains the function $2n/V$.

The last column is given so that the reader may be enabled to complete the solution, by drawing the orbit with reference to axes fixed in space. The integral $\frac{1}{2}\int \frac{2n}{V} ds$ would give nt, that is to say the angle turned through by the rotating axes since conjunction; then the polar coordinates with reference to Jove are ρ, $\psi + nt$, or with reference to the Sun are r, $\theta + nt$.

In the case of the oscillating bodies (families a and b) the polar coordinates are not given, but the rectangular coordinates with reference to axes fixed in space are clearly

$$x \cos nt - y \sin nt, x \sin nt + y \cos nt$$

for heliocentric origin, and

$$(x - 1) \cos nt - y \sin nt, \; (x - 1) \sin nt + y \cos nt$$

for jovicentric origin.

The last line of these tables gives the value of the arc and of φ when y vanishes. If the orbit were rigorously periodic and were computed with absolute accuracy, this angle would be 180° or 0°. It may be remarked that in some cases a small change in the initial value of x leads to a large change in the final value of φ, and in other cases the converse is true. Thus in some cases it is necessary to continue the search until the final value of φ only differs from 180° or 0° by a few minutes of arc, and in others even an error of a degree of arc is unimportant. The coordinates are certainly given with sufficient accuracy to draw the figures on a large scale.

Finally there is given the time-integral nT, being twice the angle turned through by the rotating axes between the first orthogonal crossing of SJ and the second (closely approximate) orthogonal crossing. Since the circuit is completed at the third crossing T is the period, and the ratio of nT to 360° is the ratio of the period of the body to that of Jove.

After the coordinates the discussion of the stability is given.

In order to test the sufficiency of the harmonic expansion of Φ to represent that function, a comparison is given between nine of the equidistant values of Φ with the corresponding values derived from a synthesis of the harmonic series, which has been calculated as far as the eighth order inclusive. Following this comparison is Φ_0 the mean value of Φ.

In the cases where the orbit is stable the value of c is given, and certain functions of it. The function $\Delta \sin^2 \frac{1}{2}\pi\sqrt{\Phi_0}$ or $\sin^2 \frac{1}{2}\pi c$ is what is called in the table of § 18 the Criterion of Stability. The function $2\pi\left(\frac{1}{2}c - 1\right)$ gives the retrogression of the pericentre, with respect to the rotating axes, in the synodic period. The function $nT - 2\pi\left(\frac{1}{2}c - 1\right)$ gives the advance of pericentre, with respect to fixed axes, in the synodic period. And $2\pi\left(1 - \dfrac{\frac{1}{2}c}{1 + \frac{nT}{2\pi}}\right)$ gives the advance of the pericentre, with respect to fixed axes, in the sidereal period.

Where the orbit is unstable, when the determinant Δ is negative the instability is of the even type, and when $\Delta \sin^2 \frac{1}{2}\pi\sqrt{\Phi_0}$ is greater than unity the instability is of the uneven type. The modulus of instability, or the number of synodic circuits, in which the amplitude of displacement increases to twice its primitive value, is given.

When the instability is of the even type c is of the form $2n + k\sqrt{-1}$, and when of the uneven type it is of the form $2n + 1 + k\sqrt{-1}$; in the tables c is given in one or other of these two forms.

FAMILY A OF SATELLITES.

$C = 40{\cdot}5$ $\qquad$ $x_0 = 1{\cdot}1135$

s	$x - 1$	y	φ	ρ	ψ	$\frac{2n}{V}$
·00	+ ·1135	+ ·0000	0° 0′	·1135	0° 0′	2·423
3	102	298	12° 56′	41	15° 7′	·441
6	002	580	25° 58′	58	30° 4′	·492
9	·0841	832	39° 15′	83	44° 5′	·574
·12	625	·1040	52° 56′	·1213	58° 59′	·679
5	366	189	67° 10′	44	72° 54′	·792
8	+ 078	269	82° 0′	71	86° 30′	·893
·21	− ·0222	271	π − 82° 42′	90	π − 80° 7′	·960
4	511	194	67° 20′	98	66° 51′	·975
7	769	044	52° 24′	96	53° 36′	·936
·30	981	·0833	38° 17′	87	40° 19′	·870
3	·1137	578	24° 58′	76	26° 55′	·803
6	233	+ 294	π − 12° 16′	66	π − 13° 24′	·757
·39	− ·1265	− ·0004	π + 0° 6′	·1265	π + 0° 11′	2·740
·3896		·0000	π − 0° 3′			

$nT = 61° 23′$

Family A of satellites continued.

Stability of $x_0 = 1\cdot1135$, $C = 40\cdot5$.

Comparison

	Computed Φ	Synthesis		Computed Φ	Synthesis
a_0	3·19	3·18	a_8	7·28	7·22
a_2	3·84	3·84	a_9	5·80	5·91
a_3	4·67	4·66	a_{10}	5·01	4·86
a_4	5·81	5·83	a_{12}	3·19	3·00
a_6	8·04	8·04			

$$\Phi_0 = 5\cdot479\,.$$

The harmonic series represents Φ well.

The determinant gives $\Delta \sin^2 \frac{1}{2}\pi\sqrt{\Phi_0} = \cdot1119, c = 2\cdot217$,

$$2\pi\left(\frac{1}{2}c - 1\right) = 39^\circ\ 4', nT - 2\pi\left(\frac{1}{2}c - 1\right) = 19^\circ\ 42', 2\pi\left(1 - \frac{\frac{1}{2}c}{1 + \frac{nT}{2\pi}}\right) = 22^\circ\ 19'.$$

The orbit is stable.

$C = 40\cdot25$ $\qquad$ $x_0 = 1\cdot1150$

s	$x - 1$	y	φ	$\frac{2n}{V}$
·00	+ ·1150	+ ·0000	0° 0′	2·418
3	118	298	12° 24′	·437
6	022	583	24° 53′	·496
9	·0867	839	37° 34′	·587
·12	659	·1054	50° 36′	·708
5	407	216	64° 12′	·846
8	+ 124	312	78° 29′	·978
·21	— ·0175	333	π — 86° 39′	3·079
4	469	277	71° 31′	·120
7	739	146	56° 43′	·097
·30	966	·0952	42° 42′	·033
3	·1142	710	29° 37′	2·954
6	260	435	17° 20′	886
9	320	+ 141	π — 5° 33′	854
·42	— ·1319	— ·0158	π + 6° 4′	2·855
·4042		.0000	π — 0° 1′	

$nT = 65^\circ\ 40'$

Family A of satellites continued.

Stability of $x_0 = 1{\cdot}1150, C = 40{\cdot}25.$

Comparison

	Computed Φ	Synthesis		Computed Φ	Synthesis
a_0	2·928	2·936	a_8	7·839	7·865
a_2	3·652	3·650	a_9	6·050	6·036
a_3	4·574	4·580	a_{10}	4·383	4·384
a_4	5·885	5·881	a_{12}	2·947	2·932
a_6	8·718	8·730			

$$\Phi = 5{\cdot}574.$$

The harmonic series represents Φ well.

The determinant gives $\Delta \sin^2 \frac{1}{2}\pi\sqrt{\Phi_0} = {\cdot}0630$, $c = 2{\cdot}161$,

$$2\pi\left(\frac{1}{2}c - 1\right) = 29^\circ\, 3',\ nT - 2\pi\left(\frac{1}{2}c - 1\right) = 36^\circ\, 37',\ 2\pi\left(1 - \frac{\frac{1}{2}c}{1 + \frac{nT}{2\pi}}\right) = 30^\circ\, 58'.$$

The orbit is stable.

$C = 40{\cdot}2$ $\qquad x_0 = 1{\cdot}1090$

s	$x - 1$	y	φ	$\frac{2n}{V}$
·00	+ ·1090	+ ·0000	0° 0′	2·276
3	058	298	12° 30′	·298
6	·0961	581	25° 1′	·362
9	806	837	37° 38′	·467
·12	598	·1052	50° 28′	·609
5	346	215	63° 45′	·780
8	+ 064	314	77° 41′	·958
·21	— ·0234	340	π — 87° 41′	3·119
4	529	289	72° 35′	·225
7	800	163	57° 36′	·255
·30	·1031	·0972	43° 22′	·219
3	210	732	30° 10′	·155
6	331	458	17° 55′	·092
9	394	+ 166	π — 6° 18′	·055
·42	— ·1397	— ·0134	π + 5° 6′	3·053
·4066		·0000	π — 0° 0′	

$$nT = 66^\circ\, 52'$$

Family A of satellites continued.

Stability of $x_0 = 1\cdot1090, C = 40\cdot2$.

	Computed Φ	Synthesis	Comparison	Computed Φ	Synthesis
a_0	2·627	2·573	a_8	8·640	8·753
a_2	3·300	3·296	a_9	6·692	6·635
a_3	4·186	4·139	a_{10}	4·760	4.758
a_4	5·498	5·601	a_{12}	3·093	3·033
a_6	8·184	8·345			

$$\Phi_0 = 5\cdot593$$

The harmonic series represents Φ fairly well.

The determinant gives $\Delta \sin^2 \frac{1}{2}\pi\sqrt{\Phi_0} = \cdot0636, c = 2\cdot162$,

$$2\pi\left(\frac{1}{2}c - 1\right) = 29^\circ\ 14',\ nT - 2\pi\left(\frac{1}{2}c - 1\right) = 37^\circ\ 38',\ 2\pi\left(1 - \frac{\frac{1}{2}c}{1 + \frac{nT}{2\pi}}\right) = 31^\circ\ 44'.$$

The orbit is stable.

$C = 40\cdot0$ $\qquad x_0 = 1\cdot03341$.

s	$x - 1$	y	ϕ	ρ	ψ	$\frac{2n}{V}$
·00	+ ·03441	+ ·00000	0° 0′	·03341	0° 0′	·939
1	3257	0995	9° 40′	3406	17° 0′	·950
2	3010	1963	18° 52′	3594	33° 7′	·981
3	2617	2882	27° 16′	3893	47° 46′	1·031
4	2101	3738	34° 44′	4288	60° 40′	·096
5	1484	4525	41° 17′	4762	71° 50′	·172
6	+ 0787	5241	47° 0′	5300	81° 28′	·259
8	− ·00785	6472	56° 20′	6519	π − 83° 5′	·458
·10	2518	7467	63° 41′	7880	71° 22′	·690
2	4355	8256	69° 39′	9334	62° 11′	·958
4	6259	8866	74° 42′	·10852	54° 47′	2·269
6	8207	9316	79° 12′	2416	48° 37′	·640
8	·10184	9617	83° 29′	4007	43° 22′	3·093
·20	2178	9769	87° 49′	5613	38° 44′	·664
2	4177	9762	π − 87° 17′	7213	34° 33′	4·410
·24	− ·16166	·09564	π − 81° 3′	·18783	π − 30° 37′	5·406

Family A of satellites continued.

s	$x - 1$	y	φ	ρ	ψ	$\frac{2n}{V}$
·26	— ·18111	·09111	72° 11′	·20274	π — 26° 42′	6·745
7	9046	8758	66° 12′	0963	24° 42′	7·525
8	9934	8300	58° 59′	1594	22° 36′	8·330
9	·20752	7726	50° 39′	2143	20° 25′	9·030
·30	1474	7035	41° 46′	2596	18° 8′	516
1	2081	6241	33° 11′	2946	15° 47′	·730
2	2570	5369	25° 34′	3200	13° 23′	·710
3	2949	4444	19° 9′	3375	10° 58′	·563
4	3231	3485	13° 50′	3491	8° 32′	·376
5	3431	2505	9° 21′	3564	6° 6′	·209
6	3559	1514	5° 26′	3608	3° 41′	·095
7	3622	+ ·0516	π — 1° 49′	3628	π — 1° 15′	·032
·38	— ·23623	— ·00484	π + 1° 41′	·23628	π + 1° 10′	9·032
·37516		·00000	π — 0° 1′			

$$nT = 97° 58'.$$

Stability of $x_0 = 1·03341$, $C = 40·0$.

Comparison

	Computed Φ	Synthesis		Computed Φ	Synthesis
a_0	— 2·49	— 0·95	a_8	17·58	20·28
a_2	2·32	2·21	a_9	41·05	39·50
a_3	2·74	3·89	a_{10}	33·03	32·56
a_4	2·93	1·61	a_{12}	0·48	— 2·63
a_6	4·70	5·88			

$$\Phi_0 = 10·124.$$

The representation of Φ by the harmonic series is not very satisfactory, nevertheless it will serve to give the result with some approach to accuracy, for the following shows the gradual approximation to a definite value as the number of rows of the determinant is increased: —

No. of rows	Value of Δ
5	·000
9	·052
13	·233
15	·243
17	·246

Family A of satellites continued.

The determinant gives $\Delta \sin^2 \frac{1}{2}\pi\sqrt{\Phi_0} = \cdot 2264$, and $c = 3\cdot 684$;

$$2\pi\left(\frac{1}{2}c-1\right)=303^\circ 10', \; nT-2\pi\left(\frac{1}{2}c-1\right)=-205^\circ 12', \; 2\pi\left(1-\frac{\frac{1}{2}c}{1+\frac{nT}{2\pi}}\right)=-161^\circ 18'.$$

The margin of stability is obviously small.

$C = 39\cdot 5$ Figure-of-eight orbit, $x_0 = 1\cdot 065$.

s	$x-1$	y	φ	ρ	ψ	$\frac{2n}{V}$
·00	+ ·0650	+ ·0000	0° 0′	·0650	0° 0′	1·434
2	631	199	10° 52′	662	17° 29′	·452
4	576	390	21° 17′	696	34° 9′	·508
6	487	570	31° 0′	750	49° 27′	·598
8	371	732	39° 52′	821	63° 7′	·718
·10	233	876	47° 53′	907	75° 8′	·870
2	+ 076	·1000	55° 11′	·1004	85° 40′	2·053
4	− ·0095	104	61° 57′	109	π − 85° 6′	·268
6	276	188	68° 21′	220	76° 55′	·522
8	466	252	74° 35′	336	69° 36′	·820
·20	661	294	80° 56′	453	62° 57′	3·167
2	860	315	87° 38′	571	56° 48′	·573
4	·1060	310	π − 85° 0′	685	51° 2′	4·039
6	257	279	76° 41′	793	45° 29′	·543
8	447	217	67° 20′	891	40° 4′	5·054
·30	624	125	57° 15′	975	34° 42′	·463
2	783	002	47° 17′	·2045	29° 20′	·738
4	917	·0855	38° 17′	100	24° 2′	·856
6	·2030	690	30° 49′	145	18° 47′	·882
8	123	513	24° 58′	184	13° 36′	·876
·40	200	329	20° 30′	224	8° 30′	·922
2	265	+ 139	17° 14′	269	π − 3° 31′	6·053
4	320	− ·0053	15° 1′	321	π + 1° 18′	·287
6	369	247	13° 54′	382	5° 56′	·690
8	417	441	14° 4′	458	10° 20′	7·425
9	442	538	14° 47′	501	12° 25′	·950
·50	− ·2469	− ·0634	π − 16° 8′	·2550	π + 14° 24′	8·688

Family A of satellites continued.

s	$x - 1$	y	φ	ρ	ψ	$\frac{2n}{V}$
·51	−·2498	−·0729	$\pi - 18^\circ\, 27'$	·2603	$\pi + 16^\circ\, 16'$	9·685
2	533	823	$22^\circ\, 26'$	663	$18^\circ\, 0'$	11·243
3	577	913	$30^\circ\, 4'$	733	$19^\circ\, 31'$	13·953
·535	604	955	$36^\circ\, 54'$	773	$20^\circ\, 8'$	16·633
4	637	992	$48^\circ\, 27'$	818	$20^\circ\, 36'$	19·562
425	657	·1007	$56^\circ\, 52'$	841	$20^\circ\, 45'$	21·083
45	679	019	$67^\circ\, 54'$	866	$20^\circ\, 49'$	23·755
475	703	025	$\pi - 82^\circ\, 2'$	891	$20^\circ\, 46'$	24·553
5	728	026	$+ 83^\circ\, 33'$	915	$20^\circ\, 36'$	·220
525	753	020	$78^\circ\, 35'$	935	$20^\circ\, 20'$	22·752
·555	775	009	$60^\circ\, 13'$	954	$19^\circ\, 59'$	21·190
·56	814	·0979	$46^\circ\, 34'$	979	$19^\circ\, 11'$	17·987
65	848	942	$37^\circ\, 23'$	999	$18^\circ\, 19'$	15·257
7	876	901	$31^\circ\, 33'$	·3014	$17^\circ\, 23'$	13·597
75	900	857	$27^\circ\, 13'$	024	$16^\circ\, 28'$	12·425
8	922	812	$23^\circ\, 55'$	033	$15^\circ\, 31'$	11·445
9	958	719	$18^\circ\, 51'$	044	$13^\circ\, 40'$	9·910
·60	987	624	$15^\circ\, 15'$	052	$11^\circ\, 47'$	·280
1	·3011	526	$12^\circ\, 22'$	057	$9^\circ\, 55'$	8·666
2	030	428	$9^\circ\, 56'$	060	$8^\circ\, 3'$	·220
4	058	230	$5^\circ\, 50'$	067	$4^\circ\, 18'$	7·727
6	072	−·0031	$+\ 2^\circ\, 22'$	072	$\pi + 0^\circ\, 35'$	·540
8	−·3074	+·0169	$-\ 0^\circ\, 53'$	·3079	$\pi - 3^\circ\, 9'$	7·595
·66308	−·3073	·0000	$+\ 1^\circ\, 49'$.			

$$nT = 229^\circ\, 19'.$$

The above is not strictly periodic, since the final value of φ is $1^\circ\, 49'$; but I find that when $x_0 = 1{\cdot}066$ the final value of φ is $-62^\circ\, 24'$, hence the periodic orbit should be $x_0 = 1{\cdot}065028$. Since the above only differs from the true periodic in the fifth place of decimals of x_0, I accept it as periodic. It would seem however as if the final value of $x - 1$ in the periodic orbit is about $-{\cdot}305$ instead of $-{\cdot}3073$, as in the above.

Stability of $x_0 = 1{\cdot}065$, $C = 39{\cdot}5$.

The determinantal method fails, because Φ varies from about -20 in one part of the orbit to more than 3000 in another, and the harmonic

Family A of satellites continued.

series gives so insufficient a representation of Φ, when we stop with the term of the eighth order, that it does not seem worth while to form and evaluate the determinant.

The orbit is clearly very unstable, with instability of the even type, as appears below in the case when $C = 39{\cdot}0$.

$C = 39{\cdot}0$ Figure-of-eight orbit, $x_0 = 1{\cdot}0941$.

It appeared from various computations that the periodic orbit should commence with $x_0 = 1{\cdot}0941$.

Accordingly after the latter part of the orbit had been computed the first part was calculated.

s	$x - 1$	y	φ	ρ	ψ	$\frac{2n}{V}$
·00	+ ·0941	+ ·0000	0° 0′	·0941	0° 0′	1·875
2	927	200	7° 56′	948	12° 9′	·888
4	886	395	15° 42′	970	24° 2′	·928
6	819	583	23° 20′	·1005	35° 27′	·991
8	728	761	30° 38′	054	46° 17′	2·081
·12	485	·1077	44° 27′	181	65° 49′	·340
6	+ 174	329	57° 29′	340	82° 32′	·717
·20	− ·0184	504	70° 41′	515	$\pi -$ 83° 0′	3·227
4	574	589	84° 51′	690	70° 8′	·880
8	971	565	$\pi -$ 76° 44′	842	58° 10′	4·562
·32	·1337	407	56° 48′	942	46° 26′	·904
6	633	139	39° 50′	991	34° 53′	·807
·40	853	·0806	27° 55′	·2020	23° 30′	·606
4	·2013	440	19° 35′	061	12° 20′	·525
8	127	+ 057	13° 59′	128	$\pi -$ 1° 32′	·639
·52	− ·2211	− ·0334	$\pi -$ 10° 49′	·2237	$\pi +$ 8° 36′	5·048

$\int_0^{\cdot 52} \frac{2n}{V} ds = 109° \, 10'$. Also the value of φ where the curve crosses the axis of x for the second time is $\pi - 13° \, 22'$.

Family A of satellites continued.

The following results in square parentheses were found by interpolation, between $x_0 = 1\cdot09$ and $x_0 = 1\cdot10$. Starting from these values the remainder of the orbit was computed as follows: —

s	$x - 1$	y	φ	ρ	ψ	$\frac{2n}{V}$
[·44	—·2020	·0437	π — 20° 5′]			
[6	084	244	17° 4′]			
[8	138	+ 055	14° 42′]			
[·50	186	— ·0139	12° 50′]	·2190	π + 3° 38′	4·847
2	228	334	11° 46′	252	8° 32′	5·104
4	268	530	11° 30′	329	13° 9′	·504
6	309	726	12° 19′	420	17° 27′	6·117
8	356	920	14° 50′	530	21° 20′	7·092
9	383	·1017	17° 10′	591	23° 6′	·839
·60	416	111	20° 46′	660	24° 42′	8·888
·605	435	158	23° 18′	695	25° 25′	9·565
1	456	203	26° 37′	734	26° 6′	10·476
15	480	247	31° 4′	776	26° 41′	11·582
2	508	288	37° 16′	820	27° 11′	13·008
25	541	326	46° 17′	866	27° 33′	14·945
3	580	356	59° 46′	914	27° 43′	16·959
35	627	374	π — 78° 46′	965	27° 37′	19·068
4	677	374	+ 79° 10′	·3009	27° 11′	18·399
45	724	357	62° 30′	043	26° 29′	16·379
5	765	330	51° 3′	068	25° 41′	14·408
55	801	295	42° 57′	087	24° 49′	12·815
6	833	257	37° 7′	099	23° 55′	11·582
65	862	216	32° 41′	109	23° 1′	10·638
7	888	173	29° 13′	117	22° 6′	9·932
75	911	129	26° 22′	121	21° 12′	·189
8	932	084	23° 58′	126	20° 17′	8·732
85	952	038	21° 52′	129	19° 22′	·305
9	969	·0991	20° 3′	131	18° 27′	7·986
·70	·3001	896	16° 56′	133	16° 38′	·379
1	028	800	14° 22′	132	14° 48′	6·953
2	050	702	12° 8′	130	12° 58′	·647
3	070	604	10° 8′	129	11° 8′	·380
5	099	406	6° 41′	126	7° 28′	·053
7	117	207	3° 37′	124	3° 48′	5·862
9	124	— 007	+ 0° 43′	124	π + 0° 8′	·789
·81	—·3122	+ ·0193	— 2° 8′	·3128	π — 3° 32′	5·847

Family A of satellites continued.

Integrating $\frac{2n}{V}$ from the completion of the half circuit to $s = \cdot 52$, I find $\int_{\cdot 52}^{\frac{1}{2}s} \frac{2n}{V} ds = 130° \, 33'$, and combining this with the previous integral, we have $nT = 239° \, 43'$.

Stability of $x_0 = 1 \cdot 0941$, $C = 39 \cdot 0$.

Comparison

	Computed Φ	Synthesis		Computed Φ	Synthesis
a_0	2·59	1·76	a_6	5·51	8·34
a_1	4·27	5·24	a_7	− 8·43	− 11·01
a_2	8·89	7·68	a_8	− 13·95	− 13·86
a_3	18·68	19·65	a_9	− 0·87	+ 3·55
a_4	44·10	44·18	a_{10}	+ 31·93	+ 39·87
a_5	41·49	39·87	a_{11}	− 18·92	− 4·86
			a_{12}	− 18·28	− 33·96

$$\Phi_0 = 8 \cdot 74 .$$

The computed and synthetic values of Φ present some concordance, but the representation of Φ by the harmonic series is unsatisfactory.

The harmonic constituents being however used in the determinant give $\Delta \sin^2 \frac{1}{2} \pi \sqrt{\Phi_0} = - 1 \cdot 063$, $c = \cdot 46 \sqrt{-1}$, modulus $= \cdot 48$.

The orbit is very unstable with even instability.

$C = 38 \cdot 5$ Figure-of-eight orbit, $x_0 = 1 \cdot 1164$.

This orbit was exceedingly troublesome, and the coordinates were found by several interpolations. After the calculations were completed an error was discovered which may be substantially corrected by increasing all the arcs by $\cdot 0001$. The following figures to three places of decimals suffice for drawing the curve with fairly close accuracy. I have not thought it worth while to recompute the whole, and only give the interpolated coordinates and function $\frac{2n}{V}$.

Family A of satellites continued.

s	$x - 1$	y	$\frac{2n}{V}$
·00	+ ·1164	+ ·000	2·20
4	12	40	·25
8	·099	78	·39
·12	79	·112	·63
6	52	41	·99
·20	+ 19	65	3·49
4	— ·017	80	4·13
8	57	85	·81
·32	96	77	5·12
6	·129	55	4·85
·40	56	25	·39
4	75	·090	·07
8	90	53	3·90
·52	·201	+ 15	·92
6	09	— ·024	4·13
·60	16	64	·63
4	22	·103	5·65
8	32	42	8·30
·70	42	59	11·83
2	60	67	15·43
4	76	58	10·58
6	90	43	8·20
8	98	24	6·88
·80	·304	06	·08
2	09	·086	5·59
4	13	66	·26
6	15	47	·05
8	17	27	4·88
·90	— ·318	— ·007	4·86

When y vanishes between $s = \cdot 52$ and $\cdot 56$, $\varphi = \pi - 12^\circ 6'$.

$$nT = 258^\circ.$$

The stability was not worked out, but the orbit is obviously evenly unstable.

Family A of satellites continued.

$C = 38{\cdot}0$ Figure-of-eight orbit, $x_0 = 1{\cdot}1305$.

The calculation of this orbit proved excessively troublesome, and the results given below are only obtained with sufficient accuracy to draw a good figure.

Two sets of curves were traced; in the first set I travelled in a positive direction, starting from points on the line SJ for which x_0 is greater than unity; in the second set I travelled in a negative direction, starting from points on the line SJ for which x_0 is less than unity. One member of each of these two families was finally selected, such that they might be approximately parts of a single orbit.

The first of these two orbits is found by interpolation between the two, namely $x_0 = 1{\cdot}126$ and $x_0 = 1{\cdot}134$.

(arc increasing)			(arc diminishing)		
s	$x - 1$	y	s	$x - 1$	y
·00	+ ·1305	+ ·000	·00	− ·3225	− ·000
4	27	·040	− ·04	21	40
8	16	·078	8	16	80
·12	·098	·114	·12	07	·119
6	75	47	6	·294	56
·20	47	75	8	83	73
4	+ 14	97	·20	70	88
8	− ·023	·211	1	61	93
·32	63	12	2	52	94
6	99	·196	3	42	92
·40	·128	68	4	34	85
4	50	34	5	29	77
8	67	·098	6	24	68
·52	81	61	7	21	59
6	90	+ 22	8	18	49
·60	97	− ·017	− ·30	− .214	− ·129
4	·201	57			
8	05	96			
·72	− ·210	− ·135			

The period of the whole periodic orbit is given in round numbers by $nT = 299°$.

The orbit is obviously very unstable, and the instability is doubtless of the even type.

FAMILIES B AND C OF SATELLITES.

$C = 39{\cdot}3$

These are two orbits which nearly coalesce. It would have been more interesting to find the orbits for that critical value of C for which they exactly coalesce, but on account of the difficulty of the search I have only found two orbits nearly coalescent.

Four orbits were computed viz. $x_0 = 1{\cdot}15, 1{\cdot}16, 1{\cdot}17, 1{\cdot}18$; the values of $\varphi - \pi$ after a semi-circuit were found to be $-6'{\cdot}5, +1'{\cdot}5, +2'{\cdot}8, -5'{\cdot}4$.

If u_0, u_1, u_2, u_3 denote any functions connected respectively with the four orbits $x_0 = 1{\cdot}15, 1{\cdot}16, 1{\cdot}17, 1{\cdot}18$ it appears that the two orbits for which the value of $\varphi - \pi$ is exactly zero are given by

$$u_1 + {\cdot}1188\,(u_0 - u_1) + {\cdot}2127\,(u_1 - u_2) + {\cdot}0394\,(u_3 - u_1),$$

and

$$u_2 + {\cdot}0628\,(u_0 - u_2) + {\cdot}3133\,(u_2 - u_1) + {\cdot}3193\,(u_3 - u_2).$$

Putting the u's equal to $1{\cdot}15, 1{\cdot}16, 1{\cdot}17, 1{\cdot}18$ we find $x_0 = 1{\cdot}15747$, $x_0 = 1{\cdot}17506$ for the two periodics.

The four computed orbits gave nT equal to $87^\circ\,15'$, $87^\circ\,52'$, $88^\circ\,46'$, $89^\circ\,51'$ respectively.

On applying the formulæ of interpolation to the values of $x - 1$, y and nT I find the two periodics as follows: —

	orbit B		orbit C	
	$x - 1$	y	$x - 1$	y
·00	+ ·15747	+ ·00000	+ ·17506	+ ·00000
3	5499	2986	7257	2986
6	4756	5889	6512	5888
9	3526	8620	5270	8614
·12	1825	·11085	3539	·11058
5	·09675	3172	1348	3098
8	7136	4756	·08761	4604
·21	·04299	·15717	·05893	·15462

Families B and C of satellites continued.

s	$x-1$	y	$x-1$	y
·24	+ ·01317	+ ·15962	+ ·02902	+ ·15616
7	− ·01638	5475	− ·00043	5082
·30	4398	4315	2807	3923
3	6845	2588	5279	2234
6	8902	0412	7384	0102
9	·10519	·07889	9053	·07615
·42	1658	5119	·10232	4860
5	2296	+ 2191	0877	+ 1936
·48	− ·12418	− ·00802	− ·10961	− ·01058
	$nT = 87^\circ\ 41'$.		$nT = 89^\circ\ 1S'$.	

The semi-arc of the periodic orbit B is ·47197, and that of C is ·46941.

The fifth place of decimals in the coordinates has been given, although it is perhaps frequently inaccurate.

Stability of orbit B, $x_0 = 1·15747$, $C = 39·3$.

Comparison

	Computed Φ	Synthesis		Computed Φ	Synthesis
a_0	2·887	2·879	a_8	7·427	7·418
a_2	4·240	4·243	a_9	4·594	4·602
a_3	6·165	6·152	a_{10}	2·676	2·677
a_4	9·024	9·042	a_{12}	1·209	1·215
a_6	12·925	12·931			

$$\Phi_0 = 6·393.$$

The harmonic expansion represents Φ well.

The determinant Δ is negative, and $\Delta \sin^2 \frac{1}{2}\pi\sqrt{\Phi_0} = -·0612$.

The modulus is $1·415$, and the instability is not great; $c = ·156\sqrt{-1}$. The orbit is unstable.

Stability of orbit C, $x_0 = 1·17506$, $C = 39·3$.

Comparison

	Computed Φ	Synthesis		Computed Φ	Synthesis
a_0	3·736	3·725	a_8	6·123	6·119
a_2	5·507	5·517	a_9	3·948	3·956
a_3	7·862	7·834	a_{10}	2·430	2·431
a_4	10·715	10·749	a_{12}	1·199	1·185
a_6	11·641	10·663			

$$\Phi_0 = 6·489.$$

Families *B* and *C* of satellites continued.

The harmonic expansion represents Φ well.

The determinant gives, $\Delta \sin^2 \frac{1}{2}\pi\sqrt{\Phi_0} = \cdot 0644$, $c = 2\cdot 163$,

$$2\pi\left(\frac{1}{2}c-1\right)=80^\circ\, 57',\ nT-2\pi\left(\frac{1}{2}c-1\right)=30^\circ\, 31',\ 2\pi\left(1-\frac{\frac{1}{2}c}{1+\frac{nT}{2\pi}}\right)=24^\circ\, 27'.$$

The orbit is stable.

FAMILY *B* OF SATELLITES.

$C = 39\cdot 0$ $\qquad x_0 = 1\cdot 1500.$

s	$x-1$	y	φ	ρ	ψ	$\frac{2n}{V}$
·00	+ ·1500	+ ·0000	0° 0′	·1500	0° 0′	2·975
4	459	397	11° 50′	512	15° 13′	3·016
8	337	777	23° 54′	546	30° 10′	·135
·12	136	·1122	36° 34′	597	44° 39′	·340
6	·0862	412	50° 29′	654	58° 36′	·611
·20	523	622	66° 27′	704	72° 8′	·876
4	+ 137	723	84° 44′	728	85° 27′	4·093
8	− ·0260	691	$\pi -$ 75° 30′	711	$\pi -$ 81° 16′	·021
·32	624	529	57° 27′	651	67° 48′	3·696
6	928	271	42° 13′	574	53° 52′	·335
·40	·1159	·0946	29° 3′	496	39° 13′	·174
4	316	579	17° 12′	438	23° 45′	2·832
8	395	+ 188	$\pi -$ 5° 28′	408	$\pi -$ 7° 41′	·738
.52	− ·1392	− ·0212	$\pi +$ 5° 56′	·1408	$\pi +$ 8° 40′	2·738
·4991		·0000	$\pi +$ 0° 1′			

$nT = 96^\circ\, 56'.$

Family B of satellites continued.

Stability of $x_0 = 1\cdot1500, C = 39\cdot0$.

	Computed Φ	Synthesis	Comparison	Computed Φ	Synthesis
a_0	1·861	2·012	a_8	9·599	9·602
a_2	3·087	3·078	a_9	4·994	4·926
a_3	5·045	5·202	a_{10}	2·206	2·274
a_4	8·405	8·166	a_{12}	0·538	0·588
a_6	17·315	17·124			

$$\Phi_0 = 6\cdot924.$$

The harmonic expansion represents Φ with fair accuracy.

The determinant Δ is negative, and $\Delta \sin^2 \frac{1}{2}\pi\sqrt{\Phi_0} = -\cdot4019$.

The instability is of the even type, the modulus is 0·58 and c is $0\cdot38\sqrt{-1}$. The orbit is therefore very unstable.

$C = 38\cdot5$ $\qquad x_0 = 1\cdot1497$.

The comparison of the orbits $x_0 = 1\cdot1500$ with a neighbouring orbit showed that the exactly periodic orbit would correspond with $x_0 = 1\cdot1497$, but the results here given will be sufficiently exact.

s	$x - 1$	y	φ	ρ	ψ	$\frac{2n}{V}$
·00	+ ·1500	+ ·0000	0° 0′	·1500	0° 0′	2·835
4	464	398	10° 24′	517	15° 12′	·880
8	356	782	20° 50′	566	29° 59′	3·020
·12	181	·1141	31° 27′	643	44° 1′	·264
6	·0941	460	42° 46′	737	57° 12′	·626
·20	639	721	55° 52′	837	69° 38′	4·119
4	+ 282	897	72° 23′	919	81° 33′	·668
8	− ·0113	950	π − 86° 53′	953	π − 86° 41′	·972
·32	500	854	65° 19′	920	74° 54′	·708
·36	− ·0831	+ ·1631	π − 48° 18′	·1830	63° 0′	4·106

Family B of satellites continued.

s	$x-1$	y	φ	ρ	ψ	$\frac{2n}{V}$
·40	— ·1095	+ ·1333	$\pi - 35° 24'$	·1725	$\pi - 50° 36'$	3·574
4	294	·0987	24° 28′	628	37° 20′	·191
8	427	610	14° 36′	552	23° 9′	2·946
·52	495	+ 217	$\pi - 5° 2'$	511	$\pi - 8° 16'$	·826
·56	— ·1498	— ·0183	$\pi + 4° 25'$	·1509	$\pi + 6° 58'$	2·821
·5418		·0000	$\pi + 0° 4'$			

$$nT = 113° 20'.$$

Stability of $x_0 = 1·1497$, $C = 38·5$.

The values of Φ were computed for $x_0 = 1·1500$, and were corrected by interpolation with values computed for $x_0 = 1·1475$, but the corrections were so small that they might have been omitted.

Comparison

	Computed Φ	Synthesis		Computed Φ	Synthesis
a_0	0·68	1·15	a_8	11·79	11·97
a_2	1·83	1·81	a_9	4·53	4·29
a_3	3·77	4·18	a_{10}	1·21	1·34
a_4	8·03	7·39	a_{12}	— 0·82	— 0·90
a_6	29·34	28·97			

$$\Phi_0 = 8·60.$$

The representation of Φ by the harmonic series is fairly good.

The determinant is negative, and $\Delta \sin^2 \frac{1}{2}\pi\sqrt{\Phi_0} = -1·815$.

The orbit is very unstable with even instability; the modulus is ·313 and $c = ·70\sqrt{-1}$.

Family B of satellites continued.

$C = 38{\cdot}0$ $x_0 = 1{\cdot}1470.$

s	$x-1$	y	φ	ρ	ψ	$\frac{2n}{V}$
·00	+ ·1470	+ ·0000	0° 0′	·1470	0° 0′	2·660
4	437	398	9° 26′	491	15° 29′	·706
8	340	786	18° 40′	553	30° 23′	·850
·12	183	·1153	27° 39′	652	44° 17′	3·106
6	·0970	492	36° 36′	779	56° 58′	·497
·20	706	791	46° 19′	926	68° 29′	4·089
2	556	922	51° 58′	·2001	73° 54′	·482
4	391	·2036	58° 41′	073	79° 8′	·957
6	213	128	67° 4′	139	84° 16′	5·504
8	+ 023	189	77° 46′	189	89° 23′	6·042
·30	− ·0175	209	π − 89° 7′	216	π − 85° 28′	·397
2	373	182	74° 56′	213	80° 18′	·353
4	558	108	62° 3′	181	75° 10′	5·932
6	735	·1998	51° 53′	126	70° 3′	·352
8	873	864	44° 6′	059	64° 53′	4·805
·40	·1004	713	37° 55′	·1986	59° 37′	·336
4	221	378	28° 22′	841	48° 28′	3·653
8	385	014	20° 7′	717	36° 13′	·214
·52	496	·0630	12° 19′	624	22° 51′	2·944
·56	·1555	+ ·0235	π − 4° 30′	·1573	π − 8° 36′	2·814
·5836	− ·1564	·0000	π − 0° 8′			

$nT = 131° 45'$.

The final value of φ changes rapidly with the initial value of x, and therefore this is a very close approximation to the periodic orbit.

Stability of $x_0 = 1{\cdot}1470$, $C = 38{\cdot}0$.

Comparison

	Computed Φ	Synthesis		Computed Φ	Synthesis
a_0	− 0·402	2·265	a_8	12·358	13·083
a_2	0·670	− 0·363	a_9	3·160	1·931
a_3	2·899	5·403	a_{10}	− 0·241	1·439
a_4	6·413	2·487	a_{12}	− 2·174	2·271
a_6	59·339	56·777			

$\Phi_0 = 12{\cdot}237$.

Family B of satellites continued.

The representation of Φ by the harmonic series is poor, but it will suffice to give some idea of the degree of instability.

The determinant is negative, and $-\Delta \sin^2 \frac{1}{2}\pi\sqrt{\Phi_0} = 4\cdot 55$.

The orbit is very unstable, with even instability; the modulus is about $\cdot 23$ and $c = \cdot 96\sqrt{-1}$.

FAMILY C OF SATELLITES.

$C = 39\cdot 0$ $\qquad\qquad x_0 = 1\cdot 2338$.

The periodic orbit was found by interpolation between $x_0 = 1\cdot 230$ and $x_0 = 1\cdot 235$, by the formula $\cdot 24\,[x_0 = 1\cdot 230] + \cdot 76\,[x_0 = 1\cdot 235]$. The following are the two computations,

s	$x - 1$	y	φ	ρ	ψ	$\frac{2n}{V}$
·00	+ ·2300	+ ·0000	0° 0′	·2300	0° 0′	6·219
4	258	397	12° 10′	293	9° 59′	·259
8	128	774	25° 50′	265	19° 59′	·302
·12	·1905	·1105	42° 5′	202	30° 6′	·180
6	594	354	60° 24′	092	40° 21′	5·679
·20	221	494	77° 56′	·1929	50° 45′	4·833
4	·0824	526	$\pi -$ 87° 10′	735	61° 38′	3·961
8	430	461	74° 5′	523	73° 36′	·223
·32	+ 060	311	61° 52′	312	87° 23′	2·652
6	− ·0269	085	49° 14′	118	$\pi -$ 76° 4′	·221
·40	538	·0790	35° 13′	·0956	55° 46′	1·914
4	721	436	19° 19′	843	31° 11′	·719
8	795	+ 045	$\pi -$ 1° 44′	795	$\pi -$ 3° 14′	·643
·52	− ·0745	− ·0351	$\pi +$ 16° 6′	·0823	$\pi +$ 25° 12′	1·687
·4846		·0000	$\pi +$ 0° 19′			

$$nT = 112^\circ\, 26'.$$

Family C of satellites continued.

s	$x - 1$	y	φ	ρ	ψ	$\frac{2n}{V}$
·00	+ ·2350	+ ·0000	0° 0′	·2350	0° 0′	6·594
4	306	397	12° 31′	340	9° 46′	·616
8	173	773	26° 41′	307	19° 35′	·640
·12	·1944	·1099	43° 32′	233	29° 30′	·434
6	627	340	62° 8′	108	39° 29′	5·780
·20	249	470	79° 9′	·1928	49° 39′	4·805
4	·0851	495	$\pi -$ 86° 43′	720	60° 21′	3·888
8	458	428	74° 15′	500	72° 13′	·147
·32	+ 087	281	62° 21′	284	86° 7′	2·581
6	− ·0244	059	49° 46′	087	$\pi -$ 77° 2′	·161
·40	516	·0766	35° 34′	·0945	56° 3′	1·859
4	700	413	19° 10′	813	30° 32′	·670
8	771	+ 021	$\pi -$ 1° 3′	771	$\pi -$ 1° 31′	·603
·52	− ·0715	− ·0374	$\pi +$ 17° 10′	·0807	$\pi +$ 27° 37′	1·661
·4821		·0000	$\pi -$ 0° 6′			

$$nT = 114° 4'.$$

The interpolated coordinates for the periodic orbit are

$x - 1$	y
·2338	+ ·0000
294	397
162	773
·1935	·1100
619	343
242	476
·0845	502
451	436
+ 081	288
− ·0250	065
521	·0772
705	418
777	+ 026
− ·0722	− ·0369

$$nT = 113° 41'.$$

The arcs with which these orbits are computed are rather longer than is desirable, nor was quite sufficient pains taken to make the second ap-

Family C of satellites continued.

proximations satisfactory. Thus the order of accuracy attained is not very high. It seemed however to be sufficient for the purpose.

Stability of $x_0 = 1{\cdot}2338$, $C = 39{\cdot}0$.

The values of Φ and of the determinant were computed for the two orbits between which the periodic orbit lies; the following are the results: —

$$x_0 = 1{\cdot}230.$$

Comparison

	Computed Φ	Synthesis		Computed Φ	Synthesis
a_0	5·40	5·57	a_8	4·47	4·58
a_2	10·65	10·71	a_9	3·06	3·04
a_3	16·30	16·40	a_{10}	1·93	1·99
a_4	18·44	18·38	a_{12}	0·47	0·47
a_6	9·69	9·70			

$$\Phi_0 = 8{\cdot}065.$$

The determinant gives $\Delta \sin^2 \frac{1}{2}\pi\sqrt{\Phi_0} = {\cdot}421$, $c = 2{\cdot}450$,

$$2\pi\left(\frac{1}{2}c-1\right)=80^\circ\ 57',\ nT-2\pi\left(\frac{1}{2}c-1\right)=31^\circ\ 29',\ 2\pi\left(1-\frac{\frac{1}{2}c}{1+\frac{nT}{2\pi}}\right)=23^\circ\ 59'.$$

$$x_0 = 1{\cdot}235.$$

Comparison

	Computed Φ	Synthesis		Computed Φ	Synthesis
a_0	6·04	6·20	a_8	4·26	4·27
a_2	11·94	12·00	a_9	2·98	2·95
a_3	17·77	17·81	a_{10}	1·88	1·89
a_4	18·65	18·55	a_{12}	0·43	0·42
a_6	9·13	9·04			

$$\Phi_0 = 8{\cdot}176.$$

The determinant gives $\Delta \sin^2 \frac{1}{2}\pi\sqrt{\Phi_0} = {\cdot}439$, $c = 2{\cdot}462$,

$$2\pi\left(\frac{1}{2}c-1\right)=83^\circ\ 10',\ nT-2\pi\left(\frac{1}{2}c-1\right)=30^\circ\ 54',\ 2\pi\left(1-\frac{\frac{1}{2}c}{1+\frac{nT}{2\pi}}\right)=23^\circ\ 28'.$$

Family C of satellites continued.

By interpolation between these two for $x_0 = 1{\cdot}2338$,

$$\Delta \sin^2 \frac{1}{2}\pi\sqrt{\Phi_0} = {\cdot}435\,,\ c = 2{\cdot}459,\ 2\pi\left(\frac{1}{2}c - 1\right) = 82^\circ\ 38',$$

$$nT - 2\pi\left(\frac{1}{2}c - 1\right) = 31^\circ\ 2',\ 2\pi\left(1 - \frac{\frac{1}{2}c}{1 + \frac{nT}{2\pi}}\right) = 23^\circ\ 35'.$$

The orbit is stable.

$C = 38{\cdot}75$ $\qquad x_0 = 1{\cdot}28733$.

s	$x - 1$	y	φ	ρ	ψ	$\frac{2n}{V}$
·00	+ ·28733	+ ·00000	0° 0′	·28733	0° 0′	10·472
2	8693	1999	2° 18′	8763	3° 59′	·610
4	8568	3995	4° 55′	8846	7° 58′	11·044
6	8340	5982	8° 24′	8964	11° 55′	·862
7	8174	6968	10° 46′	9023	13° 53′	12·471
8	7962	7945	13° 54′	9069	15° 52′	13·239
9	7688	8906	18° 8′	9085	17° 50′	14·168
·10	7330	9839	24° 14′	9047	19° 48′	15·216
1	6856	·10719	32° 51′	8916	21° 46′	16·241
2	6237	1502	44° 17′	8647	23° 41′	·623
3	5465	2136	57° 2′	8210	25° 29′	15·899
4	4576	2590	68° 19′	7615	27° 8′	14·171
5	3621	2887	76° 31′	6907	28° 37′	12·217
6	2638	3068	82° 9′	6140	30° 0′	10·533
7	1644	3168	86° 8′	5335	31° 19′	9·156
8	0645	3210	89° 0′	4509	32° 37′	8·048
·20	·18648	3172	$\pi -$ 87° 1′	2830	35° 14′	6·421
2	6655	3018	84° 11′	1138	38° 1′	5·289
4	4670	2774	81° 47′	·19453	41° 3′	4·457
6	2697	2448	79° 27′	7781	44° 26′	3·815
8	0739	2040	76° 59′	6133	48° 16′	·302
·30	·08802	1544	74° 14′	4517	52° 41′	2·881
2	6893	0949	71° 4′	2938	57° 49′	·527
4	5023	0241	67° 22′	1406	63° 53′	·225
·36	·03208	·09403	$\pi -$ 62° 56′	·09935	71° 10′	1·962

Family C of satellites continued.

s	$x - 1$	y	φ	ρ	ψ	$\frac{2n}{V}$
·38	+ ·01471	+ ·08413	π — 57° 34′	·08541	80° 5′	1·731
·40	— ·00154	7248	50° 56′	7250	π — 88° 47′	·528
2	1614	5884	42° 39′	6101	74° 40′	·353
4	2833	4303	32° 13′	5152	56° 38′	·209
5	3321	3431	26° 5′	4774	45° 56′	·152
6	3707	2509	19° 20′	4477	34° 5′	·106
7	3978	1547	12° 3′	4269	21° 15′	·074
8	4122	0558	4° 24′	4159	π — 7° 43′	·057
·485	— ·04143	+ ·00059	π — 0° 30′	·04143		1·054
·48559		·00000	π — 0° 2′			

$$nT = 179° 31'.$$

Stability of $x_0 = 1 \cdot 28733$, $C = 38 \cdot 75$.

Comparison

	Computed Φ	Synthesis		Computed Φ	Synthesis
a_0	— 4·38	8·08	a_8	3·42	— 9·75
a_2	18·34	43·45	a_9	3·08	11·03
a_3	185·33	155·74	a_{10}	2·57	0·49
a_4	46·39	79·81	a_{12}	— 3·08	8·88
a_6	6·22	15·65			

$$\Phi_0 = 23 \cdot 02 .$$

The representation of Φ by the harmonic series is bad, but it may serve to give some idea of the degree of instability.

The determinant gives $\Delta \sin^2 \frac{1}{2} \pi \sqrt{\Phi_0} = 1 \cdot 946$.

The instability is uneven; $c = 1 + \cdot 55 \sqrt{-1}$; modulus $= \cdot 40$.

Family C of satellites continued.

$C = 38{\cdot}5$ $x_0 = 1{\cdot}2760$.

s	$x - 1$	y	φ	ρ	ψ	$\frac{2n}{V}$
·00	+ ·2760	+ ·0000	0° 0′	·2760	0° 0′	7·516
2	759	200	0° 34′	766	4° 9′	·590
4	756	400	1° 3′	785	8° 15′	·829
6	752	600	1° 25′	816	12° 18′	8·258
8	746	800	1° 34′	861	16° 14′	·984
·10	741	·1000	1° 27′	918	20° 2′	10·212
2	737	200	1° 2′	988	23° 40′	12·467
3	735	300	0° 49′	·3028	25° 25′	14·561
4	734	400	1° 2′	071	27° 7′	18·411
45	732	450	1° 47′	093	27° 57′	22·00
5	730	500	4° 32′	115	28° 47′	29·20
525	727	524	8° 27′	124	29° 12′	36·46
55	721	549	21° 17′	131	29° 39′	53·80
5625	715	560	38° 47′	131	29° 52′	67·34
5750	705	567	72° 47′	126	30° 5′	81·66
5875	693	567	π — 71° 23′	115	30° 11′	62·13
6000	681	561	63° 45′	103	30° 12′	46·22
6125	671	555	58° 52′	090	30° 13′	37·74
6250	660	549	56° 23′	078	30° 12′	32·84
650	640	534	54° 39′	053	30° 10′	26·41
675	619	520	54° 4′	028	30° 7′	22·50
70	599	505	54° 2′	005	30° 7′	20·315
75	558	476	54° 53′	·2954	29° 59′	16·355
80	517	448	56° 8′	904	29° 54′	14·083
9	433	394	58° 59′	804	29° 49′	11·217
·20	346	344	61° 29′	704	29° 49′	9·406
2	167	256	65° 50′	505	30° 6′	7·150
4	·1982	179	69° 0′	306	30° 46′	5·748
8	603	050	72° 42′	·1916	33° 14′	4·027
·32	220	·0936	73° 32′	537	37° 29′	2·974
6	·0838	818	71° 41′	171	44° 18′	·234
·40	464	677	66° 35′	·0821	55° 36′	1·611
2	283	591	62° 9′	655	64° 24′	·412
4	+ 112	488	55° 11′	500	77° 4′	·182
6	— ·0041	360	44° 30′	362	π — 83° 27′	0·971
7	107	285	36° 53′	304	69° 25′	·876
·48	— ·0160	·0200	π — 27° 7′	·0256	π — 51° 20′	0·795

Family C of satellites continued.

s	$x - 1$	y	φ	ρ	ψ	$\frac{2n}{V}$
·49	— ·0196	·0107	$\pi -$ 14° 56′	·0223	$\pi -$ 28° 36′	·737
·50	210	+ 008	$\pi -$ 0° 51′	210	$\pi -$ 2° 14′	·713
·51	— ·0199	— ·0091	$\pi +$ 13° 29′	·0219	$\pi +$ 24° 33′	·729
·50084	— ·02102	·0000	$\pi +$ 0° 21′			

$$nT = 210° \, 52'.$$

A small change in x_0 makes a large change in the final value of φ, and it is therefore unnecessary to seek a more exact representation of the periodic orbit.

The stability was not computed, since the method would fail, but the orbit is obviously very unstable with uneven instability.

$C = 38{\cdot}0$ $\qquad\qquad$ $x_0 = 1{\cdot}2480.$

s	$x - 1$	y	φ	ρ	ψ	$\frac{2n}{V}$
·00	+ ·2480	+ ·0000	0° 0′	·2480	0° 0′	5·047
4	475	400	1° 32′	507	9° 11′	·176
8	460	800	2° 27′	586	18° 1′	·591
·12	444	·1199	+ 1° 50′	723	26° 9′	6·479
6	444	599	— 2° 30′	921	33° 12′	8·470
8	461	798	8° 1′	·3048	36° 10′	10·593
·20	510	991	22° 22′	204	38° 25′	15·63
1	561	·2076	41° 49′	297	39° 2′	22·07
15	599	108	60° 44′	345	39° 1′	25·62
2	646	122	— 87° 11′	389	38° 44′	27·81
25	695	113	$\pi +$ 65° 34′	424	38° 6′	26·60
3	736	084	46° 34′	440	37° 18′	22·64
35	768	046	34° 36′	442	36° 28′	19·60
4	793	003	25° 52′	437	35° 38′	17·03
5	827	·1908	$\pi +$ 14° 12′	410	34° 2′	14·16
7	847	708	$\pi -$ 0° 26′	320	30° 58′	11·02
·29	·2824	·1512	$\pi -$ 13° 15′	·3204	28° 9′	9·286

Family C of satellites continued.

s	$x-1$	y	ϕ	ρ	ψ	$\frac{2n}{V}$
·31	·2759	·1323	$\pi - 24°\,36'$	·3060	$25°\,37'$	8·070
3	660	150	$34°\,42'$	·2898	$23°\,23'$	7·072
7	384	·0862	$51°\,48'$	535	$19°\,53'$	5·462
·41	043	655	$64°\,26'$	145	$17°\,46'$	4·197
5	·1670	512	$73°\;\;2'$	·1747	$16°\,50'$	3·218
9	282	416	$78°\,16'$	348	$17°\,58'$	2·452
·53	·0889	343	$80°\,14'$	·0953	$21°\;\;6'$	1·824
5	692	309	$79°\,56'$	758	$24°\;\;4'$	·541
7	495	271	$78°\,21'$	565	$28°\,45'$	·266
8	398	250	$76°\,52'$	470	$32°\;\;9'$	·127
9	301	226	$74°\,39'$	376	$36°\,56'$	0·986
·60	205	196	$71°\,11'$	284	$43°\,45'$	·839
1	112	160	$65°\,24'$	195	$55°\;\;0'$	·682
2	+ 026	110	$54°\;\;0'$	113	$76°\,57'$	·510
25	− ·0012	077	$42°\,46'$	0783	$\pi - 81°\;\;6'$	·420
30	039	036	$23°\;\;4'$	0535	$42°\,22'$	·347
325	− ·0047	+ ·0012	$\pi - \;\;9°\;\;0'$	·00481	$\pi - 14°\,32'$	0·328
·63371	− ·0048	·0000	$\pi - \;\;1°\,37'$	·00478	$\pi - \;\;0°\;\;0'$	0·327

$$nT = 235°\,17'.$$

This orbit was not computed with high accuracy. As far as can be judged from other computations, the exactly periodic orbit would correspond to $x_0 = 1\cdot 2465$, but the calculations from which this is inferred were not conducted with the closest accuracy.

A very small difference in the initial value of x makes a considerable change in the size of the loop described. It would be very laborious to obtain the exact periodic orbit for this value of C, and the above appears to suffice.

The orbit is obviously very unstable, with uneven instability.

FAMILY A OF PLANETS.

$C = 40·0$ $\qquad x_0 = -·414.$

s	x	y	φ	r	θ	$\frac{2n}{V}$
·0	—·4140	—·0000	π + 0° 0′	·414	π + 0° 0′	1·809
·1	032	992	12° 22′	152	13° 49′	·820
·2	·3715	·1938	24° 49′	191	27° 34′	·851
·3	199	·2793	37° 28′	246	41° 7′	·899
·4	·2507	·3512	50° 24′	314	54° 29′	·960
·5	·1670	·4055	63° 47′	385	67° 38′	2·030
·6	—·0728	385	π + 77° 42′	445	π + 80° 34′	·093
·7	+·0265	474	— 87° 53′	482	— 86° 37′	·135
·8	·1249	309	73° 9′	486	73° 50′	·141
·9	·2159	·3901	58° 34′	459	61° 3′	·109
1·0	939	280	44° 27′	405	48° 8′	·045
·1	·3549	·2490	31° 9′	336	35° 4′	1·967
·2	969	·1585	18° 45′	274	21° 46′	·897
·3	·4191	·0612	7° 5′	235	8° 19′	·856
1·35	+·4228	—·0114	— 1° 24′	·4229	— 1° 32′	1·848
1·3614	+·423	·000	— 0° 6′			

$$nT = 154° 13'.$$

Although this is not strictly periodic, since the final value of φ is $-0° 6'$, it is sufficiently nearly so to be accepted as such.

Stability of $x_0 = -·414$, $C = 40·0$.

Comparison

	Computed Φ	Synthesis		Computed Φ	Synthesis
a_0	5·476	5·490	a_8	11·027	11·021
a_2	6·184	6·180	a_9	9·104	9·106
a_3	7·069	7·088	a_{10}	6·700	6·696
a_4	8·356	8·327	a_{12}	3·801	3·793
a_6	11·463	11·438			

$$\Phi_0 = 8·051.$$

Family A of planets continued.

The harmonic series represents Φ well. The determinant gives

$$\Delta \sin^2 \frac{1}{2}\pi\sqrt{\Phi_0} = \cdot 9096,\ c = 2\cdot 806,\ 2\pi\left(\frac{1}{2}c - 1\right) = 145^\circ\, 0',$$

$$nT - 2\pi\left(\frac{1}{2}c - 1\right) = 9^\circ\, 13',\ 2\pi\left(1 - \frac{\frac{1}{2}c}{1 + \frac{nT}{2\pi}}\right) = 6^\circ\, 27'.$$

The orbit is stable.

$C = 39\cdot 5$ $\qquad x_0 = -\cdot 4240.$

The periodic orbit is found by interpolation between $x_0 = -\cdot 426$ and $x_0 = -\cdot 4$, by the formula $\cdot 92228\,[x_0 = -\cdot 426] + \cdot 07772\,[x_0 = -\cdot 4]$.

The following are the two computations:

s	x	y	φ	r	θ	$\frac{2n}{V}$
0·0	$-\cdot 4260$	$-\cdot 0000$	$\pi + 0^\circ\ 0'$	·4260	$\pi + 0^\circ\ 0'$	1·861
·1	157	993	$11^\circ\ 52'$	275	$13^\circ\ 26'$	·874
·2	·3851	·1943	$23^\circ\ 49'$	314	$26^\circ\ 46'$	·905
·3	354	·2809	$35^\circ\ 51'$	374	$39^\circ\ 57'$	·959
·4	·2686	·3550	$48^\circ\ 18'$	451	$52^\circ\ 53'$	2·031
·5	·1871	·4127	$61^\circ\ 14'$	531	$65^\circ\ 37'$	·111
·6	$-\cdot 0947$	501	$74^\circ\ 45'$	600	$\pi + 78^\circ\ 8'$	·189
·7	$+\cdot 0041$	644	$\pi + 88^\circ\ 52'$	644	$-89^\circ\ 30'$	·242
·8	·1032	538	$-76^\circ\ 35'$	654	$77^\circ\ 11'$	·256
·9	965	185	$62^\circ\ 5'$	624	$64^\circ\ 50'$	·221
1·0	·2783	·3614	$48^\circ\ 6'$	560	$52^\circ\ 24'$	·144
·1	·3443	·2866	$34^\circ\ 59'$	481	$39^\circ\ 47'$	·052
·2	924	·1991	$22^\circ\ 48'$	399	$26^\circ\ 54'$	1·962
·3	·4218	037	$11^\circ\ 30'$	343	$13^\circ\ 49'$	·900
1·4	$+\cdot 4324$	$-\cdot 1044$	$-0^\circ\ 43'$	·4324	$-0^\circ\ 35'$	1·878
1·4044	$+\cdot 4324$	·0000	$-0^\circ\ 15'$			

$$nT = 165^\circ\, 0'.$$

Family A of planets continued.

s	x	y	φ	r	θ	$\frac{2n}{V}$
·0	− ·4000	− ·0000	π + 0° 0′	·4000	π + 0° 0′	1·686
·1	·3899	993	11° 39′	024	14° 17′	·701
·2	599	·1945	23° 20′	091	28° 23′	·748
·3	111	·2817	35° 7′	197	42° 9′	·825
·4	·2455	·3570	47° 8′	333	55° 29′	·934
·5	·1654	·4165	59° 39′	481	68° 20′	2·067
·6	− ·0742	568	72° 57′	627	π + 80° 47′	·218
·7	+ ·0241	740	π + 87° 19′	746	− 87° 6′	·354
·8	·1234	655	− 77° 20′	817	75° 9′	·448
·9	·2167	304	61° 22′	819	63° 16′	·444
1·0	970	·3712	46° 7′	754	51° 20′	·348
·1	·3598	·2937	32° 15′	644	39° 13′	·204
·2	·4035	040	19° 51′	522	26° 49′	·063
·3	278	·1072	− 8° 33′	410	14° 4′	1·950
1·4	+ ·4334	− ·0075	+ 2° 10′	·4335	− 1° 0′	1·887
1·4075	+ ·4331	·0000	+ 2° 58′			

$$nT = 167^\circ\, 31'.$$

The interpolated coordinates for the periodic orbit are,

x	y
− ·4240	− ·0000
137	993
·3831	·1943
335	·2810
·2668	·3552
·1854	·4130
− ·0931	506
+ ·0057	651
·1048	547
981	194
·2798	·3622
·3455	·2872
933	·1995
·4223	040
+ ·4325	− ·0046

$$nT = 165^\circ\, 12'.$$

Family A of planets continued.

Stability of $x_0 = -\cdot 426$, $C = 39\cdot 5$.

The orbit $x_0 = -\cdot 426$ was treated for stability in place of the interpolated orbit $x_0 = -\cdot 424$.

Comparison

	Computed Φ	Synthesis		Computed Φ	Synthesis
a_0	5·73	5·73	a_8	11·94	11·94
a_2	6·54	6·54	a_9	9·42	9·42
a_3	7·59	7·59	a_{10}	6·57	6·57
a_4	9·14	9·14	a_{12}	3·25	3·25
a_6	12·71	12·71			

$$\Phi_0 = 8\cdot 565 .$$

The harmonic series represents Φ perfectly. The determinant gives

$$\Delta \sin^2 \frac{1}{2}\pi\sqrt{\Phi_0} = \cdot 976\,,\ c = 2\cdot 901\,,\ 2\pi\left(\frac{1}{2}c - 1\right) = 162^\circ\ 15',$$

$$nT - 2\pi\left(\frac{1}{2}c - 1\right) = 2^\circ\ 47',\ 2\pi\left(1 - \frac{\frac{1}{2}c}{1 + \frac{nT}{2\pi}}\right) = 1^\circ\ 52'.$$

The orbit is stable, but approaches very near to instability.

The results would have been somewhat modified if we had operated on the true periodic orbit $x_0 = -\cdot 424$.

$C = 39\cdot 0$ $x_0 = -\cdot 434 .$

(Computed with 3-figured logarithms and to tenths of degree).

s	x	y	φ	r	$\theta + nt$
0	$-\cdot 434$	$-\cdot 000$	$\pi + 0^\circ\ 0'$	·434	$\pi + 0^\circ\ 0'$
·1	24	99	$11^\circ\ 18'$	36	$18^\circ\ 36'$
·2	395	195	$22^\circ\ 36'$	42	$37^\circ\ 12'$
·3	$-\cdot 348$	$-.282$	$\pi + 34^\circ\ 12'$	·449	$\pi + 55^\circ\ 36'$

Family A of planets continued.

s	x	y	φ	r	$\theta + nt$
·4	$-$·284	$-$·359	$\pi + 46°\ 0'$	·457	$\pi + 74°\ 6'$
·5	04	·420	$58°\ 24'$	67	$-87°\ 24'$
·6	114	63	$71°\ 30'$	78	$68°\ 54'$
·7	$-$·016	83	$\pi + 85°\ 24'$	84	$52°\ 24'$
·8	$+$·083	78	$-80°\ 0'$	85	$31°\ 30'$
·9	·179	49	$65°\ 24'$	83	$-12°\ 42'$
1·0	·264	·396	$51°\ 12'$	76	$+\ 5°\ 54'$
·1	·334	25	$38°\ 12'$	67	$24°\ 24'$
·2	87	·241	$26°\ 6'$	56	$42°\ 42'$
·3	·422	148	$15°\ 6'$	47	$61°\ 6'$
·4	·440	$-$·048	$-\ 4°\ 48'$	43	$+\ 79°\ 36'$
1·45	$+$·442	$+$·001	$+\ 0°\ 12'$	·442	
1·446		·000	$+\ 0°\ 6'$		

$$nT = 177°\ 0'.$$

Stability of $x_0 = -·434$, $C = 39·0$.

Comparison

	Computed Φ	Synthesis		Computed Φ	Synthesis
a_0	5·489	5·434	a_8	13·595	13·609
a_2	6·507	6·527	a_9	10·271	10·247
a_3	7·442	7·529	a_{10}	6·507	6·527
a_4	9·721	9·637	a_{12}	2·627	2·638
a_6	14·870	14·828			

$$\Phi_0 = 9·156.$$

The harmonic expansion represents Φ well.

The determinant Δ is positive and $\Delta \sin^2 \frac{1}{2} \pi \sqrt{\Phi_0}$ is 1·027, and $c = 3 + ·10\sqrt{-1}$.

The modulus of instability is 2·1.

The orbit is unstable, with uneven instability, but the instability is slight.

Family A of planets continued.

$C = 38{\cdot}5$ $\qquad$ $x_0 = -{\cdot}4440$.

s	x	y	ϕ	r	θ	$\frac{2n}{V}$
·00	− ·4440	− ·0000	$\pi +$ 0° 0′	·4440	+ 0° 0′	1·916
·08	380	797	8° 33′	452	10° 19′	925
·16	203	·1576	17° 8′	489	20° 33′	955
·24	·3911	·2320	25° 47′	547	30° 41′	2·004
·32	509	·3011	34° 34′	624	40° 38′	071
·40	006	632	43° 35′	714	50° 23′	157
8	·2410	·4164	52° 57′	811	59° 57′	258
·56	·1733	589	62° 48′	906	69° 19′	368
·64	·0993	889	73° 15′	989	78° 31′	474
·72	− ·0210	·5045	$\pi +$ 84° 22′	·5049	$\pi +$ 87° 37′	560
·80	589	043	− 83° 58′	077	− 83° 20′	605
8	·1370	·4877	72° 5′	066	74° 18′	592
·96	·2101	555	60° 26′	016	65° 14′	523
1·04	755	095	49° 25′	·4935	56° 4′	413
·12	·3312	·3523	39° 14′	835	46° 46′	286
·20	765	·2864	29° 54′	730	37° 16′	163
8	·4109	143	21° 15′	634	27° 33′	058
·36	345	·1379	13° 13′	559	17° 37′	1·980
·44	475	− ·0591	− 5° 36′	514	− 7° 31′	935
1·52	− ·4502	+ ·0208	+ 1° 49′	·4507	+ 2° 39′	1·927
1·4992		·0000	− 0° 6′			

$nT = 191° \, 21'$.

Stability of $x_0 = -{\cdot}4440$, $C = 38{\cdot}5$.

After the computation of the stability had been completed a small mistake in the calculation of the orbit was detected in consequence of which the semi-arc of the periodic orbit was taken to be 1·4987 (instead of 1·4992 as above); it was not however thought to be worth while to recompute the stability.

Family A of planets continued.

Comparison

	Computed Φ	Synthesis		Computed Φ	Synthesis
a_0	5·084	5·084	a_8	15·319	15·346
a_2	6·174	6·155	a_9	10·517	10·516
a_3	7·695	7·724	a_{10}	6·157	6·121
a_4	10·183	10·160	a_{12}	2·029	1·952
a_6	17·402	17·418			

$$\Phi_0 = 9{\cdot}786.$$

The harmonic series represents Φ well.

The determinant Δ is positive and $\Delta \sin^2 \frac{1}{2} \pi \sqrt{\Phi_0} = 1{\cdot}078$, and $c = 3 + {\cdot}176\sqrt{-1}$.

The modulus is 1·25. The orbit is unstable, with uneven instability, but the instability is not great.

$C = 38{\cdot}0$ $\qquad x_0 = -{\cdot}455.$

s	x	y	φ	r	θ	$\frac{2n}{V}$
·00	$-$·4550	$-$·0000	$\pi +$ 0° 0′	·4550	$\pi +$ 0° 0′	1·954
·08	494	·0797	8° 4′	563	10° 4′	1·964
·16	326	·1579	16° 10′	606	20° 3′	2·000
·24	050	·2329	24° 19′	672	29° 54′	·056
·32	·3669	·3032	32° 35′	760	39° 34′	·133
·40	190	·3672	41° 4′	864	49° 1′	·234
8	·2621	·4233	49° 50′	978	58° 14′	·354
·56	·1970	·4697	59° 14′	·5092	67° 14′	·496
·64	251	·5044	69° 18′	193	76° 4′	·631
·72	$-$·0480	·5255	$\pi +$ 80° 15′	282	$\pi +$ 84° 47′	·770
·80	$+$·0316	·5310	$-$ 87° 58′	310	$-$ 86° 35′	·825
·88	·1107	·5197	75° 48′	316	77° 59′	·841
·96	856	·4921	63° 47′	259	69° 20′	·753
1·04	$+$·2535	$-$·4498	$-$ 52° 34′	·5164	$-$ 60° 36′	2·611

Family *A* of planets continued.

s	x	y	φ	r	θ	$\frac{2n}{V}$
1.12	·3122	·3957	$-42°15'$	·5042	$-51°44'$	2·445
1.20	611	·3324	$33°\ 7'$	·4908	$42°38'$	·281
8	·3996	·2624	$24°43'$	730	$33°17'$	·140
·36	·4280	·1877	$16°58'$	673	$23°41'$	·031
·44	464	·1099	$9°39'$	597	$13°50'$	1·958
·52	550	$-$·0304	$-\ 2°45'$	561	$-\ 4°\ 0'$	·921
1·60	$+$·4540	$+$·0495	$+\ 4°\ 6'$	·4567	$+\ 6°14'$	1·929
1·5505		·0000	$-\ 0°\ 8'$			

$$nT = 207°\ 9'.$$

Stability of $x_0 = -·455$, $C = 38·0$.

Comparison

	Computed Φ	Synthesis		Computed Φ	Synthesis
a_0	4·722	4·886	a_8	17·052	17·170
a_2	5·941	5·927	a_9	10·602	10·491
a_3	7·767	7·821	a_{10}	5·618	5·649
a_4	10·991	10·898	a_{12}	0·952	0·990
a_6	21.495	21·508			

$$\Phi_0 = 10·666.$$

The representation of Φ by the harmonic series is good.

The determinant Δ is positive, and $\Delta \sin^2 \frac{1}{2}\pi\sqrt{\Phi_0}$ is 1·095.

The orbit is unstable and the instability is of the uneven type.

The modulus of instability is 1·14, and $c = 1 + ·193\sqrt{-1}$.

FAMILY a OF OSCILLATING SATELLITES.

$C = 40{\cdot}0$

$$x_0 = {\cdot}705.$$

s	x	y	φ	$\frac{2n}{V}$
·00	+ 7050	+ ·0000	— 0° 0′	14·622
1	053	100	3° 12′	·867
2	061	200	6° 43′	15·674
3	077	298	11° 3′	17·354
4	101	395	17° 30′	20·872
5	118	442	22° 44′	24·319
45	141	487	31° 8′	31·098
525	155	507	40° 55′	37·31
550	174	524	57° 32′	47·14
5625	185	529	— 71° 59′	54·71
5750	197	531	π + 88° 27′	55·66
5875	210	529	69° 30′	54·34
6000	220	523	54° 9′	47·14
6125	230	514	44° 29′	41·87
6250	238	505	37° 8′	37·50
6375	245	495	31° 59′	33·98
6500	251	484	28° 7′	30·28
675	262	461	22° 43′	26·92
700	271	438	18° 55′	24·14
75	285	390	13° 17′	20·59
80	295	341	10° 9′	18·47
85	303	292	7° 55′	16·99
90	309	242	6° 10′	·027
95	313	192	4° 43′	15·335
·100	317	142	3° 27′	14·810
05	319	093	2° 20′	·516
10	321	+ 043	1° 18′	·329
·115	+ ·7322	— ·0007	π + 0° 18′	14·276
·11427		·0000	π + 0° 27′	

$$nT = 138^\circ 20'.$$

Family *a* of oscillating satellites continued.

Stability of $x_0 = \cdot 705$, $C = 40\cdot 0$.

The thirteen equidistant values of Φ show great irregularity. The values numbered 0, 1, 2, 3, 4 and 8, 9, 10, 11, 12 are all negative and lie between $-2\cdot 6$ and $-3\cdot 0$; the values numbered 5 and 7 are about $+8$, and the value numbered 6 is about $+800$.

The harmonic analysis led to results which showed that the representation of Φ by the series would be so bad that it would not be worth while to continue the calculation.

The orbit is obviously very unstable.

$C = 39\cdot 0$ $x_0 = \cdot 6871$.

The coordinates for the periodic orbit were derived from the following by interpolation, as explained below.

s	x	y	φ	$\frac{2n}{V}$
·00	+ ·6870	+ ·0000	− 0° 0′	5·773
4	890	399	5° 44′	6·008
8	954	794	12° 58′	·893
·10	·7007	987	18° 4′	7·834
1	040	·1081	21° 29′	8·570
2	080	172	25° 58′	9·634
3	129	260	32° 31′	11·293
35	157	301	37° 14′	12·511
40	190	339	43° 46′	14·174
45	227	372	53° 38′	16·688
475	248	386	60° 40′	18·12
500	271	396	69° 41′	19·72
525	295	403	− 81° 8′	21·26
550	320	404	π + 85° 19′	·96
575	344	399	71° 19′	·62
·1600	·7367	·1388	π + 58° 48′	20·36

Family a of oscillating satellites continued.

s	x	y	φ	$\frac{2n}{V}$
·1625	+ ·7387	+ ·1373	π + 48° 46′	18·66
650	404	355	41° 4′	17·04
675	420	336	35° 6′	15·64
70	433	315	30° 25′	14·45
75	456	270	23° 37′	12·58
80	474	224	18° 59′	11·243
85	488	176	15° 26′	10·235
90	501	127	12° 47′	9·467
·20	519	029	9° 1′	8·350
1	533	·0930	6° 22′	7·584
2	542	830	4° 25′	·027
4	553	631	1° 54′	6·294
6	556	431	π + 0° 11′	5·875
8	555	231	π − 1° 7′	·653
·30	549	+ 031	2° 18′	·585
·32	+ ·7538	− ·0169	π − 3° 37′	5·656
·3031		·0000	π − 2° 23′	

$$nT = 146° \, 36'.$$

The following are coordinates interpolated between the preceding and the loop of the figure-of-8 $x_0 = 1·0941$, in such a way as to give a periodic orbit: —

s	x	y
·00	+ ·6871	+ ·0000
4	892	400
8	956	795
·10	·7010	987
1	045	·1081
2	085	172
3	135	259
35	164	300
4	196	337
475	252	381
55	320	399
6	368	386
·165	+ ·7407	+ ·1355

Family a of oscillating satellites continued.

s	x	y
·17	+ ·7437	+ ·1316
75	461	273
8	481	227
85	497	180
9	510	142
·20	534	027
1	549	·0929
4	573	645
6	583	446
8	587	246
·30	+ ·7588	+ ·0047

$$nT = 145° 40'.$$

Stability of $x_0 = \cdot 6870$, $C = 39 \cdot 0$.

In order to try the determinantal process on one orbit which is obviously very unstable, I treated the first of the above as though it were periodic with the following results: —

Comparison

	Computed Φ	Synthesis		Computed Φ	Synthesis
a_0	— 2·7	+ 38·6	a_7	+ 18·2	......
a_1	— 2·7	......	a_8	— 2·2	+ 87·0
a_2	— 2·9	— 3·2	a_9	— 3·3	+ 34·7
a_3	— 2·9	+ 38·3	a_{10}	— 3·3	+ 2·6
a_4	— 2·4	+ 82·9	a_{11}	— 3·3	......
a_5	+ 3·7	......	a_{12}	— 3·3	+ 35·8
a_6	+ 498·9	+ 379·5			

$$\Phi_0 = 41 \cdot 2 .$$

The function Φ is obviously one which would require a very large number of terms of an harmonic series for adequate representation, and the above is very bad.

However with 17 rows I find $\Delta \sin^2 \frac{1}{2}\pi\sqrt{\Phi_0} = -148 \cdot 4$; $c = 2 \cdot 0 \sqrt{-1}$, modulus $= \cdot 11$.

I think it is certain that the instability is of the even type, and is very great.

Family a of oscillating satellites continued.

$C = 38{\cdot}5$ $\qquad x_0 = {\cdot}6814.$

Two orbits were computed, namely $x_0 = {\cdot}6817$, giving the final value of φ equal to $\pi + 5^\circ\, 11'$ and $nT = 147^\circ\, 46'$, and ${\cdot}6810$, giving final $\varphi = \pi - 6^\circ\, 26'$ and $nT = 151^\circ\, 53'$. The arcs in the latter orbit were shorter than in the former throughout a portion of the curve. Interpolation between these two by the formula ${\cdot}446\,(x_0 = {\cdot}6810) + {\cdot}554\,(x_0 = {\cdot}6817)$ gives the following results: —

s	x	y	$\frac{2n}{V}$
·00	+ ·6814	+ ·0000	4·85
4	831	400	·98
8	884	796	5·44
·12	982	·1183	6·53
4	·7055	369	7·62
6	153	543	9·70
7	217	620	11·63
8	295	675	14·46
9	390	699	17·44
·20	482	662	15·27
1	543	581	11·69
2	584	491	9·61
3	615	396	8·28
4	637	299	7·36
6	666	102	6·22
8	682	·0903	·50
·30	691	703	5·07
2	695	504	4·79
4	698	304	·61
6	698	+ 105	·52
·38	+ ·7698	— ·0094	4·52
·37054	·7698	·0000	

$$nT = 149^\circ\, 36'.$$

The orbit is obviously unstable, and the instability is of the even type.

Family a of oscillating satellites continued.

$C = 38{\cdot}0$ $x_0 = {\cdot}676.$

This orbit was exceedingly troublesome, and the coordinates were found by several interpolations amongst the same orbits as those used in finding the figure-of-8 orbit $x_0 = 1{\cdot}1305$. Two sets of curves were traced; in the first set I started from one side of the oval, and in the second from the other side. The two curves were so selected that they might join one another as nearly as may be. The period of this orbit was not determined.

(arc increasing) x	(arc increasing) y	(arc diminishing) x	(arc diminishing) y
+ ·676	+ ·000	+ ·778	— ·009
77	40	78	+ ·011
82	80	79	31
90	·119	79	51
·704	56	79	71
13	74	78	·111
19	82	77	31
26	90	+ ·774	+ ·151
34	95		
43	98		
53	96		
60	89		
65	80		
68	71		
71	61		
73	51		
+ ·774	+ ·141		

nT undetermined.

FAMILY b OF OSCILLATING SATELLITES.

$C = 38{\cdot}5$ $\qquad x_0 = 1{\cdot}2919.$

The following was computed, —

s	$x - 1$	y	φ	$\frac{2n}{V}$
·00	+ ·29215	+ ·0000	— 0° 0′	8·52
4	932	400	2° 54′	9·00
8	971	797	9° 14′	10·84
·10	·3014	993	16° 10′	13·02
1	046	·1087	21° 56′	14·70
2	091	177	31° 49′	17·19
25	120	217	39° 5′	19·54
30	155	254	48° 21′	20·60
35	195	283	59° 56′	22·21
40	241	303	73° 40′	23·21
45	290	311	— 87° 38′	·00
50	340	307	π + 79° 36′	21·83
55	388	293	69° 0′	20·27
60	433	272	60° 51′	18·70
65	475	245	54° 15′	17·32
70	514	214	48° 59′	16·21
8	584	143	40° 54′	14·40
9	645	064	34° 51′	13·09
·20	699	·0980	30° 5′	12·15
2	787	801	22° 20′	10·89
4	853	612	16° 22′	·12
6	900	418	11° 13′	9·63
8	931	220	6° 28′	·36
·30	+ ·3945	+ ·0021	π + 1° 55′	9·25
·30209		·0000	π + 1° 27′	

$$nT = 213° \, 52'.$$

Family *b* of oscillating satellites continued.

The above, not being exactly periodic, was corrected by extrapolation from the orbit $x_0 = 1 \cdot 295$, which gave $\pi + 7° 58'$ as the final value of φ. The corrected coordinates are,

s	$x - 1$	y
·00	+ ·2919	+ ·0000
4	929	400
8	968	797
·10	·3009	993
1	041	·1088
2	085	178
25	113	219
3	147	256
35	187	286
4	233	306
45	282	314
5	332	311
55	380	297
6	425	275
7	505	216
8	575	145
9	635	065
·20	687	·0979
2	772	799
4	835	609
6	879	413
8	905	214
·30	+ ·3915	+ ·0014

Family *b* of oscillating satellites continued.

$C = 38\cdot0$ $\qquad x_0 = 1\cdot25945.$

The following orbit was computed,

s	$x - 1$	y	φ	$\frac{2n}{V}$
·00	+ ·2600	+ ·0000	$-\ 0^\circ\ 0'$	5·399
8	607	800	$1^\circ\ 4'$	6·030
·12	625	·1199	$4^\circ\ 51'$	7·152
6	693	592	$16^\circ\ 33'$	9·480
8	772	776	$29^\circ\ 45'$	11·822
9	829	858	$40^\circ\ 37'$	13·133
·20	903	925	$55^\circ\ 9'$	14·339
1	992	970	$72^\circ\ 20'$	·822
2	·3090	986	$-\ 89^\circ\ 9'$	·306
3	190	974	$\pi +\ 77^\circ\ 10'$	13·153
4	284	943	$66^\circ\ 53'$	11·932
5	373	897	$59^\circ\ 2'$	10·935
7	532	778	$48^\circ\ 3'$	9·423
9	671	634	$40^\circ\ 12'$	8·410
·33	892	309	$28^\circ\ 46'$	7·231
7	·4056	·0945	$20^\circ\ 15'$	6·567
·41	171	563	$13^\circ\ 15'$	·202
·45	+ ·4241	+ ·0169	$\pi +\ 6^\circ\ 59'$	6·005
·4670	+ ·4258	·0000	$\pi +\ 4^\circ\ 27'$	

$$nT = 214^\circ\ 40'.$$

Interpolation between the above and a neighbouring orbit gave the following coordinates for the periodic orbit,

s	$x - 1$	y
·00	·2595	·0000
8	600	800
·12	616	·1199
6	681	593
8	757	778
9	812	861
·20	884	929
·21	·2973	·1975

Family *b* of oscillating satellites continued.

s	$x - 1$	y
·22	·3071	·1992
3	170	980
4	264	948
5	352	900
7	508	777
9	642	630
·33	852	299
7	·4001	·0931
·41	095	546
·45	·4139	·0154
·4656	·4149	·0000

$$nT = 208°.$$

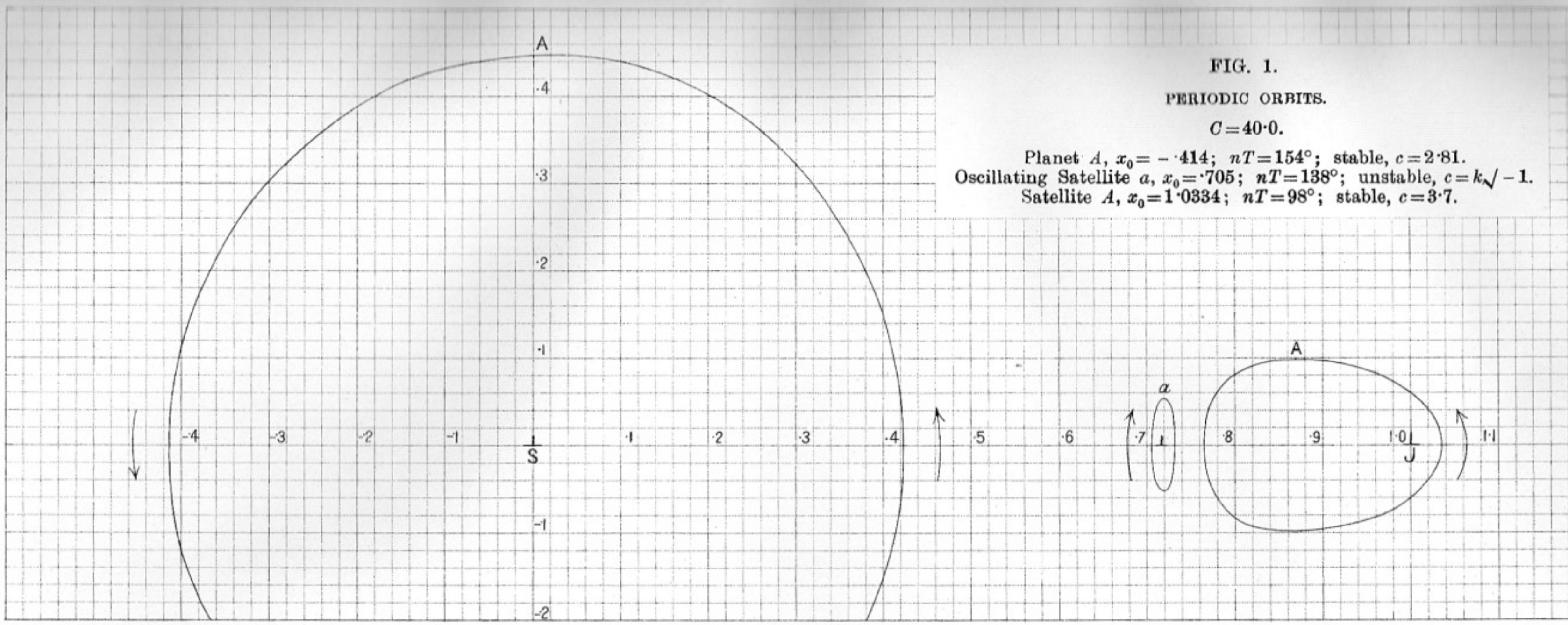

FIG. 1.

PERIODIC ORBITS.

$C=40\cdot 0$.

Planet A, $x_0=-\cdot 414$; $nT=154°$; stable, $c=2\cdot 81$.
Oscillating Satellite a, $x_0=\cdot 705$; $nT=138°$; unstable, $c=k\sqrt{-1}$.
Satellite A, $x_0=1\cdot 0334$; $nT=98°$; stable, $c=3\cdot 7$.

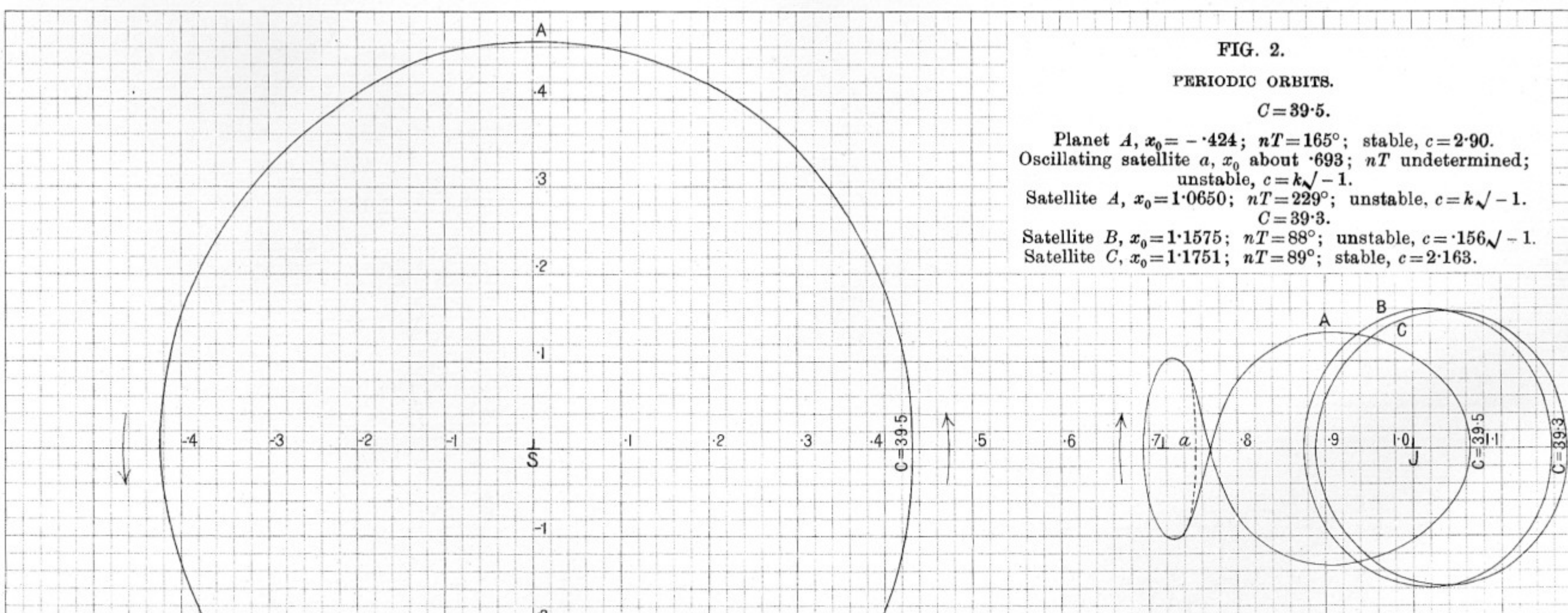

FIG. 2.

PERIODIC ORBITS.

$C=39\cdot 5$.

Planet A, $x_0=-\cdot 424$; $nT=165°$; stable, $c=2\cdot 90$.
Oscillating satellite a, x_0 about $\cdot 693$; nT undetermined; unstable, $c=k\sqrt{-1}$.
Satellite A, $x_0=1\cdot 0650$; $nT=229°$; unstable, $c=k\sqrt{-1}$.

$C=39\cdot 3$.

Satellite B, $x_0=1\cdot 1575$; $nT=88°$; unstable, $c=\cdot 156\sqrt{-1}$.
Satellite C, $x_0=1\cdot 1751$; $nT=89°$; stable, $c=2\cdot 163$.

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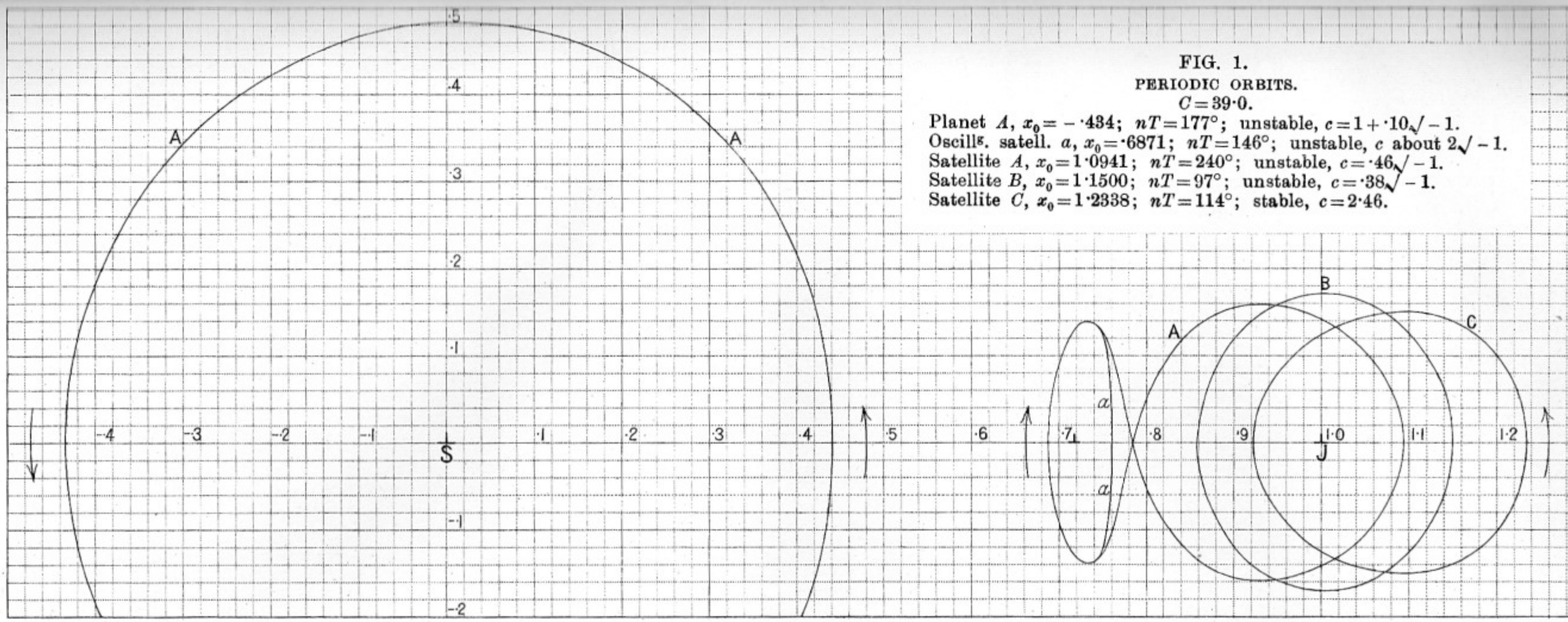

FIG. 1.

PERIODIC ORBITS.

$C = 39{\cdot}0$.

Planet A, $x_0 = -{\cdot}434$; $nT = 177°$; unstable, $c = 1 + {\cdot}10\sqrt{-1}$.
Oscill^g. satell. a, $x_0 = {\cdot}6871$; $nT = 146°$; unstable, c about $2\sqrt{-1}$.
Satellite A, $x_0 = 1{\cdot}0941$; $nT = 240°$; unstable, $c = {\cdot}46\sqrt{-1}$.
Satellite B, $x_0 = 1{\cdot}1500$; $nT = 97°$; unstable, $c = {\cdot}38\sqrt{-1}$.
Satellite C, $x_0 = 1{\cdot}2338$; $nT = 114°$; stable, $c = 2{\cdot}46$.

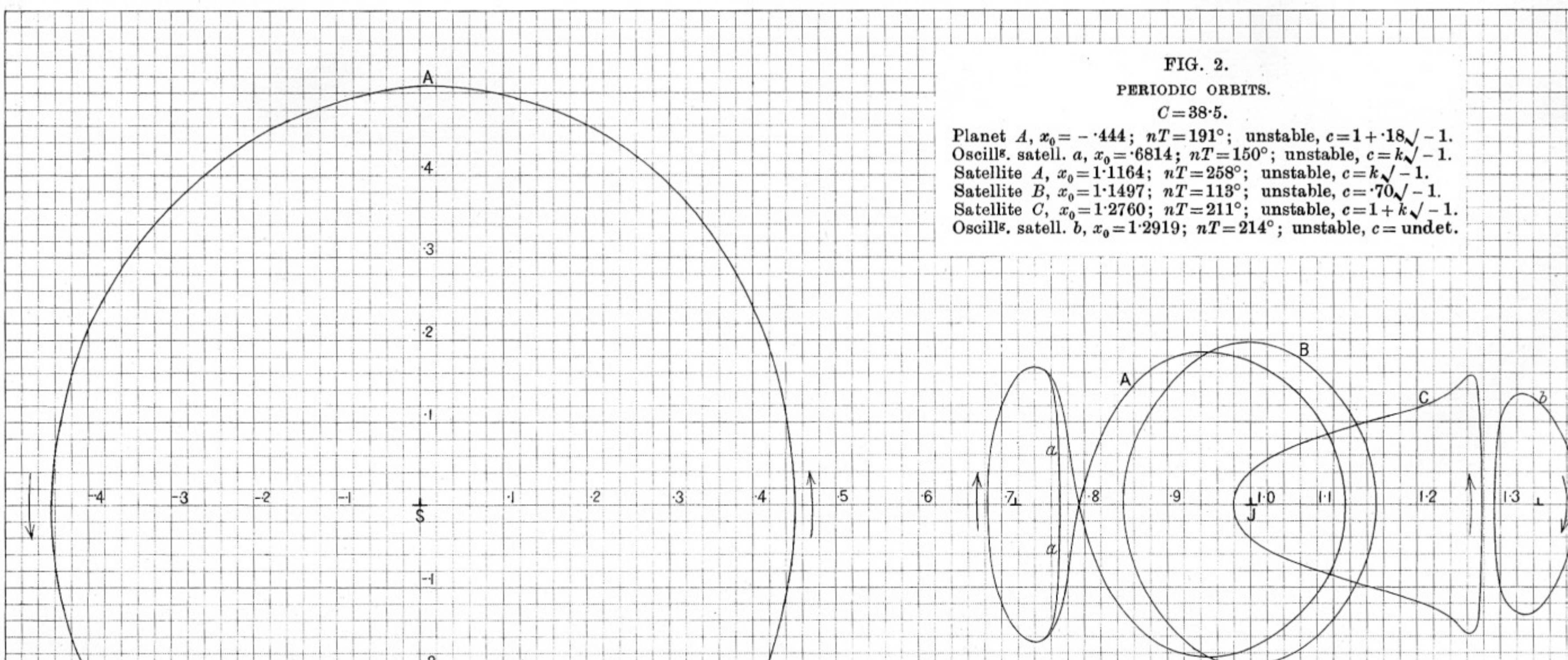

FIG. 2.

PERIODIC ORBITS.

$C = 38{\cdot}5$.

Planet A, $x_0 = -{\cdot}444$; $nT = 191°$; unstable, $c = 1 + {\cdot}18\sqrt{-1}$.
Oscill^g. satell. a, $x_0 = {\cdot}6814$; $nT = 150°$; unstable, $c = k\sqrt{-1}$.
Satellite A, $x_0 = 1{\cdot}1164$; $nT = 258°$; unstable, $c = k\sqrt{-1}$.
Satellite B, $x_0 = 1{\cdot}1497$; $nT = 113°$; unstable, $c = {\cdot}70\sqrt{-1}$.
Satellite C, $x_0 = 1{\cdot}2760$; $nT = 211°$; unstable, $c = 1 + k\sqrt{-1}$.
Oscill^g. satell. b, $x_0 = 1{\cdot}2919$; $nT = 214°$; unstable, $c =$ undet.

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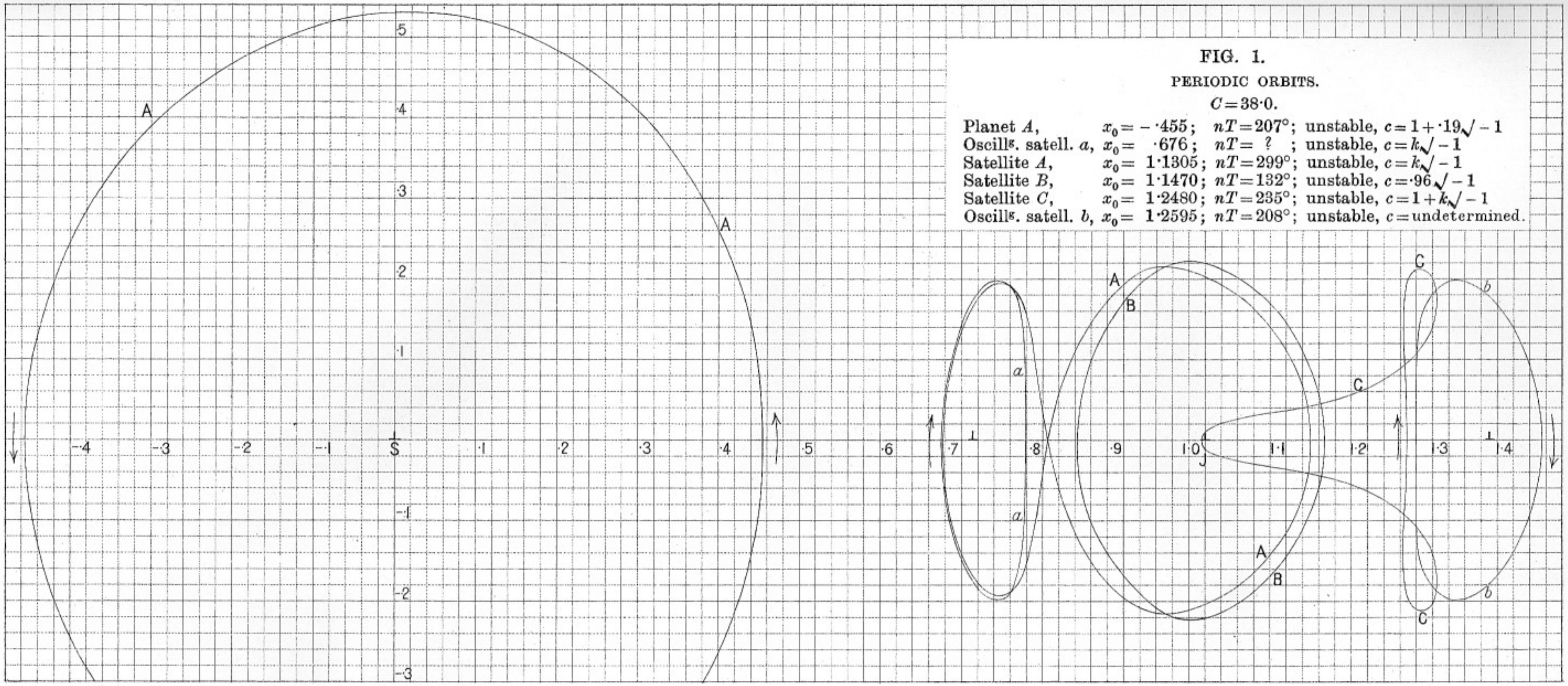

FIG. 1.

PERIODIC ORBITS.

$C = 38{\cdot}0$.

Planet A,	$x_0 = -{\cdot}455$;	$nT = 207°$;	unstable,	$c = 1 + {\cdot}19\sqrt{-1}$
Oscill^g. satell. a,	$x_0 = {\cdot}676$;	$nT = ?$;	unstable,	$c = k\sqrt{-1}$
Satellite A,	$x_0 = 1{\cdot}1305$;	$nT = 299°$;	unstable,	$c = k\sqrt{-1}$
Satellite B,	$x_0 = 1{\cdot}1470$;	$nT = 132°$;	unstable,	$c = {\cdot}96\sqrt{-1}$
Satellite C,	$x_0 = 1{\cdot}2480$;	$nT = 235°$;	unstable,	$c = 1 + k\sqrt{-1}$
Oscill^g. satell. b,	$x_0 = 1{\cdot}2595$;	$nT = 208°$;	unstable,	$c =$ undetermined.

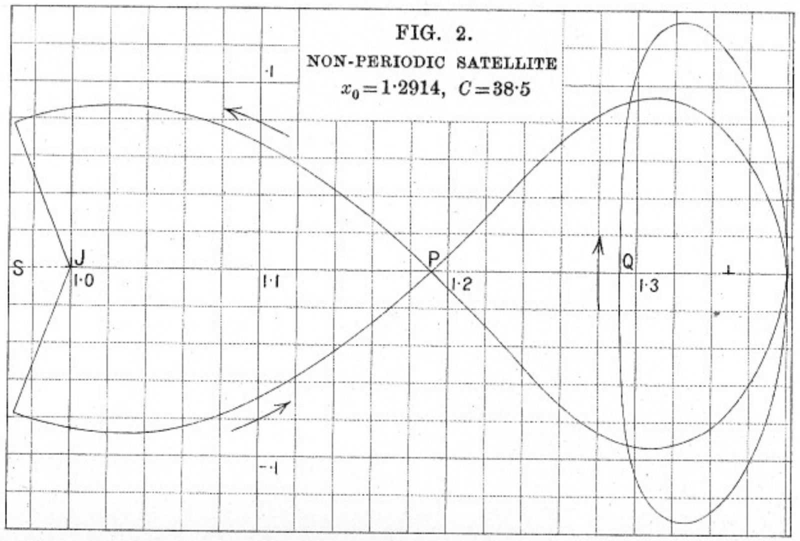

FIG. 2.

NON-PERIODIC SATELLITE

$x_0 = 1{\cdot}2914,\ C = 38{\cdot}5$

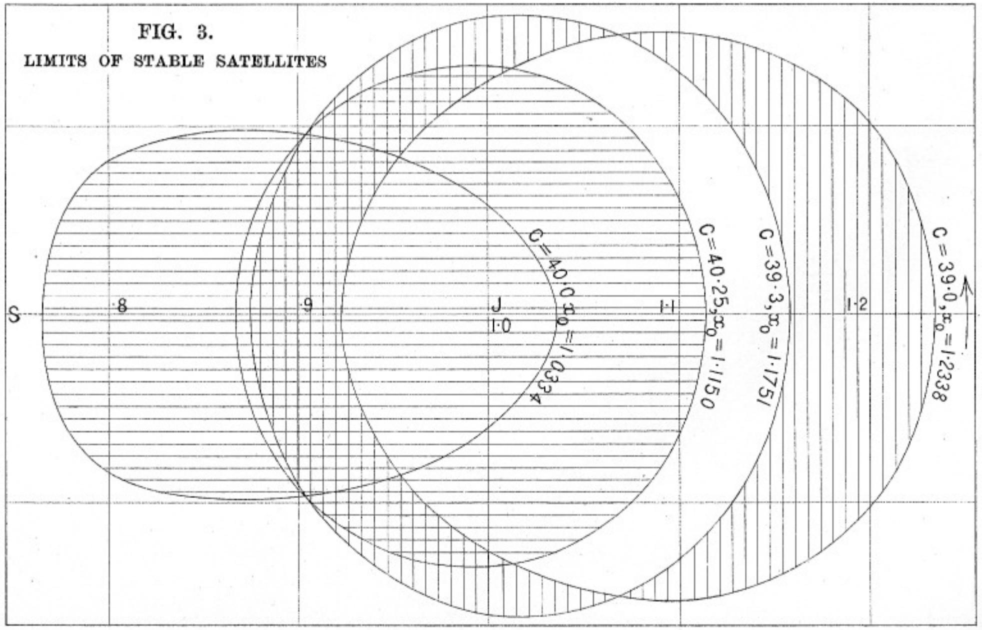

FIG. 3.

LIMITS OF STABLE SATELLITES

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FIG. 1.
FAMILY *A* OF SATELLITES
The values of *C* are given for each curve.

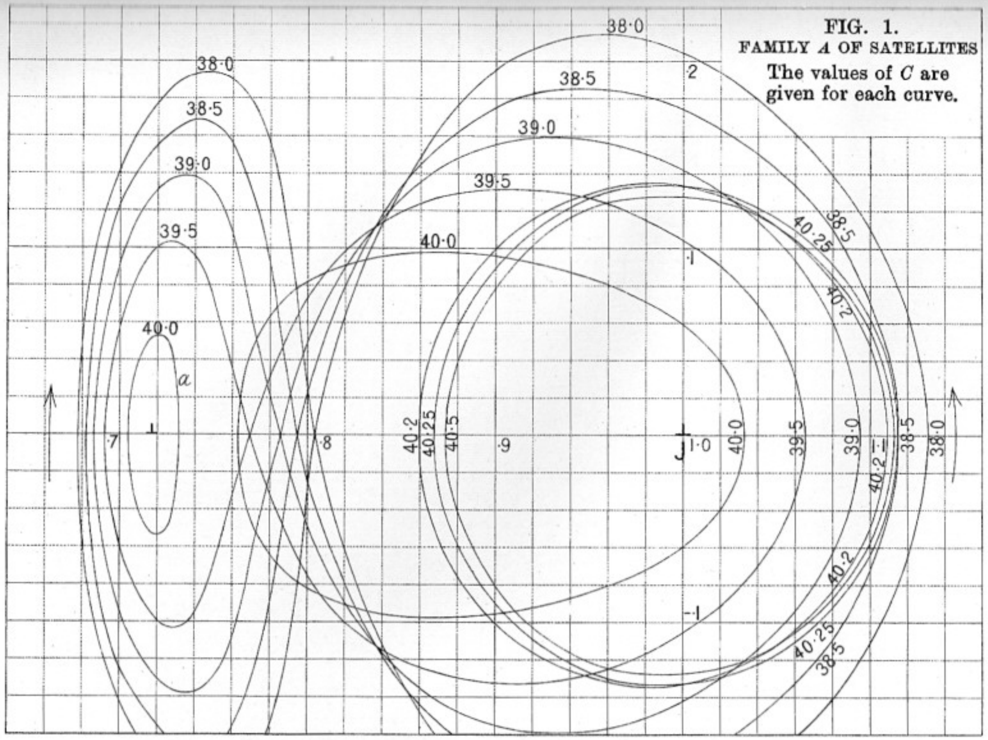


FIG. 2.
FAMILY *B* OF SATELLITES
The values of *C* are given for each curve.

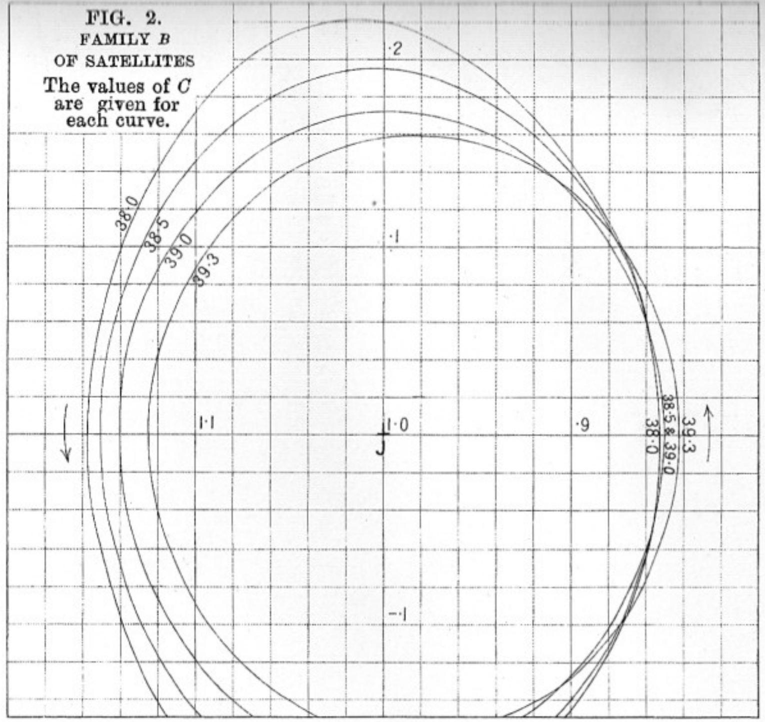


FIG. 3.
FAMILY *C* OF SATELLITES
The values of *C* are given for each curve.

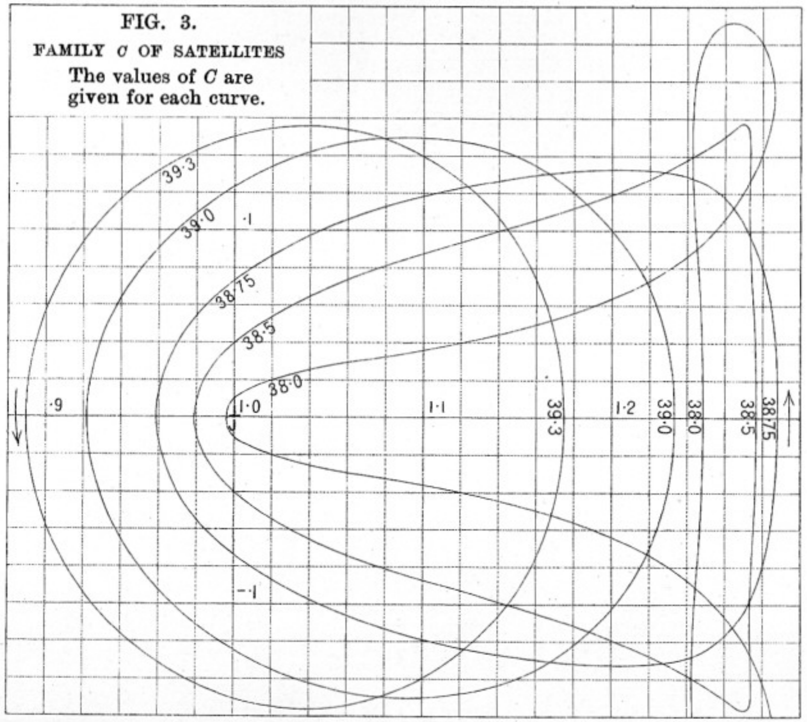
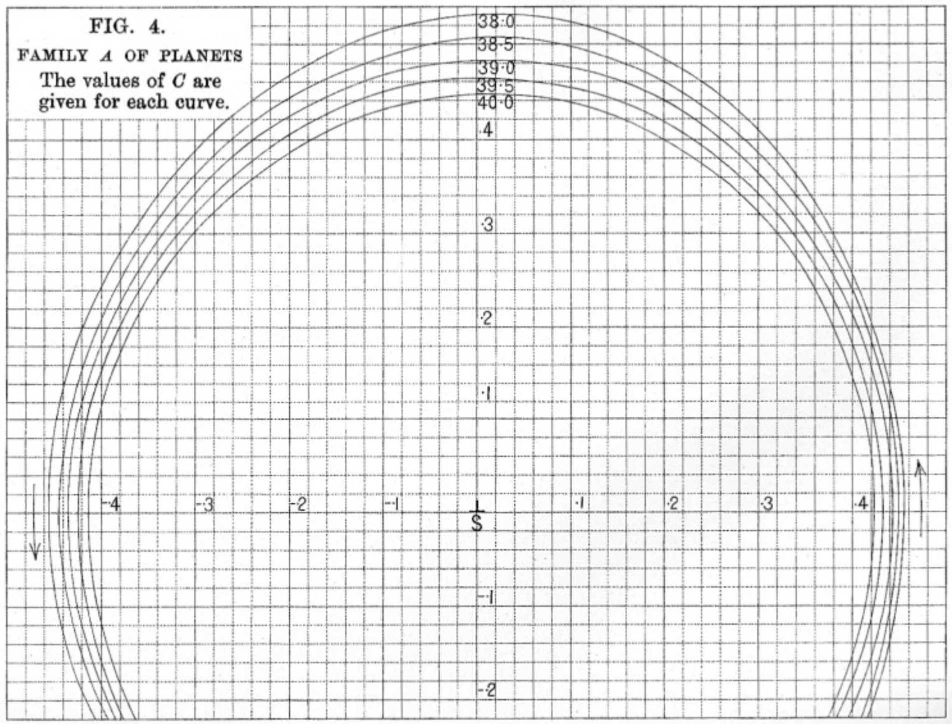


FIG. 4.
FAMILY *A* OF PLANETS
The values of *C* are given for each curve.



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