DISCONTINUOUS MARKOFF PROCESSES

BY

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Summary

The present paper is devoted to the theory of discontinuous Markoff processes, that is processes where the transitions between states take place either by "jumps" of some specified kind, or by other means. States are taken as point x in an abstract space; *phases* are points (x, t) in the product state $\times$ time space; sets of states are denoted by X, sets of phases by S.

It is shown in § 2 that such a process is specified by *two* functions: the probability $\chi_0(X, t \mid x_0, t_0)$ of a transition $x_0 \to X$ *without "jumps"* in the time interval $[t_0, t)$, and the probability distribution $\psi(S \mid x_0, t_0)$ of the first jump time and the consequent state, given an initial phase (x_0, t_0). The total transition probability $\chi(X, t \mid x_0, t_0)$ is required to satisfy the integral equation

$$\chi(X, t \mid x_0, t_0) = \chi_0(X, t \mid x_0, t_0) + \int \chi(X, t \mid \xi, \tau)\, \psi(d\xi, d\tau \mid x_0, t_0). \qquad \text{(I.E.)}$$

The main problem is to study the existence and uniqueness of the solutions of I.E. which also satisfy the conditions (stated in § 1) for being transition probabilities of a Markoff process.

Previous work (cf. § 4) on this subject relates to special cases, mainly to processes where transitions occur *only* by jumps. In § 5, two auxiliary sets of functions are introduced: the distributions $\psi_n(S \mid x_0, t_0)$ of the nth jump time and consequent state (which form a Markoff chain), and the transition probabilities $\chi_n(X, t \mid x_0, t_0)$

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involving *exactly* n jumps. It is shown in § 6 that the probability $\chi_R = \sum_0^\infty \chi_n$ of a "*regular*" transition involving a *finite* number of jumps satisfies I.E., and that this is properly normalized to unity if and only if the cumulative distribution $\sigma_\infty(t \mid x_0, t_0)$ of the time of occurrence of a "singular jump", i.e. one involving an infinity of "ordinary jumps", is identically zero. It turns out in § 8 that the condition $\sigma_\infty \equiv 0$ is also necessary and sufficient for χ_R to be the *unique* solution of I.E. The remainder of § 8 is devoted to the study of a class of solutions of I.E. for *unstable processes* (namely, those for which $\sigma_\infty \not\equiv 0$) which is obtained by applying the foregoing theory to the process viewed in terms of the "singular jumps", with χ_R (the transition probability involving only "ordinary jumps") playing the role of χ_0 and a postulated distribution $\psi^{(1)}$ for the first "singular jump" time and consequent state instead of ψ. This procedure can be repeated if the process thus viewed is again unstable, the numbers of "ordinary jumps" at each stage being multiples of the successive powers of the first transfinite ordinal. § 9 is devoted to certain properties of those *trapping phases* (x_0, t_0) at which there is a probability unity of an instantaneous "singular jump": i.e. $\sigma_\infty(t \mid x_0, t_0) = 1$ for all $t > t_0$. In § 10 a tentative study is made of processes where the Markoff chain of distributions $\{\psi_n\}$ tends to an ergodic limit. A set of examples illustrating various features of the theory is given in § 11.

1. Markoff processes

The concept of a Markoff process is obtained by abstraction from physical processes involving systems whose state x changes with the time t according to some chance law, such that the probability of a transition from a given state x_0 at time t_0 to a state x at a later time t depends only on the state x_0 at t_0, and is independent of the states of the system at times prior to t_0.

The *state space* $\mathcal{X}$ is the set of all possible states x of the systems; it is assumed that a *Borel field* $\mathcal{B}_x$ of subsets of $\mathcal{X}$ is defined. Let $\mathcal{T}$ be the time axis, $\mathcal{B}_t$ the Borel field of subsets T of $\mathcal{T}$ generated by the intervals of $\mathcal{T}$; in order to avoid trivial complications it will be assumed that $\mathcal{T} = (0, \infty)$, but the subsequent considerations remain valid if $\mathcal{T}$ is any real interval, finite or infinite. An ordered pair $s = (x, t)$ is a *phase* of the process, the cartesian product space $\mathcal{S} = \mathcal{X} \times \mathcal{T}$ its *phase-space*; $\mathcal{B}_s = \mathcal{B}_x \times \mathcal{B}_t$ denotes the minimal Borel field of subsets S of $\mathcal{S}$ containing all rectangle sets $X \times T$ such that $X \in \mathcal{B}_x$ and $T \in \mathcal{B}_t$. The qualification measurable applied to sets X, T, S means that respectively $X \in \mathcal{B}_x$, $T \in \mathcal{B}_t$, $S \in \mathcal{B}_s$; it will often be omitted when this is unlikely to cause confusion, as only measurable sets are considered in this paper.

The notation X^+, T^+, S^+ will be used for the complements of X, T and S respectively. A *measurable function* on $\mathfrak{X}$, $\mathfrak{T}$ or $\mathfrak{S}$ is always a Borel measurable function. There will also be occasion to consider measurable subsets of and measurable functions on $\mathfrak{X}\times\mathfrak{T}^2$, defined analogously. A *distribution* $\varrho(X)$ on $\mathcal{B}_x$ is a measure on $\mathcal{B}_x$ such that $\varrho(\mathfrak{X})\leqslant 1$; it is a *probability distribution* if and only if it is normalized to unity: i.e. in case $\varrho(\mathfrak{X})=1$. A *conditional distribution* $\chi(X\,|\,x)$ on $\mathcal{B}_x\times\mathfrak{X}$ is a distribution on $\mathcal{B}_x$ for fixed x(1) and a measurable function on $\mathfrak{X}$ for fixed X(1); it is a *conditional probability distribution* if and only if $\chi(\mathfrak{X}\,|\,x)\equiv 1$. Similar definitions avail for distributions on $\mathcal{B}_t$, $\mathcal{B}_s$ and conditional distributions on $\mathcal{B}_t\times\mathfrak{T}$, $\mathcal{B}_s\times\mathfrak{S}$.

The *instantaneous* state of the system in Markoff process is specified by its *instantaneous distribution* $\varrho(X,t)$: i.e. the probability that $x\in X$ at t; this is a function on $\mathcal{B}_x\times\mathfrak{T}$ which for fixed t is a distribution on $\mathcal{B}_x$; this specification is *incomplete* unless $\varrho(\mathfrak{X},t)\equiv 1$. The *temporal evolution* of the process is specified by its *transition distribution* $\chi(X,t\,|\,x_0,t_0)$: i.e. the probability of a transition $x_0\to X$ in $[t_0,t)$, or in other words, the probability that $x\in X$ at t conditional on x_0 at t_0. This is a function on $\mathcal{B}_x\times\mathfrak{T}\times\mathfrak{S}$ satisfying the following conditions:

(1) $\chi(X,t\,|\,x_0,t_0)$ *is a distribution on* $\mathcal{B}_x$ *for fixed* t, x_0, t_0, *a measurable function on* $\mathfrak{S}$ *for fixed* X, t; hence it is a conditional distribution on $\mathcal{B}_x\times\mathfrak{X}$ for fixed t, t_0.

(2) χ satisfies the *Chapman-Kolmogoroff equation* (briefly C.K. equation):

$$\chi(X,t\,|\,x_0,t_0)=\int_{\mathfrak{X}}\chi(X,t\,|\,\xi,\tau)\,\chi(d\xi,\tau\,|\,x_0,t_0),\quad (t\geqslant\tau\geqslant t_0). \tag{1.1}$$

(3) $$\chi(X,t\,|\,x_0,t_0)=\delta(X\,|\,x_0)=\begin{cases}1 \text{ if } x_0\in X\\ 0 \text{ otherwise}\end{cases}\text{ if } t\leqslant t_0. \tag{1.2}$$

These will be called the *incomplete Markoff process conditions* (briefly, I.M.P. conditions) because the specification of the process is incomplete unless in addition χ satisfies:

(4) $$\chi(\mathfrak{X},t\,|\,x_0,t_0)\equiv 1, \tag{1.3}$$

in which case it will be termed a *transition probability* and will be said to satisfy the *complete Markoff process conditions* (briefly, C.M.P. conditions). The transition distribution of a process determines the transformation with time of its instantaneous distribution by the relation

$$\varrho(X,t)=\int_{\mathfrak{X}}\chi(X,t\,|\,x_0,t_0)\,\varrho(dx_0,t_0),\quad (t\geqslant t_0). \tag{1.4}$$

The process will be termed *time-homogeneous* if χ depends only on $t-t_0$.

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(1) By "fixed x", "fixed X" etc. we shall always mean "each fixed $x\in\mathfrak{X}$", "each fixed $X\in\mathcal{B}_x$" and so on.

A few relevant remarks may be added here:

1) As is evident from (1.1) and (1.4), χ specifies the evolution of the process for increasing time only, and hence need be defined only for $t \geqslant t_0$; condition (3) is a convention introduced to complete the definition of χ for all values of $t \in \mathcal{J}$.

2) Contrary to the usual practice in the literature, the additional continuity condition

$$\lim_{t \downarrow t_0} \chi(X, t \mid x_0, t_0) = \delta(X \mid x_0) \tag{1.5}$$

will *not* be imposed in the present paper; instead, the behaviour of χ as $t \downarrow t_0$ will be investigated (cf. § 6).

3) No attempt is made to set up a complete "probabilistic" scheme for the processes studied: i.e. to define a probability measure for the space of all "realized functions" $x(t)$. This precludes the use of "probability arguments" in proofs, which will consequently be purely analytical; however, such arguments or interpretations will sometimes be briefly sketched as an aid to intuition.

2. Discontinuous Markoff processes

Discontinuous Markoff processes are taken here to be loosely speaking *the class of Markoff processes where the state of the system can change by sudden chance jumps*; the precise definition is given later in this section. Previous work (Feller [4, 5], Pospišil [14], Doeblin [2], Doob [3]) was concerned mainly with the more restricted class where the state remains unchanged between jumps, and moreover a probability rate $q(x, t)$ (probability per unit time) is defined for the jumps. The application of the present theory to this sub-class and hence its connection with previous work are discussed in § 4. A discontinuous Markoff process is specified by two functions:

1) The probability $\chi_0(X, t \mid x_0, t_0)$ of a transition $x_0 \to X$ in $[t_0, t)$ with *no* jumps.

2) The probability $\psi(S \mid x_0, t_0)$ that $(x, t) \in S$, where t is the *first jump time* and x is the state to which the system is taken by this first jump (*the consequent state*), given the *initial phase* (x_0, t_0). It is important to notice that "jumps" in the above may refer to jumps of a specified kind, and hence that a transition with no jumps of this kind, whose probability is given by χ_0, may occur partly as a result of jumps of some other kind. We write $\varkappa_0(t \mid x_0, t_0) = \chi_0(\mathcal{X}, t \mid x_0, t_0)$ for the probability of no jumps in $[t_0, t)$ given (x_0, t_0); $\psi(X, t \mid x_0, t_0) = \psi(X \times [t_0, t) \mid x_0, t_0)$ for the probability that the first jump time lies in $[t_0, t)$ and the consequent state $x \in X$ given (x_0, t_0); $\sigma(t \mid x_0, t_0) = \psi(\mathcal{X}, t \mid x_0, t_0)$ for the cumulative distribution of the firt jump time given (x_0, t_0).

The two functions χ_0, ψ are postulated to satisfy the *$\chi_0\psi$-conditions*:

(1) $\chi_0(X, t \mid x_0, t_0)$ *satisfies the I.M.P. conditions of* § 1.

(2) $\psi(S \mid x_0, t_0)$ *is a conditional distribution on* $\mathcal{B}_s \times S$.

(3) $$\psi(S \mid x_0, t_0) = \lim_{t\to\infty} \sigma(t \mid x_0, t_0) = 1 - \lim_{t\to\infty} \varkappa_0(t \mid x_0, t_0).$$

(4) $$\psi(X, t \mid x_0, t_0) = \psi(X, \tau \mid x_0, t_0) + \int_{\mathfrak{X}} \psi(X, t \mid \xi, \tau)\, \chi_0(d\xi, \tau \mid x_0, t_0),$$
$$(t \geqslant \tau \geqslant t_0). \tag{2.1}$$

(5) *For fixed* x_0, t_0, $\sigma(t \mid x_0, t_0)$ *is continuous to the left in* t *and vanishes for* $t \leqslant t_0$.

LEMMA 2.1 $$\sigma(t \mid x_0, t_0) = 1 - \varkappa_0(t \mid x_0, t_0). \tag{2.2}$$

For $t \leqslant t_0$, this follows immediately from (5) and I.M.P. condition (3), which implies that $\varkappa_0 = 1$ for $t \leqslant t_0$. It follows from (4) and I.M.P. condition (2) that

$$\sigma(t \mid x_0, t_0) = \sigma(\tau \mid x_0, t_0) + \int_{\mathfrak{X}} \sigma(t \mid \xi, \tau)\, \chi_0(d\xi, \tau \mid x_0, t_0) \tag{2.3}$$

$$\varkappa_0(t \mid x_0, t_0) = \int_{\mathfrak{X}} \varkappa_0(t \mid \xi, \tau)\, \chi_0(d\xi, \tau \mid x_0, t_0) \tag{2.4}$$

for all $t \geqslant \tau \geqslant t_0$. Making $t \to \infty$ in both equations, it follows from (3) that

$$1 = \sigma(\tau \mid x_0, t_0) + \varkappa_0(\tau \mid x_0, t_0).$$

Hence the lemma is also true for $t > t_0$. This result has an obvious probability interpretation: for if $\varkappa_0$ is the probability of no jumps in $[t_0, t)$, then $\sigma = 1 - \varkappa_0$ is the probability that the first jump time lies in $[t_0, t)$ (and is also the probability of one or more jumps in $[t_0, t)$).

It follows immediately from the $\chi_0\psi$-conditions and lemma 2.1 that (1) for fixed x_0, t_0, $\varkappa_0(t \mid x_0, t_0)$ is a non-increasing function on $\mathfrak{T}$ continuous to the left; (2) ψ is a conditional *probability* distribution if and only if $\sigma(\infty \mid x_0, t_0) \equiv 1$; (3) $\psi(X, t \mid x_0, t_0)$ is a non-decreasing function on $\mathfrak{T}$ continuous to the left for X, x_0, t_0 fixed, a distribution on $\mathcal{B}_x$ for t, x_0, t_0 fixed, a measurable function on S for X, t fixed, and vanishes for $t \leqslant t_0$.

A precise definition of a discontinuous Markoff process can now be given: *A discontinuous Markoff process specified by a pair of functions* χ_0, ψ *satisfying the* $\chi_0\psi$-*conditions is a process whose transition probability* χ *satisfies the C.M.P. conditions of* § 1 *and the integral equation.*

$$\chi(X, t \mid x_0, t_0) = \chi_0(X, t \mid x_0, t_0) + \int_{\mathfrak{X} \times [t_0, t)} \chi(X, t \mid \xi, \tau)\, \psi(d\xi, d\tau \mid x_0, t_0). \tag{2.5}$$

This equation (referred to briefly henceforth as I.E.) is suggested by the following heuristic "probability argument": the probability $\chi(X, t \mid x_0, t_0)$ of a transition $x_0 \to X$ in $[t_0, t)$ must be the sum of the probabilities $\chi_0(X, t \mid x_0, t_0)$ and say $\chi_J(X, t \mid x_0, t_0)$ of such a transition with *no* jumps and *at least one* jump respectively. The second in its turn must be the sum over $\mathcal{X}$ and $[t_0, t)$ of the product of $\psi(d\xi, d\tau \mid x_0, t_0)$, the probability that the first jump occur in $(\tau, \tau + d\tau)$ with consequent state in $\{d\xi\}$, and $\chi(X, t \mid \xi, \tau)$, the probability of a transition $\xi \to X$ in $[\tau, t)$, and is hence equal to the second term in the right-hand-side of I.E. It was introduced independently by a number of authors for various special processes (see e.g. Bartlett [1], where further references will be found); in particular it was used by Doob [3] in the case of time-homogeneous processes with no change of state between jumps and state space $\mathcal{R}_1$ (i.e. the real line).

A solution of I.E. satisfying the I.M.P. conditions only may be interpreted as a *transition distribution giving an "incomplete description" of a discontinuous process.* It remains to be shown that discontinuous processes defined as above exist, to investigate their properties and the conditions under which they are uniquely and completely defined. In other words, our problem is: *given the functions χ_0, ψ, to inquire into existence, uniqueness and properties of solutions of I.E. satisfying the I.M.P. or C.M.P. conditions.*

3. Singular and regular phases. Compositions.

The specification of discontinuous Markoff processes given in § 2 does not preclude positive probabilities for instantaneous jumps: i.e. it is possible that for certain initial phases (x_0, t_0)

$$\lim_{t \downarrow t_0} \sigma(t \mid x_0, t_0) = \sigma(t_0 + 0 \mid x_0, t_0) > 0.$$

A phase (x, t) will be called *singular* if $\sigma(t+0 \mid x, t) = 1$, *regular* if $\sigma(t+0 \mid x, t) = 0$. $\mathcal{D}_n = \{(x, t) \mid \sigma(t + 1/n \mid x, t) = 1\}$ is a non-increasing sequence of measurable subsets of $\mathcal{S}$ (because $\sigma(t_1 \mid x, t)$ is non-decreasing in t_1); clearly the *set of all singular phases* $\mathcal{D} = \lim_{n\to\infty} \mathcal{D}_n$. Similarly $\mathcal{R}_{nk} = \{(x, t) \mid \sigma(t + 1/n \mid x, t) \leqslant 1/k\}$ is a non-decreasing sequence for fixed k, and $\lim_{n\to\infty} \mathcal{R}_{nk}$ is clearly a non-increasing sequence; the *set of all regular* phases $\mathcal{R} = \lim_{k\to\infty} \lim_{n\to\infty} \mathcal{R}_{nk}$.

Lemma 3.1. *$\mathcal{D}$ and $\mathcal{R}$ are measurable subsets of $\mathcal{S}$, and $\mathcal{R} \subset \mathcal{D}^+$.*

Let $\eta(X \mid x_0, t_0) = \psi(X, t_0 + 0 \mid x_0, t_0)$; $\eta(\mathcal{X} \mid x_0, t_0) = 1$ if and only if $(x_0, t_0) \in \mathcal{D}$.

LEMMA 3.2. *If* $(x_0, t_0) \in \mathcal{D}$, *then*

$$\psi(X, t \mid x_0, t_0) = \eta(X \mid x_0, t_0)\, \varepsilon(t - t_0), \quad \text{where } \varepsilon(t) = \begin{cases} 1 \text{ if } t > 0, \\ 0 \text{ if } t \leqslant 0. \end{cases} \tag{3.1}$$

Since $\psi(X, t \mid x_0, t_0)$ is non-decreasing in t, it is $\geqslant \eta(X \mid x_0, t_0)$ for $t \geqslant t_0$; suppose that it is $>$; then if $(x_0, t_0) \in \mathcal{D}$, $t > t_0$

$$\begin{aligned} \sigma(t \mid x_0, t_0) &= \psi(X, t \mid x_0, t_0) + \psi(X^+, t \mid x_0, t_0) \\ &> \eta(X \mid x_0, t_0) + \eta(X^+ \mid x_0, t_0) \\ &= \sigma(t_0 + 0 \mid x_0, t_0) = 1, \end{aligned}$$

which is impossible; hence $\psi(X, t \mid x_0, t_0) = \eta(X \mid x_0, t_0)$ for $t \geqslant t_0$; this proves the lemma.

It is convenient to gather at this stage a few lemmas relative to conditional distributions that will be used repeatedly in the sequel. Since they are more or less well known, their proofs are relegated to an Appendix.

LEMMA 3.3. *Let* $\beta(S \mid x_0, t_0)$ *be a conditional distribution on* $\mathcal{B}_s \times \mathcal{S}$. *If* $\alpha(x, t)$ *is a bounded measurable function on* $\mathcal{S}$, *then for every* $S \in \mathcal{B}_s$

$$\gamma(x_0, t_0) = \int_S \alpha(x, t)\, \beta(dx, dt \mid x_0, t_0) \tag{3.2}$$

exists and is a bounded measurable function on $\mathcal{S}$. *If* $S = X \times T$, *where* $X \in \mathcal{B}_x$ *and* $T \in \mathcal{B}_t$, *then*

$$\gamma(x_0, t_0) = \int_T \int_X \alpha(x, t)\, \beta(dx, dt \mid x_0, t_0) = \int_X \int_T \alpha(x, t)\, \beta(dx, dt \mid x_0, t_0). \tag{3.3}$$

If $\alpha(S \mid x, t)$ *is a conditional distribution on* $\mathcal{B}_s \times \mathcal{S}$, *then for every* $S' \in \mathcal{B}_s$

$$\gamma(S \mid x_0, t_0) = \int_{S'} \alpha(S \mid x, t)\, \beta(dx, dt \mid x_0, t_0) \tag{3.4}$$

exists and is likewise a conditional distribution on $\mathcal{B}_s \times \mathcal{S}$.

Note that the conclusions of the lemma apart from (3.2) are still true if we substitute $(\mathcal{X}, \mathcal{B}_x)$ or $(\mathcal{T}, \mathcal{B}_t)$ for $(\mathcal{S}, \mathcal{B}_s)$. We call γ the *composition* of α and β.

LEMMA 3.4. *Compositions are associative: let* $\alpha(x, t)$ *be a bounded measurable function on* $\mathcal{S}$, $\beta(S \mid x, t)$ and $\gamma(S \mid x, t)$ *conditional distributions on* $\mathcal{B}_s \times \mathcal{S}$; *then for every* $S_1 \in \mathcal{B}_s$, $S_2 \in \mathcal{B}_s$

$$\begin{aligned} &\int_S \left\{ \int_{S_2} \alpha(x_2, t_2)\, \beta(dx_2, dt_2 \mid x_1, t_1) \right\} \gamma(dx_1, dt_1 \mid x_0, t_0) \\ &\qquad = \int_{S_2} \alpha(x_2, t_2) \left\{ \int_{S_1} \beta(dx_2, dt_2 \mid x_1, t_1)\, \gamma(dx_1, dt_1 \mid x_0, t_0) \right\}. \end{aligned} \tag{3.5}$$

It will be convenient to introduce the symbolic notation $\gamma = \alpha * \beta$ for the particular compositions

$$\gamma(t, x_0, t_0) = \int_{\mathfrak{X}\times[t_0,t)} \alpha(t, \xi, \tau)\,\beta(d\xi, d\tau \mid x_0, t_0) = \int_{t_0}^{t}\int_{\mathfrak{X}} \alpha(t, \xi, \tau)\,\beta(d\xi, d\tau \mid x_0, t_0).$$

The symbol $\int_{t_0}^{t}$ in such compositions will always be taken to mean integration over $[t_0, t)$; i.e. strictly $\int_{t_0-0}^{t-0}$.

LEMMA 3.5. *Let α, β, γ be as in Lemma 3.4., and let $\zeta = \beta * \gamma$. If $\beta(X, t \mid x_0, t_0)$ and $\gamma(X, t \mid x_0, t_0)$ vanish for $t \leqslant t_0$, then so does $\zeta(X, t \mid x_0, t_0)$. It then follows that*

$$(\alpha * \beta) * \gamma = \alpha * (\beta * \gamma). \tag{3.6}$$

To prove this, note that

$$\zeta(X, t \mid x_0, t_0) = \int_{t_0}^{t}\int_{\mathfrak{X}} \beta(X, t \mid \xi, \tau)\,\gamma(d\xi, d\tau \mid x_0, t_0) = \int_{S} \beta(X, t \mid \xi, \tau)\,\gamma(d\xi, d\tau \mid x_0, t_0)$$

because $\beta = 0$ for $\tau \geqslant t$ and $\gamma = 0$ for $\tau \leqslant t_0$ in the integrand; hence clearly $\zeta = 0$ for $t \leqslant t_0$. Similarly

$$\int_{t_0}^{t}\int_{\mathfrak{X}} \alpha(\xi, \tau)\,\beta(d\xi, d\tau \mid x_0, t_0) = \int_{0}^{t}\int_{\mathfrak{X}} \alpha(\xi, \tau)\,\beta(d\xi, d\tau \mid x_0, t_0);$$

(3.6) then follows if we let $S_1 = \mathbf{S}$ and $S_2 = \mathfrak{X} \times [0, t)$ in (3.5). In this symbolic notation, I.E. becomes

$$\chi * (I - \psi) = \chi_0 \tag{3.7}$$

where

$$I(S \mid x, t) = \begin{cases} 1 & \text{if } (x, t) \in S \\ 0 & \text{otherwise} \end{cases}$$

is the "unit" phase-space conditional distribution; note that

$$I(X, t \mid x_0, t_0) = \delta(X \mid x_0)\,\varepsilon(t - t_0),$$

where $\delta(X \mid x_0)$ was defined in (1.2), $\varepsilon(t)$ in (3.1).

LEMMA 3.6. *Let $\{\alpha_n\}$ be a sequence of conditonal distributions on $\mathfrak{B}_s \times \mathbf{S}$, converging to $\alpha(S \mid x, t)$ for every $S \in \mathfrak{B}_s$ and $(x, t) \in \mathbf{S}$; then α is a conditional distribution on $\mathfrak{B}_s \times \mathbf{S}$. Let β be any conditional distribution on $\mathfrak{B}_s \times \mathbf{S}$; then for every $S \in \mathfrak{B}_s$ and $(x, t) \in \mathbf{S}$*

$$\lim_{n\to\infty} \alpha_n * \beta = \alpha * \beta, \tag{3.8}$$

$$\lim_{n\to\infty} \beta * \alpha_n = \beta * \alpha. \tag{3.9}$$

The notation $S(t)$, $S(x)$ will be used for respectively t- and x-sections of S.

LEMMA 3.7. *Let $\beta(X \mid x, t)$ be a function on $\mathcal{B}_x \times S$, which is a distribution on $\mathcal{B}_x$ for fixed x, t and a measurable function on S for fixed X; then $\beta'(S \mid x, t) = \beta(S(t) \mid x, t)$ is a conditional distribution on $\mathcal{B}_s \times S$, with $\beta'(X, t \mid x_0, t_0) = \beta(X \mid x_0, t_0)\, \varepsilon(t - t_0)$. Similarly let $\mu(T \mid x, t)$ be a function on $\mathcal{B}_t \times S$, which is a distribution on $\mathcal{J}$ for fixed x, t and a measurable function on S for fixed T; then $\mu'(S \mid x, t) = \mu(S(x) \mid x, t)$ is a conditional distribution on $\mathcal{B}_s \times S$, with $\mu'(X, t \mid x_0, t_0) = \mu([t_0, t))\, \delta(X \mid x_0)$.*

LEMMA 3.8. *Let $\alpha(x, t)$ be a bounded measurable function on S, β and μ as in Lemma 3.7; then*

$$\int_X \alpha(x, t_0)\, \beta(dx \mid x_0, t_0) = \int_{X \times \mathcal{J}} \alpha(x, t)\, \beta'(dx, dt \mid x_0, t_0), \tag{3.10}$$

$$\int_T \alpha(x_0, t)\, \mu(dt \mid x_0, t_0) = \int_{\mathfrak{X} \times T} \alpha(x, t)\, \mu'(dx, dt \mid x_0, t_0). \tag{3.11}$$

These two lemmas (3.7 and 3.8) show that *Lemmas* 3.3, 3.4, *and* 3.6 *apply to compositions like* (3.10) *and* (3.11) *involving only one of the variables x, t.*

4. Step processes and q-processes

The class of discontinuous Markoff processes where there is no change of state between jumps is clearly the class for which χ_0 is of the form

$$\chi_0(X, t \mid x_0, t_0) = \varkappa_0(t \mid x_0, t_0)\, \delta(X \mid x_0) \tag{4.1}$$

where $\delta(X \mid x_0)$ was defined in (1.2) and $\varkappa_0$ is a function $\mathcal{J} \times S$ satisfying the *$\varkappa_0$-conditions*:

(1) $0 \leqslant \varkappa_0 \leqslant 1$; *$\varkappa_0(t \mid x_0, t_0)$ is a measurable functions on S for fixed t, is continuous to the left in t for fixed x_0, t_0 and is equal to* 1 *for $t \leqslant t_0$.*

(2) $$\varkappa_0(t \mid x_0, t_0) = \varkappa_0(t \mid x_0, \tau)\, \varkappa_0(\tau \mid x_0, t_0), \quad (t \geqslant \tau \geqslant t_0). \tag{4.2}$$

It follows from (2) that $\varkappa_0(t \mid x_0, t_0)$ is non-increasing in t when x_0, t_0 are fixed and non-decreasing in t_0 when x_0, t_0 are fixed. Any realization of such a process has the character of a step function; hence processes of this class (i.e. with χ_0 of the form (4.1)) will be called *step processes*. It will now be shown that for step processes

$$\psi(X, t \mid x_0, t_0) = \int_{t_0}^{t} \phi(X \mid x_0, \tau)\, \sigma(d\tau \mid x_0, t_0), \tag{4.3}$$

where $\sigma(t \mid x_0, t_0) = 1 - \varkappa_0(t \mid x_0, t_0)$, and is therefore a non-negative non-decreasing, left-continuous function of t when x_0, t_0 are fixed, inducing a distribution $\sigma(T \mid x_0, t_0)$ on $\mathcal{B}_t$; ϕ is a function on $\mathfrak{X} \times S$, which will be said to satisfy the *ϕ-conditions* if:

$\phi(X \mid x_0, t_0)$ is a probability distribution on $\mathcal{B}_x$ for fixed x_0, t_0, a measurable function on S for fixed X.

THEOREM 4.1. *Let $\chi_0(X, t \mid x_0, t_0)$ be as in (4.1), with $\varkappa_0(t \mid x_0, t_0)$ satisfying the $\varkappa_0$-conditions; let*

$$\psi(S \mid x_0, t_0) = \int_0^\infty \phi(S(t) \mid x_0, t)\, \sigma(dt \mid x_0, t_0) \tag{4.4}$$

where $S(t)$ is a t-section of S, ϕ satisfies the ϕ-conditions and $\sigma = 1 - \varkappa_0$; then the pair χ_0, ψ satisfy the $\chi_0 \psi$-conditions of § 2. Conversely, if a pair of functions χ_0, ψ, where χ_0 is of the form (4.1) satisfy the $\chi_0 \psi$-conditions, then $\varkappa_0$ satisfies the $\varkappa_0$-conditions, and for every X, x_0, t, t_0 a measurable non-negative function $\phi(X \mid x_0, \tau)$ is defined for almost all $(\sigma)\, \tau \in \mathcal{J}$ such that ψ satisfies (4.3); for every fixed x_0 and almost all $(\sigma)\, t \in \mathcal{J}$,

$$\phi(\mathfrak{X} \mid x_0, t) = 1 \quad \text{and} \quad \phi\left(\bigcup_{n=1}^{\infty} X_n \mid x_0, t\right) = \sum_1^\infty \phi(X_n \mid x_0, t), \tag{4.5}$$

for every sequence $\{X_n\}$ of disjoint measurable subsets of $\mathfrak{X}$.

It is easily seen that a function χ_0 defined as in (4.1) satisfies the I.M.P. conditions (1) and (3) if and only if $\varkappa_0$ satisfies the $\varkappa_0$-conditions (1), apart from left-continuity in t, which however is necessary and sufficient for σ to be left-continuous; furthermore, χ_0 satisfies the C.K. equation (I.M.P. condition (2)) if and only if $\varkappa_0$ satisfies (4.2); hence the $\varkappa_0$-conditions are necessary and sufficient for χ_0 and σ to satisfy respectively the $\chi_0 \psi$-conditions (1) and (5). If ϕ satisfies the ϕ-conditions and and $\sigma = 1 - \varkappa_0$, then by Lemma 3.7

$$\phi'(S \mid x_0, t_0) = \phi(S(t_0) \mid x_0, t_0) \quad \text{and} \quad \sigma'(S \mid x_0, t_0) = \sigma(S(x_0) \mid x_0, t_0)$$

are conditional distributions on $\mathcal{B}_s \times S$. If now ψ is defined by (4.4), then by lemma 3.8

$$\psi(S \mid x_0, t_0) = \int_S \phi'(S \mid \xi, \tau)\, \sigma'(d\xi, d\tau \mid x_0, t_0);$$

hence by lemma 3.3 ψ is a conditional distribution on $\mathcal{B}_s \times S$; it is further obvious that $\psi(\mathfrak{X}, t \mid x_0, t_0) = \sigma(t \mid x_0, t_0)$. It follows from (4.2) that

$$\sigma(t \mid x_0, t_0) = \sigma(\tau \mid x_0, t_0) + \sigma(t \mid x_0, \tau)\, \varkappa_0(\tau \mid x_0, t_0), \tag{4.6}$$

which substituted in (4.3) shows that

$$\psi(X, t \mid x_0, t_0) = \psi(X, \tau \mid x_0, t_0) + \psi(X, t \mid x_0, \tau)\, \varkappa_0(\tau \mid x_0, t_0); \tag{4.7}$$

this is identical with (2.1) when χ_0 is of the form (4.1). Hence ψ satisfies the $\chi_0 \psi$-conditions (2), (3) and (4). Thus χ_0 and ψ satisfy the $\chi_0 \psi$-conditions, and this completes the proof of the first part of the theorem. In the converse direction, suppose now that χ_0 and ψ are two functions satisfying the $\chi_0\psi$-conditions, χ_0 being of the form (4.1). It has already been shown that the facts that χ_0 satisfies the I.M.P. conditions, that σ is left-continuous in t and that by lemma 2.1 $\sigma = 1 - \varkappa_0$ imply that $\varkappa_0$ satisfies the $\varkappa_0$-conditions. Since for X, x_0, t_0 fixed $\psi(X, t \mid x_0, t_0)$ is absolutely continuous with respect to $\sigma(T \mid x_0, t_0)$, it follows by the Radon-Nykodim theorem that there exists a non-negative measurable function $\phi(X \mid x_0, \tau)$ defined for almost all (σ) $\tau \in \mathcal{J}$ such that (4.3) is true. Let $\{X_n\}$ be a sequence of disjoint measurable subsets of $\mathfrak{X}$, let O_n be the exceptional subset of $\mathcal{J}$ where $\phi(X_n \mid x_0, \tau)$ is not defined, $n = 1, 2, \ldots$; then $\sigma\left(\bigcup_{n=1}^{\infty} O_n \mid x_0, t_0\right) = 0$; hence it is legitimate to write

$$
\begin{aligned}
\psi\left(\bigcup_{n=1}^{\infty} X_n, t \mid x_0, t_0\right) &= \int_{t_0}^{t} \phi\left(\bigcup_{n=1}^{\infty} X_n \mid x_0, \tau\right) \sigma(d\tau \mid x_0, t_0) \\
&= \sum_{n=1}^{\infty} \psi(X_n, t \mid x_0, t_0) \\
&= \sum_{n=1}^{\infty} \int_{t_0}^{t} \phi(X_n \mid x_0, \tau)\, \sigma(d\tau \mid x_0, t_0) \\
&= \int_{t_0}^{t} \sum_{n=1}^{\infty} \phi(X_n \mid x_0, \tau)\, \sigma(d\tau \mid x_0, t_0),
\end{aligned}
$$

where the last step is justified by Lebesgue's bounded convergence theorem; hence for almost all (σ) $t \in \mathcal{J}$

$$\phi\left(\bigcup_{n=1}^{\infty} X_n \mid x_0, t\right) = \sum_{n=1}^{\infty} \phi(X_n \mid x_0, t).$$

This completes the proof of Theorem 4.1. Note that though the function ϕ in the 2nd part of the theorem has the normalization and complete additivity properties (4.5) it does *not* follow that it satisfies the ϕ-conditions: this is due of course in the first place to the possible existence for every X, x_0 of exceptional subsets of $\mathcal{J}$ where ϕ is not defined, so that considered as a function of all three variables, ϕ might not be defined over any appreciable subset of $\mathfrak{X} \times \mathcal{S}$. The theorem shows essentially that the class of step processes consists of those processes specified by functions χ_0, ψ of the form (4.1) and (4.3).

Another important class of discontinuous Markoff processes, which we shall call *q-processes*, consists of those processes for which a *jump rate*

$$q(x,t)=\lim_{\delta t\downarrow 0}\frac{1}{\delta t}\{1-\varkappa_0(t+\delta t\,|\,x,t)\}\tag{4.8}$$

is defined. With certain additional measurability and continuity restrictions, they are characterized by the following theorem:

THEOREM 4.2. *Let $\chi_0(X,t\,|\,x_0,t_0)$ be a function satisfying the I.M.P. conditions and such that:* (i) *χ_0 is a measurable function on $\mathfrak{J}\times S$ for fixed X, a continuous function on $\mathfrak{J}$ for fixed X, x_0, t_0;* (ii) *for fixed x_0, t_0, $\varkappa_0$ has a continuous derivative $\varkappa_0'(t\,|\,x_0,t_0)$ for all $t\neq t_0$ and a right-hand derivative $-q(x_0,t_0)$ at $t=t_0$;* (iii) *$q(x,t)$ is non-negative and continuous on $\mathfrak{J}$ for fixed x. If $\phi(X\,|\,x,t)$ is a function satisfying the ϕ-conditions and continuous on $\mathfrak{J}$ for fixed X, x, then χ_0 and the function*

$$\psi(S\,|\,x_0,t_0)=\int_{t_0}^{\infty}dt\int_{X}\phi(S(t)\,|\,x,t)\,q(x,t)\,\chi_0(dx,t\,|\,x_0,t_0)\tag{4.9}$$

jointly satisfy the $\chi_0\psi$-conditions; furthermore, $\psi(X,t\,|\,x_0,t_0)$ is measurable on $\mathfrak{J}\times S$ for fixed X and has a continuous derivative

$$\psi'(X,t\,|\,x_0,t_0)=\begin{cases}\int_X\phi(X\,|\,x,t)\,q(x,t)\,\chi_0(dx,t\,|\,x_0,t_0) & \text{for } t\geqslant t_0\\ 0 & \text{for } t<t_0\end{cases}\tag{4.10}$$

for fixed X, x_0, t_0. Conversely, suppose that χ_0 and ψ satisfy the $\chi_0\psi$-conditions; χ_0 satisfies conditions (i), (ii) *and* (iii) *above; $\psi(X,t\,|\,x_0,t_0)$* (a) *is a measurable function on $\mathfrak{J}\times S$ for fixed X,* (b) *for fixed X, x_0, t_0 has a continuous derivative $\psi'(X,t\,|\,x_0,t_0)$ for all $t\neq t_0$, and* (c) *$\psi'(X,t+0\,|\,x,t)$ exists and is continuous on $\mathfrak{J}$ for fixed X, x; then ψ' satisfies* (4.10) *and ψ satisfies* (4.9) *with*

$$\phi(X\,|\,x,t)\,q(x,t)=\psi'(X,t+0\,|\,x,t),\tag{4.11}$$

where ϕ and q have the properties stated in the first part of the theorem.

Conditions (i), (ii) and (iii) imply that $\varkappa'(t\,|\,x_0,t_0)$ is a measurable and non-positive function on $\mathfrak{J}\times S$: for

$$\varkappa_0(t\,|\,x_0,t_0)=1+\int_{t_0}^{t}\varkappa_0'(\theta\,|\,x_0,t_0)\,d\theta\tag{4.12}$$

$$q(x,t)=-\varkappa_0'(t+0\,|\,x,t),\tag{4.13}$$

and that q is a non-negative measurable function on S. Substituting (4.12) in the C.K. equation for χ_0, one finds that

$$\begin{aligned}\varkappa_0(t\,|\,x_0,t_0) &= \int_X \varkappa_0(t\,|\,\xi,\tau)\,\chi_0(d\xi,\tau\,|\,x_0,t_0)\\ &= \varkappa_0(\tau\,|\,x_0,t_0)+\int_\tau^t d\theta\int_X \varkappa_0'(\theta\,|\,\xi,\tau)\,\chi_0(d\xi,\tau\,|\,x_0,t_0)\\ &= \varkappa_0(\tau\,|\,x_0,t_0)+\int_\tau^t \varkappa_0'(\theta\,|\,x_0,t_0)\,d\theta,\quad (t\geqslant\tau\geqslant t_0),\end{aligned}\tag{4.14}$$

where the change in the order of integration in the 2nd line is justified by Fubini's theorem, and the 3rd line is an immediate consequence of (4.12). Hence

$$\varkappa_0'(t\,|\,x_0,t_0)=\int_X \varkappa_0'(t\,|\,\xi,\tau)\,\chi_0(d\xi,\tau\,|\,x_0,t_0)=-\int_X q(\xi,t)\,\chi_0(d\xi,t\,|\,x_0,t_0)\tag{4.15}$$

where the 3rd expression follows by (4.13) on making $t\to\tau$ in the 2nd. If ϕ satisfies the ϕ-conditions, then ψ defined by (4.9) is a conditional distribution on $\mathcal{B}_s\times S$; the proof of this assertion is similar to the proof in theorem 4.1 that ψ defined by (4.4) is such a distribution and will therefore be omitted. It follows that

$$\psi(X,t\,|\,x_0,t_0)=\int_{t_0}^t d\tau\int_X \phi(X\,|\,\xi,\tau)\,q(\xi,\tau)\,\chi_0(d\xi,\tau\,|\,x_0,t_0)\tag{4.16}$$

for $t\geqslant t_0$, and vanishes for $t\leqslant t_0$; hence by (4.15)

$$\sigma(t\,|\,x_0,t_0)=\psi(\mathfrak{X},t\,|\,x_0,t_0)=\int_{t_0}^t d\tau\int_X q(x,\tau)\,\chi_0(dx,\tau\,|\,x_0,t_0)=1-\varkappa_0(t\,|\,x_0,t_0).\tag{4.17}$$

Finally, substituting the C.K. equation for χ_0 in the right-hand side of (4.16),

$$\begin{aligned}\psi(X,t\,|\,x_0,t_0) &= \psi(X,\tau\,|\,x_0,t_0)+\int_\tau^t d\theta\int_X \phi(X\,|\,\zeta,\theta)\,q(\zeta,\theta)\int_X \chi_0(d\zeta,\theta\,|\,\xi,\tau)\,\chi_0(d\xi,\tau\,|\,x_0,t_0)\\ &= \psi(X,\tau\,|\,x_0,t_0)+\int_X \psi(X,t\,|\,\xi,\tau)\,\chi_0(d\xi,\tau\,|\,x_0,t_0),\end{aligned}\tag{4.18}$$

where the passage to the 2nd line is justified by an argument similar to that used in the proof of lemma 3.4. This completes the proof that χ_0 and ψ satisfy the $\chi_0\psi$-conditions. If $\phi(X\,|\,x,t)$ is continuous on $\mathcal{J}$ for X and x fixed, then the existence of the continuous derivative (4.10) follows from (4.16). If conversely $\psi(X,t\,|\,x_0,t_0)$ satisfies conditions (a), (b) and (c) in the second part of the theorem, then $\psi'(X,t\,|\,x_0,t_0)$ is a measurable function on $\mathcal{J}\times S$ for fixed X. Let $\Phi(X\,|\,x,t)=\psi'(X,t\,|\,x,t)$; then since $\sigma=1-\varkappa_0$,

$$\Phi(\mathfrak{X}\,|\,x,t)=\sigma'(t+0\,|\,x,t)=-\varkappa_0'(t+0\,|\,x,t)=q(x,t).\tag{4.19}$$

It is then easily seen that $\phi(X \mid x, t) = \Phi(X \mid x, t)/q(x, t)$ satisfies the ϕ-conditions and is continuous on $\mathcal{J}$ for X, x fixed. It follows from (2.1) that

$$\psi(X, t \mid x_0, t_0) - \psi(X, \tau \mid x_0, t_0) = \int_\tau^t d\theta \int_X \psi'(X, \theta \mid \xi, \tau)\, \chi_0(d\xi, \tau \mid x_0, t_0)$$

$$= \int_\tau^t \psi'(X, \theta \mid x_0, t_0)\, d\theta, \quad (t \geqslant \tau \geqslant t_0), \tag{4.20}$$

where the change in the order of integration in the 1st line is justified by Fubini's theorem. Hence ψ' satisfies (4.10), because

$$\psi'(X, t \mid x_0, t_0) = \int_X \psi'(X, t \mid \xi, \tau)\, \chi_0(d\xi, \tau \mid x_0, t_0)$$

$$= \int_X \phi(X \mid \xi, t)\, q(\xi, t)\, \chi_0(d\xi, t \mid x_0, t_0),$$

where the last expression follows by (4.11) on making $t \to \tau$ in the 2nd. It follows that ψ satisfies (4.9), because $\psi(S \mid x_0, t_0)$ defined by (4.9) is the extention of $\psi(X, t \mid x_0, t_0)$ to a measure on $\mathcal{B}_s$, and such an extension is unique. This completes the proof of theorem 4.2.

Consider now step processes which are also q-processes in the sense of theorem 4.2; they are characterized by the following lemma:

LEMMA 4.3. *A function $\varkappa_0$ on $\mathcal{J} \times S$ satisfies the $\varkappa_0$-conditions and conditions* (ii) *and* (iii) *of Theorem* 4.2 *if and only if it is of the form*

$$\varkappa_0(t \mid x_0, t_0) = \exp\left\{-\int_{t_0}^t q(x_0, \tau)\, d\tau\right\}, \quad (t \geqslant t_0), \tag{4.21}$$

where $q(x, t)$ is a measurable non-negative function on S, continuous on $\mathcal{J}$ for fixed x. The "if" part of the lemma is obvious; the "only if" follows from the fact that (4.21) is the unique solution with the initial condition $\varkappa_0(t_0 \mid x_0, t_0) = 1$ of

$$\frac{\partial}{\partial t} \varkappa_0(t \mid x_0, t_0) = q(x_0, t)\, \varkappa_0(t \mid x_0, t_0), \tag{4.22}$$

which in turn is implied by (4.2) and (4.8).

COROLLARY. *A step process specified by a pair of functions χ_0, ψ satisfying the conditions laid down in the first part of Theorem* 4.1 *is also a q-process in the sense of Theorem* 4.2 *if and only if $\varkappa_0$ is of the form* (4.21), *where $q(x\ t)$ has the properties stated in the lemma.*

For a process of this class, $\psi(X, t \mid x_0, t_0)$ has the continuous derivative

$$\psi'(X, t \mid x_0, t_0) = \phi(X \mid x_0, t)\, q(x_0, t) \exp\left\{-\int_{t_0}^{t} q(x_0, \tau)\, d\tau\right\}, \tag{4.23}$$

which substituted in I.E. (2.5) gives

$$\chi(X, t \mid x_0, t_0) = \exp\left\{-\int_{t_0}^{t} q(x_0, \tau)\, d\tau\right\} \delta(X \mid x_0) +$$
$$+ \int_{t_0}^{t} \exp\left\{-\int_{t_0}^{\tau} q(x_0, \theta)\, d\theta\right\} q(x_0, \tau)\, d\tau \int_{X} \chi(X, t \mid \xi, \tau)\, \phi(d\xi \mid x_0, \tau). \tag{4.24}$$

Fix X, t and x_0; it is evident that χ has the derivative

$$\frac{\partial}{\partial t_0} \chi(X, t \mid x_0, t_0) = q(x_0, t_0) \Big[\exp\left\{-\int_{t_0}^{t} q(x_0, \tau)\, d\tau\right\} \delta(X \mid x_0) +$$
$$+ \int_{t_0}^{t} \exp\left\{-\int_{t_0}^{\tau} q(x_0, \theta)\, d\theta\right\} q(x_0, \tau)\, d\tau \int_{X} \chi(X, t \mid \xi, \tau)\, \phi(d\xi \mid x_0, \tau) -$$
$$- \int_{X} \chi(X, t \mid \xi, t_0)\, \phi(d\xi \mid x_0, t_0)\Big], \tag{4.25}$$

which after substitution of (4.24) for the 1st and 2nd terms in the curly brackets of the right-hand-side becomes the so-called "backward integro-differential equation"

$$\frac{\partial}{\partial t_0} \chi(X, t \mid x_0, t_0) = q(x_0, t_0) \left\{\chi(X, t \mid x_0, t_0) - \int_{X} \chi(X, t \mid \xi, t_0)\, \phi(d\xi \mid x_0, t_0)\right\}. \tag{4.26}$$

Conversely, suppose that χ satisfies (4.26), where ϕ and q satisfy the conditions stated in theorem 4.2. Fix X, x_0 and t; then (4.26) can be written, suppressing the fixed variables

$$\left.\begin{aligned} &\frac{\partial \chi(t_0)}{\partial t_0} - q(t_0)\, \chi(t_0) = -F(t_0), \\ \text{where} \qquad &F(t_0) = q(x_0, t_0) \int_{X} \chi(X, t \mid \xi, t_0)\, \phi(d\xi \mid x_0, t_0). \end{aligned}\right\} \tag{4.27}$$

Since $q(t_0)$ is continuous, the unique solution of the 1st order differential equation (4.27) with the boundary condition $\chi(t) = \delta(X \mid x_0)$ is

$$\chi(t_0) = \exp\left\{-\int_{t_0}^{t} q(\tau)\, dt\right\} \delta(X \mid x_0) + \int_{t_0}^{t} \exp\left\{-\int_{t_0}^{\tau} q(\theta)\, d\theta\right\} F(\tau)\, d\tau, \tag{4.28}$$

which after substitution for F is seen to be identical with (4.24). Hence:

Lemma 4.4. *In the case of q-step processes with a continuous jump rate, the integral equation* (2.5) *of* § 2 *implies the backward integro-differential equation* (4.25) *and conversely.*

The restriction on the jump rate $q(x, t)$ to be continuous on $\mathcal{J}$ is quite severe: it would exclude for instance $q(x, t) = (x - t)^{-2}$ in the case were $\mathfrak{X} = \mathcal{R}_1$. It is apparent, however, that it can be considerably relaxed. Take the case of q-step processes: let $q(x, t)$ be an extended non-negative measurable function on $\mathcal{S}$; then it is easily seen that

$$\varkappa_0(t \mid x_0, t_0) = \exp\left\{-\int_{t_0-0}^{t-0} q(x_0, \tau)\, d\tau\right\},$$

where the integral is taken in the extended sense, satisfies the $\varkappa_0$-conditions, and hence can be used as in the corollary of lemma 4.3 to construct a pair χ_0, ψ specifying a q-step process; lemma 4.4 is of course no longer true in this case. The following results are easily proved. Let $t_j(x_0, t_0) = \inf\{t \mid \varkappa_0(t \mid x_0, t_0) = 0 \text{ and} t > t_0\}$ if the set is non-vacuous, $t_j(x_0, t_0) = +\infty$ if it is. Then $\varkappa_0(t \mid x_0, t_0)$ is 0 for $t > t_j(x_0, t_0)$, > 0 for $t < t_j(x_0, t_0)$; it is absolutely continuous on $[t_0, a]$ for every $a < t_j(x_0, t_0)$, and it exhibits at most one saltus, namely, a jump down to 0 at $t = t_j(x_0, t_0)$ if $\varkappa_0(t_j(x_0, t_0) \mid x_0, t_0) > 0$. Also

$$\varkappa_0(t_2 \mid x_0, t_1) = 0 \text{ if } t_2 > t_j(x_0, t_0) > t_1;$$

hence $$\varkappa_0(t_j(x_0, t_0) + 0 \mid x_0, t_j(x_0, t_0)) = 0, \text{ i.e. } (x_0, t_j(x_0, t_0)) \in \mathcal{D};$$

therefore $t_j(x_0, t_0) = t_0$ if and only if $(x_0, t_0) \in \mathcal{D}$. It follows that a phase (x, t) is either regular or singular: $\mathcal{R} = \mathcal{D}^+$.

In the case of general q-processes, suppose that $\chi_0(X, t \mid x_0, t_0)$ is a function satisfying the I.M.P. conditions and such that

$$\varkappa_0(t \mid x_0, t_0) = -\int_{t_0}^{t} d\tau \int_{\mathfrak{X}} q(x, \tau)\, \chi_0(dx, \tau \mid x_0, t_0), \tag{4.29}$$

where $q(x, t)$ is a measurable non-negative function on $\mathcal{S}$: this is true for example if for every fixed t, $[1 - \varkappa_0(t + \delta t \mid x, t)]/\delta t$ converges as $\delta t \to 0$ to $q(x, t)$ for all $x \in \mathfrak{X}$ except possibly a set X_0 such that $\chi_0(X_0, t \mid x_0, t_0) = 0$, and is dominated in some neighbourhood of t independent of x by a function which is integrable with respect to $\chi_0(X, t \mid x_0, t_0)$. Let $\phi(X \mid x, t)$ satisfy the ϕ-conditions. Then it can be shown that χ_0 and the function

$$\psi(S \mid x_0, t_0) = \int_{t_0}^{\infty} dt \int_{\mathfrak{X}} \phi(S(t) \mid x, t)\, q(x, t)\, \chi_0(dx, t \mid x_0, t_0) \tag{4.30}$$

satisfy jointly the $\chi_0 \psi$-conditions.

The special types of discontinuous Markoff processes considered in the present section form in fact a very wide class. For example the class of step processes includes most Markoff processes with a countable state space. The theory of general q-step processes was given by Feller in two basic papers [4, 5]; cf also Pospišil [14]; Feller's starting point in his second paper is the "backward equation" (4.26), which as shown in Lemma 4.4 is equivalent to I.E for this particular class of processes. The point of view adopted in the present paper is similar to that of Feller's, and the results of §§ 5 and 6, as well as Lemma 8.3 are generalizations of Feller's [5] results. Step processes were also considered by Doeblin [2] from the "probabilistic" point of view (cf. the remarks at the end of § 1). There is of course a wide literature on special processes coming within the purview of the present general theory, in particular processes with a countable state space (see Bartlett [1], Feller [6] and Doob [3] for further references).

5. The distributions of states, jump-times and jump-numbers

In the present section, we introduce the *Markoff chain* $\{\psi_n(S \mid x, t)\}$ *of the n-th jump time and consequent state distributions*. From this chain we construct the transition distribution $\chi_n = \chi_0 \ast \psi_n$, where $\chi_n(X, t \mid x_0, t_0)$ is interpreted as *the joint distribution of the number of jumps n and the transition* $x_0 \to X$ *in* $[t_0, t)$, which is shown to satisfy the I.M.P. conditions. The reason for introducing these concepts is that, as is shown in § 6, $\chi_R = \sum_0^\infty \chi_n$ is *a solution of the integral equation* (2.5) *which satisfies the I.M.P. conditions*. A necessary and sufficient condition for χ_R to satisfy the full C.M.P. conditions is given in § 6, and it turns out in § 8 that this is also the necessary and sufficient condition for χ_R to be the *unique* solution of I.E.

LEMMA 1. *Let*

$$\psi_n(S \mid x_0, t_0) = \int_S \psi_{n-1}(S \mid \xi, \tau)\, \psi(d\xi, d\tau \mid x_0, t_0), \quad (n = 1, 2, \ldots), \tag{5.1}$$

where $\psi_0 = I$ *(cf. § 3) and hence* $\psi_1 = \psi$. *Each member of the sequence of functions defined by this iteration relation is a conditional distribution on* $\mathcal{B}_s \times S$. *Let* $\psi_n(X, t \mid x_0, t_0) = \psi_n(X \times [t_0, t) \mid x_0, t_0)$; $\psi_n(X, t \mid x_0, t_0) = 0$ *if* $t \leqslant t_0$.

COROLLARY.

$$\psi_n(X, t \mid x_0, t_0) = \int_{t_0}^{t} \int_X \psi_{n-1}(X, t \mid \xi, \tau)\, \psi(d\xi, d\tau \mid x_0, t_0) \quad (n = 1, 2, \ldots). \tag{5.2}$$

This is true for $n=1$ by the definition of ψ. Suppose it is true for $n-1$; then by Lemma 3.3 $\psi_n(S\,|\,x,t)$ is a conditional distribution on $\mathcal{B}_s \times \mathcal{S}$, and by Lemma 3.5 $\psi_n(X,t\,|\,x_0,t_0)=0$ for $t \leqslant t_0$. Hence by induction the lemma is true. The corollary follows by Lemma 3.5. $\psi_n(S\,|\,x_0,t_0)$ will be called the *n-th jump phase space distribution*; clearly it must be interpreted as the distribution of the nth jump time and consequent state conditional on (x_0, t_0), and

$$\sigma_n(t\,|\,x_0,t_0)=\psi_n(\mathcal{X},t\,|\,x_0,t_0), \quad (n=0,1,\ldots) \tag{5.3}$$

as the cumulative distribution of the nth jump time. It follows from Lemma 5.1 that $\psi_n(X,t\,|\,x_0,t_0)$ has the same properties as $\psi(X,t\,|\,x_0,t_0)$, as stated in § 2 after the $\chi_0\psi$-conditions.

LEMMA 5.2

$$\psi_n=\psi_k * \psi_{n-k} \quad \text{and} \quad \sigma_n=\sigma_k * \psi_{n-k}, \quad (k=0,1,\ldots,n;\ n=1,2,\ldots). \tag{5.4}$$

This follows immediately from (5.2) using Lemma 3.5.

Let now

$$\chi_n=\chi_0 * \psi_n, \quad (n=0,1,2,\ldots). \tag{5.5}$$

$\chi_n(X,t\,|\,x_0,t_0)$ must clearly be interpreted as the joint probability of exactly n jumps and a transition $x_0 \to X$ in $[t_0,t)$, and

$$\varkappa_n(t\,|\,x_0,t_0)=\chi_n(\mathcal{X},t\,|\,x_0,t_0), \quad (n=0,1,2,\ldots) \tag{5.6}$$

as the probability distribution of the number of jumps in $[t_0,t)$ given x_0, at t_0. It follows from Lemma 3.3 that for n, X, t fixed $\chi_n(X,t\,|\,x_0,t_0)$ is a bounded measurable function on $\mathcal{S}$; hence by Lemma 5.2, and using Lemma 3.5:

LEMMA 5.3

$$\chi_n=\chi_j * \psi_{n-j} \quad \text{and} \quad \varkappa_n=\varkappa_j * \psi_{n-j}, \quad (j=0,1,\ldots,n;\ n=1,2,\ldots). \tag{5.7}$$

Since $\varkappa_0=1-\sigma$, it follows using (5.4) that

$$\varkappa_n=(1-\sigma) * \psi_n=\sigma_n-\sigma_{n+1}, \quad (n=0,1,\ldots), \tag{5.8}$$

which has an obvious probability interpretation when we remark that $\sigma_n(t\,|\,x_0,t_0)$ is the probability of n or more jumps in $[t_0,t)$ given x_0 at t_0. It follows by iterating (5.8) that

$$\sigma_{n+1}=1-\sum_{j=0}^{n}\varkappa_j \leqslant 1, \quad (n=0,1,\ldots),$$

and hence:

LEMMA 5.4. *The sequence $\{\sigma_n\}$ is non-decreasing and converges pointwise to*

$$\sigma_\infty(t \mid x_0, t_0) = \lim_{n\to\infty} \sigma_n(t \mid x_0, t_0) = 1 - \sum_0^\infty \varkappa_j(t \mid x_0, t_0) \leqslant 1, \tag{5.9}$$

where $\sigma_\infty(t \mid x_0, t_0)$ is non-decreasing function on $\mathcal{T}$ continuous to the left for x_0, t_0 fixed, a measurable function on S for t fixed, and vanishes for $t \leqslant t_0$.

The last part of the lemma follows by Lemma 3.6. Thus $\sigma_\infty(t \mid x_0, t_0)$ has the character of a cumulative distribution for t conditional on (x_0, t_0), though it need not be normalized to 1 even if all the σ_n are; it may be interpreted, since the σ_n are the probabilities of n or more jumps in $[t_0, t)$, as the cumulative distribution of the time of occurrence of a "singular" jump involving an infinite number of "ordinary" jumps, conditional on the initial phase (x_0, t_0). A phase (x_0, t_0) will be called *stable* if it cannot be followed by such a singular jump; i.e. if $\sigma_\infty(t \mid x_0, t_0) = 0$ for all t, or equivalently, if $\sigma_\infty(\infty \mid x_0, t_0) = 0$; conversely, if $\sigma_\infty(\infty \mid x_0, t_0) > 0$, then (x_0, t_0) is an *unstable* phase. The whole process will be termed *stable* if $\sigma_\infty \equiv 0$, *unstable* otherwise.

LEMMA 5.5.

$$\psi_n(X, t \mid x_0, t_0) = \psi_n(X, \tau \mid x_0, t_0) + \\ + \sum_{j=1}^{n} \int_X \psi_j(X, t \mid \xi, \tau)\, \chi_{n-j}(d\xi, \tau \mid x_0, t_0), \quad (t \geqslant \tau \geqslant t_0;\ n = 1, 2, \ldots). \tag{5.10}$$

This is true for $n = 1$ by the $\chi_0\psi$-condition 4. Suppose that it is true for n. One finds then on substituting for both ψ_n and ψ in $\psi_{n+1} = \psi_n \ast \psi$ that

$$\begin{aligned}
&\psi_{n+1}(X, t \mid x_0, t_0) \\
&= \int_{t_0}^{\tau}\int_X \psi_n(X, t \mid \zeta, \theta)\, \psi(d\zeta, d\theta \mid x_0, t_0) + \int_{\tau}^{t}\int_X \psi_n(X, t \mid \zeta, \theta)\, \psi(d\zeta, d\theta \mid x_0, t_0) \\
&= \int_{t_0}^{\tau}\int_X \psi_n(X, \tau \mid \zeta, \theta)\, \psi(d\zeta, d\theta \mid x_0, t_0) + \\
&\qquad + \sum_{j=1}^{n} \int_{t_0}^{\tau}\int_X \left\{\int_X \psi_j(X, t \mid \xi, \tau)\, \chi_{n-j}(d\xi, \tau \mid \zeta, \theta)\right\} \psi(d\zeta, d\theta \mid x_0, t_0) + \\
&\qquad + \int_{\tau}^{t}\int_X \psi_n(X, t \mid \zeta, \theta) \int_X \psi(d\zeta, d\theta \mid \xi, \tau)\, \chi_0(d\xi, \tau \mid x_0, t_0) \\
&= \psi_{n+1}(X, \tau \mid x_0, t_0) + \sum_{j=1}^{n+1} \int_X \psi_j(X, t \mid \xi, \tau)\, \chi_{n-j+1}(d\xi, \tau \mid x_0, t_0).
\end{aligned}$$

In the passage from the 2nd to the 3rd line, the changes in the order of integration are justified by lemma (3.4); the final result is obtained by using the relations $\chi_{n-j} * \psi = \chi_{n-j+1}$, $\psi_n * \psi = \psi_{n+1}$. Lemma 5.5 follows by induction.

The function $\chi_n(X, t \mid x_0, t_0)$ may be considered as the transition distribution of a Markoff process with states (x, j) at t, where j is the *number of jumps* in $[0, t)$, the suffix n denoting the increase of j in $[t_0, t)$: i.e. $n = j - j_0 \geqslant 0$. The state space in this case is $\mathfrak{X} \times \mathfrak{N}$, where $\mathfrak{N}$ is the set of all non-negative integers.

THEOREM 5.6. *The function $\chi_n(X, t \mid x_0, t_0)$ is a transition distribution satisfying the I.M.P. conditions relative to the states (x, j).*

It follows from the definition (5.5) of χ_n by Lemma 3.3 that (a) $\chi_{j-j_0}(X, t \mid x_0, t_0)$ is for fixed j, X, t a measurable function on $S \times \mathfrak{N}$; (b) for fixed j_0, t, x_0, t_0 a distribution on $\mathcal{B}_x \times \mathcal{B}_n$ (i.e. the Borel field of subsets of $\mathfrak{X} \times \mathfrak{N}$), with a total variation which by Lemma 5.4 is

$$\sum_{n=0}^{\infty} \chi_n(\mathfrak{X}, t \mid x_0, t_0) = \sum_{n=0}^{\infty} \varkappa_n(t \mid x_0, t_0) = 1 - \sigma_\infty(t \mid x_0, t_0) \leqslant 1. \tag{5.11}$$

Hence χ_n satisfies I.M.P. condition (1). Write

$$\chi_n(X, t \mid x_0, t_0) = \int_{t_0}^{\tau} \int_{\mathfrak{X}} \chi_0(X, t \mid \zeta, \theta)\, \psi_n(d\zeta, d\theta \mid x_0, t_0) + \int_{\tau}^{t} \int_{\mathfrak{X}} \chi_0(X, t \mid \zeta, \theta)\, \psi_n(d\zeta, d\theta \mid x_0, t_0). \tag{5.12}$$

Substituting for χ_0 in the first term in the right-hand side of (5.12) from the Chapman-Kolmogoroff equation (1.1)

$$\chi_0(X, t \mid \zeta, \theta) = \int_{\mathfrak{X}} \chi_0(X, t \mid \xi, \tau)\, \chi_0(d\xi, \tau \mid \zeta, \theta) \tag{4.13}$$

(which χ_0 must satisfy since it satisfies the I.M.P. conditions), and from equation (5.10) for ψ_n in the second term, one finds after inverting the order of integration in both terms, which is again justified by Lemma 3.4, that

$$\chi_n(X, t \mid x_0, t_0) = \sum_{k=0}^{n} \int_{\mathfrak{X}} \chi_k(X, t \mid \xi, \tau)\, \chi_{n-k}(d\xi, \tau \mid x_0, t_0), \quad (t \geqslant \tau \geqslant t_0;\ n = 0, 1, \ldots). \tag{5.14}$$

This is clearly the form that the I.M.P. condition (2) takes for χ_n. For $t \leqslant t_0$, $\chi_0(X, t \mid x_0, t_0) = \delta(X \mid x_0)$ by I.M.P. condition (3), $\psi_n(X, t \mid x_0, t_0) = 0$ for all n, and hence, since by definition $\chi_n = \chi_0 * \psi_n$,

$$\chi_n(X, t \mid x_0, t_0) = \delta(X \mid x_0)\, \delta_n, \quad (t \leqslant t_0), \tag{5.15}$$

where δ_n is 1 or 0 according as $n=0$ or $n>0$, which is the form that the I.M.P. condition (3) takes for χ_n. This completes the proof of Theorem 5.6.

COROLLARY. $\sum_{n=0}^{\infty} \chi_n(\mathfrak{X}, t \mid x_0, t_0)=1$ *if and only if* $\sigma_\infty(t \mid x_0, t_0)=0$; *hence* χ_n *is a transition probability satisfying the C.M.P. condition* (4) *if and only if the process is stable* (*i.e.* $\sigma_\infty \equiv 0$). This is an immediate consequence of (5.9).

6. The regular solution of I.E.

THEOREM 6.1. *The series* $\Sigma \chi_n$ *converges to a transition distribution*

$$\chi_R(X, t \mid x_0, t_0)=\sum_{n=0}^{\infty} \chi_n(X, t \mid x_0, t_0), \tag{6.1}$$

which is a solution of I.E. satisfying the I.M.P. conditions, with

$$\varkappa_R(t \mid x_0, t_0)=\chi_R(\mathfrak{X}, t \mid x_0, t_0)=1-\sigma_\infty(t \mid x_0, t_0) \leqslant 1. \tag{6.2}$$

Convergence of the series follows at once from the fact that it is majored by $\Sigma \varkappa_n$. It follows from Theorem 5.6 and Lemma 3.6 that χ_R satisfies I.M.P. condition (1). It satisfies I.M.P. condition (2) because

$$\begin{aligned}
&\int_{\mathfrak{X}} \chi_R(X, t \mid \xi, \tau)\, \chi_R(d\xi, \tau \mid x_0, t_0) \\
&\qquad = \sum_{i=0}^{\infty} \int_{\mathfrak{X}} \chi_i(X, t \mid \xi, \tau)\, \chi_R(d\xi, \tau \mid x_0, t_0) \\
&\qquad = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \int_{\mathfrak{X}} \chi_i(X, t \mid \xi, \tau)\, \chi_j(d\xi, \tau \mid x_0, t_0) \\
&\qquad = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \int_{\mathfrak{X}} \chi_k(X, t \mid \xi, \tau)\, \chi_{n-k}(d\xi, \tau \mid x_0, t_0) \\
&\qquad = \sum_{n=0}^{\infty} \chi_n(X, t \mid x_0, t_0) \\
&\qquad = \chi_R(X, t \mid x_0, t_0), \quad (t \geqslant \tau \geqslant t_0).
\end{aligned} \tag{6.3}$$

The 1st and 2nd lines in (6.3) are justified by Lemma 3.6, the rearrangement of terms in the 2nd line leading to the 3rd by the fact that this is a convergent double series of non-negative terms; the 4th line then follows by Theorem 5.6, equation (5.14). It follows immediately from (5.15) that χ_R satisfies I.M.P. condition (3). Finally, using Lemma 3.6 and (5.7)

$$\chi_R * \psi = \left(\sum_{n=0}^{\infty} \chi_n\right) * \psi = \sum_{n=0}^{\infty} (\chi_n * \psi) = \sum_{n=0}^{\infty} \chi_{n+1} = \chi_R - \chi_0; \tag{6.4}$$

this proves that χ_R satisfies I.E., and hence completes the proof of Theorem 6.1. An immediate concequence of (6.2) is:

COROLLARY. $\varkappa_R(t \mid x_0, t_0) = \chi_R(\mathfrak{X}, t \mid x_0, t_0) = 1$ *if and only if* $\sigma_\infty(t \mid x_0, t_0) = 0$; *hence* χ_R *is a transition probability satisfying C.M.P. condition* (4) *if and only if the process is stable* (*i.e.* $\sigma_\infty \equiv 0$).

The transition distribution χ_R will be called the *regular solution* of I.E. Note that this solution is meaningful only provided that χ_R does not reductive trivially to χ_0! It is readily seen that $\chi_R \equiv \chi_0$ if and only if $\chi_1 = \chi_0 * \psi \equiv 0$, and it is important therefore to know under what conditions this can happen. The answer is provided by the following lemma.

LEMMA 6.2. $\chi_1(X, t \mid x_0, t_0) = 0$ *for all* $X \in \mathcal{B}_x$ *and* $t \in \mathcal{J}$ *if and only if* $\psi(\mathcal{D}^+ \mid x_0, t_0) = 0$; *hence* $\chi_1 \equiv 0$ *and* $\chi_R \equiv \chi_0$ *if and only if* $\psi(\mathcal{D}^+ \mid x, t_0) \equiv 0$.

$\mathcal{D}$ here is the set of all singular phases defined in § 3, $\mathcal{D}^+ = \mathcal{S} - \mathcal{D}$. If $\psi(\mathcal{D}^+ \mid x_0, t_0) = 0$, then

$$\begin{aligned} \varkappa_1(t \mid x_0, t_0) &= \int_{t_0}^{t} \int_{\mathfrak{X}} \varkappa_0(t \mid \xi, \tau)\, \psi(d\xi, d\tau \mid x_0, t_0) \\ &= \int_{\mathcal{D}^+ \cap \mathfrak{X} \times [t_0, t)} \varkappa_0(t \mid \xi, \tau)\, \psi(d\xi, d\tau \mid x_0, t_0) = 0, \end{aligned} \tag{6.5}$$

because by definition $\varkappa_0(t \mid \xi, \tau) = 1 - \sigma(t \mid \xi, \tau) = 0$ if $(\xi, \tau) \in D$ and $t \geqslant \tau$; hence $\chi_1(X, t \mid x_0, t_0) = 0$ for all $X \in \mathcal{B}_x$, $t \in \mathcal{J}$. Let

$$S_{nk} = \left\{(x, t) \,\middle|\, \frac{n-1}{k} \leqslant t \leqslant \frac{n}{k} \text{ and } \varkappa_0\left(\frac{n}{k}\middle| x, t\right) \geqslant \frac{1}{k}\right\} \quad \text{and } S_0 = \bigcup_{n,k} S_{nk}; \tag{6.6}$$

if $(x, t) \in \mathcal{D}^+$, then $(x, t) \in S_{nk}$ for some n, k; hence $\mathcal{D}^+ \subset S_0$. If conversely $\chi_1(X, t \mid x_0, t_0) = 0$ for all $X \in \mathcal{B}_x$, $t \in \mathcal{J}$, then

$$\begin{aligned} \varkappa_1(t \mid x_0, t_0) &= \int_{t_0}^{t} \int_{\mathfrak{X}} \varkappa_0(t \mid \xi, \tau)\, \psi(d\xi, d\tau \mid x_0, t_0) \\ &= \int_{0}^{t} \int_{\mathfrak{X}} \varkappa_0(t \mid \xi, \tau)\, \psi(d\xi, d\tau \mid x_0, t_0) = 0. \end{aligned} \tag{6.7}$$

Since this is true for all $t \in \mathcal{J}$, and since $\varkappa_0 \geqslant 0$, it follows that for every S_{nk}

$$0 = \int_{S_{nk}} \varkappa_0 \left(\frac{n}{k} \middle| \xi, \tau \right) \psi (d\xi, d\tau \mid x_0, t_0) \geq \frac{1}{k} \psi (S_{nk} \mid x_0, t_0); \tag{6.8}$$

hence $\psi (S_{nk} \mid x_0, t_0) = 0$, and therefore

$$\psi (S_0 \mid x_0, t_0) \leq \sum_{n,k} \psi (S_{n,k} \mid x_0, t_0) = 0; \tag{6.9}$$

hence $\psi (\mathcal{D}^+ \mid x_0, t_0) = 0$. If $\chi_1 \equiv 0$, then $\varkappa_1 \equiv 0$; it follows by induction (since $\varkappa_{n+1} = \varkappa_n * \psi$) that $\varkappa_n \equiv 0$, $n = 1, 2, \ldots$; hence $\chi_R \equiv \chi_0$. This completes the proof of the lemma. Thus *the regular solution χ_R reduces trivially to χ_0 only in the degenerate case of a process where the variation of ψ is confined so the singular phases, and the probability of a finite number of jumps in any finite time interval is zero*; notice that for such a process it follows from the fact that $\varkappa_n \equiv 0$ for all $n \geq 1$ that

$$\sigma \equiv \sigma_n, \quad n = 1, 2, \ldots \quad \text{and hence } \sigma \equiv \sigma_\infty,$$

i.e. either the system makes no jumps at all, or it executes a singular transition.

The behaviour of the regular solution $\chi_R (X, t \mid x_0, t_0)$ when $t \downarrow t_0$ will now be considered. It is clear that we cannot expect the regular solution χ_R of every discontinuous Markoff process to satisfy the continuity condition (1.5), at any rate not for all initial phases (x_0, t_0). It will however be satisfied for *regular* initial phases (i.e such that $\sigma (t_0 + 0 \mid x_0, t_0) = 0$), provided that it is satisfied by χ_0 for such phases. For

$$\sum_1^\infty \chi_n \leq \sum_1^\infty \varkappa_n = \sigma - \sigma_\infty$$

and hence:

LEMMA 6.3. *If (x_0, t_0) is a regular phase and more generally if*

$$\sigma (t_0 + 0 \mid x_0, t_0) = \sigma_\infty (t_0 + 0 \mid x_0, t_0),$$

then

$$\chi_R (X, t_0 + 0 \mid x_0, t_0) = \chi_0 (X, t_0 + 0 \mid x_0, t_0).$$

COROLLARY. *If (x_0, t_0) is a regular phase, then*

$$\chi_R (X, t_0 + 0 \mid x_0, t_0) = \delta (X \mid x_0)$$

if and only if

$$\chi_0 (X, t_0 + 0 \mid x_0, t_0) = \delta (X \mid x_0).$$

7. Existence of an "inverse" to $(I - \psi)$

Let $S_t = S \cap \mathfrak{X} \times (0, t)$, where $t < \infty$; we call S_t a *t-bounded set*. The series $\sum \psi_n (S_t \mid x_0, t_0)$ is majored by $\sum \sigma_n (t \mid x_0, t_0)$, and hence converges if and only if the latter does.

LEMMA 7.1. *$\sum \sigma_n (t \mid x_0, t_0)$ converges if and only if $\sigma_\infty (t \mid x_0, t_0) = 0$ and $\sum_n \varkappa_n (t \mid x_0, t_0)$ converges; if this is so, then*

$$\sum_1^\infty \sigma_n (t \mid x_0, t_0) = \sum_0^\infty n \varkappa_n (t \mid x_0, t_0) = \bar{n} (t \mid x_0, t_0). \tag{7.1}$$

In words, for a stable initial phase $\sum_1^\infty \sigma_n$ is equal to the mean number of jumps $\bar{n}$. For

$$\sum_{k=1}^n \sigma_k = \sum_{k=1}^n \left[1 - \sum_{j=0}^{k-1} \varkappa_j\right] = n \left[1 - \sum_{j=0}^n \varkappa_j\right] + \sum_{j=0}^n j \varkappa_j = n \sigma_{n+1} + \sum_{j=0}^n j \varkappa_j. \tag{7.2}$$

If $\bar{n} < \infty$ and $\sigma_\infty = 0$, then

$$n \sigma_{n+1} = n \left[1 - \sum_0^n \varkappa_j\right] = n \sum_{n+1}^\infty \varkappa_j < \sum_{n+1}^\infty j \varkappa_j \to 0 \quad \text{as } n \to \infty; \tag{7.3}$$

hence $n \sigma_{n+1} \to 0$, and by (7.2) $\sum_1^\infty \sigma_n = \bar{n}$. Conversely, if $\sum_1^\infty \sigma_n < \infty$, then obviously $\sigma_\infty = 0$ and by (7.2) $\sum_0^\infty j \varkappa_j = \bar{n} < \infty$; hence $\sum_1^\infty \sigma_n = \bar{n}$ and $n \sigma_{n+1} \to 0$.

Suppose $\sigma_\infty (t \mid x_0, t_0) = 0$, $\bar{n} (t \mid x_0, t_0) < \infty$, and write

$$\Omega_t (S \mid x_0, t_0) = \sum_0^\infty \psi_n (S_t \mid x_0, t_0). \tag{7.4}$$

Clearly for fixed x_0, t_0 and $t < \infty$ Ω_t is a totally finite measure on $\mathcal{B}_s$ and is non-decreasing in t. Hence

$$\Omega (S \mid x_0, t_0) = \lim_{t \to \infty} \Omega_t (S \mid x_0, t_0) \tag{7.5}$$

is a measure on $\mathcal{B}_s$ finite for every t-bounded set, but totally finite if and only if $\lim_{t \to \infty} \bar{n} (t \mid x_0, t_0) < \infty$.

LEMMA 7.2. *If $\sigma_\infty (t \mid x_0, t_0) = 0$, $\bar{n} (t \mid x_0, t_0) < \infty$ for every $(x_0, t_0) \in S$ and every finite $t \in \mathcal{T}$, and if $\Omega (S \mid x_0, t_0)$ is defined by (7.4) and (7.5), then*

$$(I - \psi) * \Omega = I \quad \text{and} \quad \chi_R = \chi_0 * \Omega. \tag{7.6}$$

This is true because obviously for every finite t we can replace Ω in

$$\int_{t_0}^t \int_X \psi (X, t \mid \xi, \tau) \, \Omega (d\xi, d\tau \mid x_0, t_0) \quad \text{and} \quad \int_{t_0}^t \int_X \chi_0 (X, t \mid \xi, \tau) \, \Omega (d\xi, d\tau \mid x_0, t_0) \tag{7.7}$$

by Ω_t, which is a totally finite measure, and hence can apply Lemma 3.6 to justify the inversion of summation and integration in

$$\left.\begin{aligned} &(I-\psi)\ast\sum_0^\infty \psi_n=\sum_0^\infty (I-\psi)\ast\psi_n=I \\ \text{and}\qquad &\chi_0\ast\sum_0^\infty \psi_n=\sum_0^\infty \chi_0\ast\psi_n=\sum_0^\infty \chi_n=\chi_R. \end{aligned}\right\} \tag{7.8}$$

Thus Ω is a kind of "right-inverse" of $(I-\psi)$, if one considers the χ's and ψ's as elements of an algebra of operators with multiplication identified as composition; in this interpretation, ψ_n is the "n-th power" of ψ, and $\Omega=\Sigma\,\psi_n$ is analogous to the Neumann series solution of integral equation theory. If furthermore $\bar{n}(t\,|\,x_0, t_0)$ is bounded, say by N, then it is easily seen that $N^{-1}\,\Omega(S\,|\,x_0, t_0)$ is a distribution on $\mathcal{B}_s\times S$, and it also a "left-inverse" of $(I-\psi)$: i.e. $\Omega\ast(I-\psi)=I$.

8. General solutions of I.E. Transition distributions of unstable processes.

A function $\chi(X, t\,|\,x_0, t_0)$ will be termed a general solution of I.E. if for fixed X, t it is a bounded measurable function on S satisfying I.E. It is immediately obvious that:

LEMMA 8.1. *The class of all general solutions of I.E. is the class of all functions of the form*

$$\chi(X, t\,|\,x_0\, t_0)=\chi_R(X, t\,|\,x_0, t_0)+\alpha(t, x_0, t_0), \tag{8.1}$$

where χ_R is the regular solution of I.E. and α is any function on $\mathcal{J}\times S$ bounded and measurable on S for fixed t, which satisfies the homogeneous integral equation, (briefly H.I.E.)

$$\alpha\ast(I-\psi)=0. \tag{8.2}$$

THEOREM 8.2. *Every solution $\alpha(t, x_0, t_0)$ of H.I.E. vanishes for a given initial phase (x_0, t_0) and all $t\leqslant a$ if and only if $\sigma_\infty(a\,|\,x_0, t_0)=0$.*

COROLLARY. *The regular solution χ_R is the unique solution of I.E. if and only if the process is stable (i.e. $\sigma_\infty\equiv 0$).* The condition of the theorem is necessary because $\sigma_\infty(t\,|\,x_0, t_0)$ is a solution of H.I.E.: for let $n\to\infty$ in both sides of the relation (5.4) $\sigma_{n+1}=\sigma_n\ast\psi$; then by lemmas 5.4 and 3.6 $\sigma_\infty=\sigma_\infty\ast\psi$. Conversely, suppose $\sigma(a\,|\,x_0, t_0)=0$; let $\alpha(t, x_0, t_0)$ be any measurable solution of H.I.E. such that for fixed t

$$|\,\alpha(t\,|\,x_0, t_0)\leqslant N(t)<\infty;$$

by iteration

$$\alpha=\alpha\ast\psi=\alpha\ast\psi_n=\lim_{n\to\infty}\alpha\ast\psi_n. \tag{8.3}$$

But for $t \leqslant a$

$$\left| \int_{t_0}^{t} \int_{\mathcal{X}} \alpha(t \mid \xi, \tau)\, \psi_n(d\xi, d\tau \mid x_0, t_0) \right| \leqslant N(t)\, \sigma_n(t \mid x_0, t_0) \leqslant N(t)\, \sigma_n(a \mid x_0, t_0), \tag{8.4}$$

hence $|\alpha(t, x_0, t_0)| \leqslant N(t)\, \sigma_n(a \mid x_0, t_0)$ for all n, and therefore

$$|\alpha(t \mid x_0, t_0)| \leqslant N(t)\, \sigma_\infty(a \mid x_0, t_0) = 0;$$

hence $\alpha(t, x_0, t_0) =$ for all $t \leqslant a$. This completes the proof of Theorem 8.2.

THEOREM 8.3. *χ_R is the minimal non-negative solution of I.E.* By this is meant that if $\chi(X, t \mid x_0, t_0)$ is a non-negative solution of I.E., then $\chi \geqslant \chi_R$. Let $\Xi_n = \sum_0^n \chi_j$; by hypothesis $\chi = \chi_0 + \chi * \psi$ and $\chi * \psi \geqslant 0$, hence $\chi \geqslant \chi_0 = \Xi_0$. Suppose $\chi \geqslant \Xi_n$; then $\chi * \psi \geqslant \Xi_n * \psi = \Xi_{n+1} - \chi_0$; hence $\chi \geqslant \Xi_{n+1}$. Thus by induction $\chi \geqslant \Xi_n$ for all n, and therefore $\chi \geqslant \lim \Xi_n = \chi_R$.

Let us now reconsider the problem formulated at the end of § 2 in the light of the results obtained so far. The regular solution χ_R is the unique solution of I.E. and satisfies the C.M.P. conditions if and only if $\sigma_\infty \equiv 0$; hence for a stable process χ_R provides a complete answer to the problem. If $\sigma_\infty \not\equiv 0$. the solution of I.E. is not unique; every solution satisfying the I.M.P. conditions must be of the form

$$\chi(X, t \mid x_0, t_0) = \chi_R(X, t \mid x_0, t_0) + \chi_S(X, t \mid x_0, t_0), \tag{8.5}$$

where χ_S satisfies H.I.E. for every fixed X (by Lemma 8.1) and is non-negative (since by Lemma 8.3 χ_R is the minimal non-negative solution); the imposition of the I.M.P. conditions 1–3 on x then implies that

(1) χ_S itself satisfies I.M.P. condition 1, with $\chi_S(\mathfrak{X}, t \mid x_0, t_0) \leqslant \sigma_\infty(t \mid x_0, t_0)$;

(2) χ_S satisfies the functional relation

$$\begin{aligned}\chi_S(X, t \mid x_0, t_0) &= \int_{\mathcal{X}} \chi_R(X, t \mid \xi, \tau)\, \chi_S(d\xi, \tau \mid x_0, t_0) + \int_{\mathcal{X}} \chi_S(X, t \mid \xi, \tau)\, \chi_R(d\xi, \tau \mid x_0, t_0) + \\ &\quad + \int_{\mathcal{X}} \chi_S(X, t \mid \xi, \tau)\, \chi_S(d\xi, \tau \mid x_0\, t_0), \quad (t \geqslant \tau \geqslant t_0);\end{aligned}$$

(3) $\chi_S(X, t \mid x_0, t_0) = 0$ for $t \leqslant t_0$.

Finally, χ satisfies the C.M.P. conditions if and only if $\chi_S(\mathfrak{X}, t \mid x_0, t_0) \equiv \sigma_\infty(t \mid x_0, t_0)$. The one question still open is therefore that of the existence and properties, in the case of an unstable process, of solutions of type (8.5) satisfying the C.M.P. condition.

Suppose that such a solution exists. Let us call (somewhat loosely for the moment) *regular* a transition involving a finite number of jumps, *singular* one that involves an infinity of jumps. Then χ_R and χ_S in (8.5) must be interpreted as the probabilities respectively of a regular and a singular transition $x_0 \to X$ in $[t_0, t)$; hence we call χ_S the *singular component* of χ. The non-uniqueness of χ arises from the fact that the basic functions χ_0, ψ cannot determine the evolution of the process in the event of singular transitions; some additional hypothesis is clearly required for this purpose. We shall now develop the theory of a general class of solutions of I.E. consequent upon what is perhaps the most natural form for such an additional hypothesis (1). Let us call the process defined by χ_0, ψ and with the regular solution χ_R, the zero-order or basic process, and its jumps the zero-order jumps. The gist of this new hypothesis is to take the singular "jumps" of the zero-order process to be the jumps of a new 1st-order process, the 1st-order jumps. The probability of a transition $x_0 \to X$ in $[t_0, t)$ *without* 1st-order jumps is by definitions $\chi_R(X, t \mid x_0, t_0)$. Let us write $\chi_R = \chi_0^{(1)}$, $\varkappa_R = \varkappa_0^{(1)}$. The additional postulate required in order to determine the 1st order process is the assumption of a 1st-order jump time and consequent state distributions $\psi^{(1)}(S \mid x_0, t_0)$ satisfying jointly with $\chi_0^{(1)}$ the $\chi_0 \psi$-conditions of § 2; this implies in particular that $\psi^{(1)}(\mathfrak{X}, t \mid x_0, t_0) = \sigma^{(1)}(t \mid x_0, t_0) = 1 - \varkappa_0^{(1)}(t \mid x_0, t_0) = \sigma_\infty(t \mid x_0, t_0)$. It is now legitimate to apply the theory developed in §§ 5 and 6 to this 1st-order process: that is, we form the sequence $\{\psi_n^{(1)}\}$ of phase-space distributions defined inductively by the relation $\psi_n^{(1)} = \psi_{n-1}^{(1)} \ast \psi^{(1)}$, the transition distributions $\chi_n^{(1)} = \chi_0^{(1)} \ast \psi^{(1)}$, and finally the regular solution $\chi_R^{(1)} = \sum_0^\infty \chi_n^{(1)}$ of the I.E. $\chi \ast (I - \psi^{(1)}) = \chi_0^{(1)}$. Since $\chi_0^{(1)} = \chi_R$, we can write $\chi_R^{(1)} = \chi_R + \chi_S^{(1)}$, where $\chi_S^{(1)} = \sum_1^\infty \chi_n^{(1)}$; hence $\chi_R^{(1)}$ will satisfy the I.E. $\chi \ast (I - \psi) = \chi_0$ provided that $\chi_S^{(1)}$ satisfies the corresponding H.I.E.; i.e. that $\chi_S^{(1)} \ast (I - \psi) = 0$. It will be shown below that this is the case if $\psi^{(1)}$ itself satisfies H.I.E. Furthermore $\chi_R^{(1)}$ satisfies the C.M.P. conditions if and only if $\sigma_\infty^{(1)} = \lim_{n \to \infty} \sigma_n^{(1)} \equiv 0$. If this is not the case, the whole procedure may be repeated: i.e. we can assume a 2nd order process defined by the functions $\psi_0^{(2)} = \chi_R^{(1)}$ and $\psi^{(2)}$, with regular solution $\chi_R^{(2)}$ and so on.

The first step in developing the theory of this class of solutions is to show that there are plenty of functions $\psi^{(1)}$ satisfying the required conditions. This is accomplished in the following lemma:

LEMMA 8.4. *If $\sigma_\infty \not\equiv 0$, there exists an infinite number of distinct solutions of H.I.E. which are conditional distributions on $\mathfrak{B}_s \times \mathfrak{S}$; let $\alpha(S \mid x_0, t_0)$ be such a solution:*

(1) A special case of this class of solutions has been given by DOOB (1945).

then $\alpha(\mathfrak{X}, t \mid x_0, t_0) \leqslant \sigma_\infty(t \mid x_0, t_0)$. *If* $\alpha(\mathfrak{X}, t \mid x_0, t_0) = \sigma_\infty(t \mid x_0, t_0)$, *it will be called maximal; there exists an infinity of such maximal solutions.*

Let $\phi(X \mid t)$ be any function on $\mathcal{B}_x \times \mathcal{T}$ which is a distribution on $\mathcal{B}_x$ for fixed t and a measurable function on τ for fixed X. Then

$$\alpha(S \mid x_0, t_0) = \int_0^\infty \phi(S(t) \mid t)\ \sigma_\infty(dt \mid x_0, t_0) \tag{8.6}$$

(a) is by Lemma 3.7 a conditional distribution on $\mathcal{B}_s \times S$;

(b) satisfies H.I.E., because σ_∞ does so, for using lemma 3.4

$$\begin{aligned}
&\int_{t_0}^{t} \int_{X} \alpha(X, t \mid \xi, \tau)\, \psi(d\xi, d\tau \mid x_0, t_0) \\
&\qquad = \int_{t_0}^{t} \int_{X} \Big\{\int_{\tau_0}^{t} \phi(X \mid \theta)\, \sigma_\infty(d\theta \mid \xi, \tau)\Big\}\, \psi(d\xi, d\tau \mid x_0, t_0) \\
&\qquad = \int_{t_0}^{t} \phi(X \mid \theta)\, \Big\{\int_{S} \sigma_\infty(d\theta \mid \xi, \tau)\, \psi(d\xi, d\tau \mid x_0, t_0)\Big\} \\
&\qquad = \int_{t_0}^{t} \phi(X \mid \theta)\, \sigma_\infty(d\theta \mid x_0, t_0) \\
&\qquad = \alpha(X, t \mid x_0, t_0).
\end{aligned}$$

Note that the converse is not true: there are solutions of H.I.E. which are distributions on $\mathcal{B}_s \times S$ and are not of the form (8.6). If α is a solution of H.I.E. and at the same time a distribution on $\mathcal{B}_s \times S$, then $\alpha(\mathfrak{X}, t \mid x_0, t_0) \leqslant 1$; it follows as in the proof of theorem 8.2 that $\alpha(\mathfrak{X}, t, x_0, t_0) \leqslant \sigma_\infty(t \mid x_0, t_0)$. That there exist an infinite of distinct solutions where the equality holds is seen by choosing ϕ in (8.6) to be a probability distribution on $\mathcal{B}_x$ (i.e. $\phi(\mathfrak{X}, t) \equiv 1$). This completes the proof of Lemma 8.4.

Let now $\psi^{(1)}(S \mid x_0, t_0)$ be such a maximal solution of H.I.E., and write $\sigma_\infty = \sigma^{(1)}$, $\chi_R = \chi_0^{(1)}$, $\varkappa_R = \varkappa_0^{(1)}$.

THEOREM 8.5. *If* $\psi^{(1)}$ *is a maximal solution of H.I.E., then* $\chi_0^{(1)}$ *and* $\psi^{(1)}$ *satisfy jointly the* $\chi_0\psi$*-conditions.*

For we know that $\chi_0^{(1)}$ satisfies the I.M.P. conditions, that $\sigma^{(1)} = 1 - \varkappa^{(1)}$, and that for fixed x_0, t_0, $\sigma^{(1)}(t \mid x_0, t_0)$ is continuous to the left in t and vanishes for $t \leqslant t_0$; by hypothesis $\psi^{(1)}$ is a conditional distribution on $\mathcal{B}_s \times S$, and $\psi^{(1)}(\mathfrak{X}, t \mid x_0, t_0) = \sigma^{(1)}(t \mid x_0, t_0)$. Hence there remains only to prove that $\chi_0^{(1)}$ and $\psi^{(1)}$ satisfy the $\chi_0\psi$-condition (4) (equation (2.1)). Since $\psi^{(1)}$ satisfies H.I.E., it follows by iteration, as in (8.3), that

$$\psi^{(1)} = \psi^{(1)} * \psi_n = \lim_{n\to\infty} \psi^{(1)} * \psi_n;$$

hence, substituting from (5.10) and using Lemma 3.3

$$\begin{aligned}
&\psi^{(1)}(X, t \mid x_0, t_0) \\
&= \int_{t_0}^{\tau} \int_{\mathfrak{X}} \psi^{(1)}(X, t \mid \zeta, \theta)\, \psi_n(d\zeta, d\theta \mid x_0, t_0) + \\
&\quad + \sum_{j=0}^{n} \int_{\mathfrak{X}} \left\{ \int_{\tau}^{t} \psi^{(1)}(X, t \mid \zeta, \theta)\, \psi_{n-j}(d\zeta, d\theta \mid \xi, \tau) \right\} \chi_j(d\xi, \tau \mid x_0, t_0) \\
&= \int_{t_0}^{\tau} \int_{\mathfrak{X}} \psi^{(1)}(X, t \mid \zeta, \theta)\, \psi_n(d\zeta, d\theta \mid x_0\, t_0) + \sum_{j=0}^{n} \int_{\mathfrak{X}} \psi^{(1)}(X, t \mid \xi, \tau) \times \\
&\quad \times \chi_j(d\xi, \tau \mid x_0, t_0), \quad (t \geqslant \tau \geqslant t_0;\ n = 1, 2, \ldots).
\end{aligned} \tag{8.7}$$

Let $n \to \infty$ in the right-hand-side of (8.7); then

$$\begin{aligned}
\psi^{(1)}(X, t \mid x_0, t_0) = \lim_{n\to\infty} \int_{t_0}^{\tau} \int_{\mathfrak{X}} \psi^{(1)}(X, t \mid \zeta, \theta)\, \psi_n(d\zeta, d\theta \mid x_0, t_0) + \\
+ \int_{\mathfrak{X}} \psi^{(1)}(X, t \mid \xi, \tau)\, \chi_0^{(1)}(d\xi, \tau \mid x_0, t_0), \quad (t \geqslant \tau \geqslant t_0);
\end{aligned} \tag{8.8}$$

hence

$$\sigma^{(1)}(t \mid x_0, t_0) = \lim_{n\to\infty} \int_{t_0}^{\tau} \int_{\mathfrak{X}} \sigma^{(1)}(t \mid \zeta, \theta)\, \psi_n(d\zeta, d\theta \mid x_0, t_0) + \int_{\mathfrak{X}} \sigma^{(1)}(t \mid \xi, \tau)\, \chi_0^{(1)}(d\xi, \tau \mid x_0, t_0). \tag{8.9}$$

On the other hand, it follows immediately from the fact that $\sigma^{(1)} = 1 - \varkappa_0^{(1)}$ and that $\chi_0^{(1)}$ satisfies the C.K. equation that

$$\sigma^{(1)}(t \mid x_0, t_0) = \sigma^{(1)}(\tau \mid x_0, t_0) + \int_{\mathfrak{X}} \sigma^{(1)}(t \mid \xi, \tau)\, \chi_0^{(1)}(d\xi, \tau \mid x_0, t_0); \tag{8.10}$$

hence, subtracting (8.10) from (8.9) and using furthermore the fact that $\sigma^{(1)} = \lim_{n\to\infty} \sigma^{(1)} * \psi_n$,

$$\lim_{n\to\infty} \int_{t_0}^{\tau} \int_{\mathfrak{X}} \{\sigma^{(1)}(t \mid \zeta, \theta) - \sigma^{(1)}(\tau \mid \zeta, \theta)\}\, \psi_n(d\zeta, d\theta \mid x_0, t_0) = 0.$$

But $0 \leqslant \psi^{(1)}(X, t \mid x_0, t_0) - \psi^{(1)}(X, \tau \mid x_0, t_0) \leqslant \sigma^{(1)}(t \mid x_0, t_0) - \sigma^{(1)}(\tau \mid x_0, t_0)$ for every $X \in \mathfrak{B}_x$ and $t \geqslant \tau \geqslant t_0$; hence

$$\begin{aligned}
\lim_{n\to\infty} \int_{t_0}^{\tau} \int_{\mathfrak{X}} \{\psi^{(1)}(X, t \mid \zeta, \theta) - \psi^{(1)}(X, \tau \mid \zeta, \theta)\}\, \psi_n(d\zeta, d\theta \mid x_0, t_0) \\
= \lim_{n\to\infty} \int_{t_0}^{\tau} \int_{\mathfrak{X}} \psi^{(1)}(X, t \mid \zeta, \theta)\, \psi_n(d\zeta, d\theta \mid x_0, t_0) - \psi^{(1)}(X, \tau \mid x_0, t_0) = 0.
\end{aligned}$$

Hence finally, substituting in (8.8)

$$\psi^{(1)}(X, t \mid x_0, t_0) = \psi^{(1)}(X, \tau \mid x_0, t_0) + \int_{\mathfrak{X}} \psi^{(1)}(X, t \mid \xi, \tau)\, \chi_0^{(1)}(d\xi, \tau \mid x_0, t_0), \quad (t \geqslant \tau \geqslant t_0). \tag{8.11}$$

This completes the proof of Theorem 8.5.

Thus $\chi_0^{(1)}$ and $\psi^{(1)}$ define jointly the 1st order process whose "jumps" are the singular transitions of the zero order process defined by χ_0 and ψ. Form then the functions $\psi_n^{(1)}$, $\sigma_n^{(1)}$, $\chi_n^{(1)}$, $\varkappa_n^{(1)}$ analogous respectively to ψ_n, σ_n, χ_n and $\varkappa_n$. By Lemma 5.4, $\sigma_\infty^{(1)} = \lim_{n\to\infty} \sigma_n^{(1)}$. By Theorem 6.1 $\chi_R^{(1)} = \sum_0^\infty \chi_n^{(1)}$ is the "regular solution" of the integral equation

$$\chi * (I - \psi^{(1)}) = \chi_0^{(1)}, \tag{8.12}$$

with $\varkappa_R^{(1)}(t \mid x_0, t_0) = \chi_R^{(1)}(\mathfrak{X}, t \mid x_0, t_0) = 1 - \sigma_\infty^{(1)}(t \mid x_0, t_0)$. It follows from $\psi^{(1)} = \psi^{(1)} * \psi$, using lemma 3.5, that

$$\psi_n^{(1)} = \psi_{n-1}^{(1)} * \psi^{(1)} = \psi_{n-1}^{(1)} * (\psi^{(1)} * \psi) = \psi_n^{(1)} * \psi,$$

and hence $$\chi_n^{(1)} = \chi_0^{(1)} * (\psi_n^{(1)} * \psi) = \chi_n^{(1)} * \psi \quad \text{for all } n \geqslant 1.$$

Hence $\chi_S^{(1)} = \chi_S^{(1)} * \psi$, where

$$\chi_S^{(1)} = \sum_1^\infty \chi_n^{(1)} = \chi_R^{(1)} - \chi_0^{(1)} = \chi_R^{(1)} - \chi_R,$$

and therefore $\chi_R^{(1)}$ is a solution of type (8.4).

THEOREM 8.6. $\chi_R^{(1)} = \sum_0^\infty \chi_n^{(1)}$ *is a solution of I.E. satisfying the I.M.P. conditions, with* $\varkappa_R^{(1)} = 1 - \sigma_\infty^{(1)}$; $\chi_R^{(1)}$ *satisfies the C.M.P. conditions if and only if* $\sigma_\infty^{(1)} \equiv 0$.

The class of solutions $\{\chi_R^{(1)}\}$ is meaningful only provided that its members do not all reduce trivially to χ_R, i.e. that not all $\chi_R^{(1)} \equiv \chi_R$. In accordance with the terminology introduced in § 3, the singular phases of the 1st order process are those phases (x, t) for which $\sigma^{(1)}(t+0 \mid x, t) = 1$; for reasons that will appear later they will be called the *trapping phases*; according to Lemma 3.1., the set K of all trapping phases is measurable. It is also obvious that $K \subset \mathfrak{D}$ and that $\chi_R(X, t \mid x_0, t_0) = 0$ for all $X \in \mathfrak{B}_x$, $t \geqslant t_0$, if $(x_0, t_0) \in K$. According to Lemma 6.2, for any given maximal solution $\psi^{(1)}$ of H.I.E., the corresponding regular solution $\chi_R^{(1)} \equiv \chi_R$ if and only if $\psi^{(1)}(K^+ \mid x_0, t_0) \equiv 0$. The question is therefore, under what conditions will this be true of the whole class $\{\psi^{(1)}\}$ of maximal solutions of H.I.E.? Call t_b a *barrier* if every phase (x, t_b) is singular; let

$$B = \{t \mid K(t) = \mathfrak{X}\} = \{t \mid \sigma_\infty(t+0 \mid x, t) = 1 \text{ for all } x \in \mathfrak{X}\} \tag{8.13}$$

be *the set of all barrier times*; let $\sigma_\infty(T \mid x_0, t_0)$ be the distribution induced on $\mathfrak{B}_t$ by $\sigma_\infty(t \mid x_0, t_0)$. Suppose that $\sigma_\infty(B^+ \mid x, t) \equiv 0$; then $\sigma_\infty(t+0 \mid x, t) = 0$ for all $x \in \mathfrak{X}$ whenever $t \in B^+$; hence $K = \mathfrak{X} \times B$, and consequently

$$\psi^{(1)}(K^+ \mid x_0, t_0) = \psi^{(1)}(\mathfrak{X} \times B^+ \mid x_0, t_0) = \sigma_\infty(B^+ \mid x_0, t_0) = 0.$$

Conversely, suppose that every maximal $\psi^{(1)}(K^+ \mid x_0, t_0) \equiv 0$; this must then be true in particular of all maximal solutions of H.I.E. of the form (8.6). Choose $\phi^{(1)}$ such that $\phi^{(1)}(K^+(t) \mid t) = 1$ whenever $t \in B^+$; then

$$0 = \psi^{(1)}(K^+ \mid x_0, t_0) = \int_0^\infty \phi^{(1)}(K^+(t) \mid t)\, \sigma_\infty(d\tau \mid x_0, t_0) \geqslant \sigma_\infty(B^+ \mid x_0, t_0);$$

hence $\sigma_\infty(B^+ \mid x_0, t_0) \equiv 0$. Thus we have proved:

LEMMA 8.7. *$\psi^{(1)}(K^+ \mid x_0, t_0) \equiv 0$ and hence $\chi_R^{(1)} \equiv \chi_R$ for the whole class of solutions $\{\psi^{(1)}\}$, $\{\chi_R^{(1)}\}$ if and only if $\sigma_\infty(B^+ \mid x_0, t_0) \equiv 0$.*

In other words, the class $\{\chi_R^{(1)}\}$ is trivial only in the case where the whole variation of σ_∞ is confined to the barrier times. Note that the class $\{\chi_R^{(1)}\}$ will yield non-trivial solutions $\chi_R^{(1)} \not\equiv \chi_0$ in the case of "degenerate" processes where $\chi_R \equiv \chi_0$ (cf. lemma 6.2), provided that $\sigma(B^+ \mid x_0, t_0) \not\equiv 0$ (since for such processes $\sigma_\infty \equiv \sigma$).

If $\sigma_\infty^{(1)} \equiv 0$, then $\chi_R^{(1)}$ satisfies the C.M.P. conditions. Suppose that this is not so; it is obviously legitimate to apply the results of § 8 to $\psi^{(1)}$ and $\chi_R^{(1)}$ instead of ψ and χ_R; hence there exists an infinite class $\{\psi^{(2)}\}$ of maximal solutions of the H.I.E. $\alpha \star \psi^{(1)} = \alpha$, each of which, together with $\chi_0^{(2)} = \chi_R^{(1)}$ defines a *2nd order process* with "regular" solution $\chi_R^{(2)}$, etc. If $\sigma_\infty^{(2)} \not\equiv 0$, this procedure can be repeated to define a *3rd order process*, and so on, leading to sequence of solutions $\{\chi_R^{(n)}\}$, which may be said to terminate at the nth step if $\chi_R^{(n+1)} \equiv \chi_R^{(n)}$. Let $B_n = \{t \mid \sigma_\infty^{(n)}(t+0 \mid x, t) = 1 \text{ for all } x \in \mathfrak{X}\}$.

THEOREM 8.8. (1) *There exists a class of non-decreasing sequences of solutions $\{\chi_R^{(n)}\}$ of I.E. satisfying the I.M.P. conditions constructed inductively by the method described above.*

(2) *A sequence terminates with a last element $\chi_R^{(n)}$ satisfying the C.M.P. conditions if and only if $\sigma_\infty^{(n)} \equiv 0$.*

(3) *A sequence terminates necessarily at the n-th step: i.e. because $\chi_R^{(n+1)} \equiv \chi_R^{(n)}$ for every $\psi^{(n+1)}$, if and only if $\sigma_\infty^{(n)}(B_n^+ \mid x_0, t_0) = 0$.*

(4) *If a sequence does not terminate, it converges to a solution $\chi_R^{(\infty)} = \lim_{n\to\infty} \chi_R^{(n)}$ of I.E. satisfying the I.M.P. conditions, with $\varkappa_R^{(\infty)}(t \mid x_0, t_0) = 1 - \sigma_\infty^{(\infty)}(t \mid x_0, t_0)$, where $\sigma_\infty^{(\infty)} = \lim_{n\to\infty} \sigma_\infty^{(n)}$; $\chi_R^{(\infty)}$ satisfies the C.M.P. conditions if and only if $\sigma_\infty^{(\infty)} \equiv 0$.*

The second part of the theorem follows from the application of Lemma 8.7 to $\chi_R^{(n)}$. To show that $\chi_R^{(\infty)}$ satisfies I.E., we merely have to make $n\to\infty$ in $\chi_R^{(n)}*(I-\psi)=\chi_0$; one shows similarly that $\chi_R^{(\infty)}$ satisfies the 2nd I.M.P. condition (i.e. the C.K. equation); the rest is easily proved. This completes the proof of Theorem 8.8. Note that the jump numbers can be well-ordered so that if we assign the finite ordinals to the 0-order jump numbers, then $\varkappa_n^{(k)}$ is the probability of $\omega^k n$ 0-order jumps, where ω is the first transfinite ordinal, and $\varkappa_n^{(\infty)}$ is the probability of $\omega^\omega n$ 0-order jumps.

The class of solutions described in Theorem 8.8. to which we shall refer as *class A*, does nos exhaust all possible solutions of I.E. satisfying the I.M.P. or C.M.P. conditions. Examples will be given later of a *class B* of solutions where χ_S in (8.5) is itself a maximal solution of H.I.E. (i.e. χ_S is a $\psi^{(1)}$). A further example will also be given of a process which is "pathological" in the sense of Lemma 6.2, i.e. where $\chi_R\equiv\chi_0$, and which yet possesses a "sensible" solution satisfying the C.M.P. conditions; the trouble with this class of processes is that their evolution is no longer properly specified by the integral equation (2.5).

9. Trapping phases

A trapping phase was defined in § 8 as a phase (x, t) such that $\sigma_\infty(t+0\,|\,x, t)=1$; the set of all trapping phases was denoted by K. It was mentioned in § 8 that K is measurable, $K\subset D$, and that if $(x_0, t_0)\in K$, then $\varkappa_R(t\,|\,x_0, t_0)=0$ for all $t>t_0$; hence

$$\chi_R(X, t\,|\,x_0, t_0)=[1-\varepsilon(t-t_0)]\,\delta(X\,|\,x_0)\quad\text{if}\quad(x_0, t_0)\in K. \tag{9.1}$$

Some further properties of trapping phases are given below. Let

$$\eta_n(X\,|\,x_0, t_0)=\psi_n(X, t_0+0\,|\,x_0, t_0),$$

with

$$\eta_0(X\,|\,x_0, t_0)=\delta(X\,|\,x_0)\quad\text{and}\quad\eta_1=\eta;$$

it is obvious that

$$\eta_n(X\,|\,x_0, t_0)=\int_{\mathfrak{X}}\eta_j(X\,|\,x, t_0)\,\eta_{n-j}(dx\,|\,x_0, t_0),\quad(j=0,\ 1,\ \ldots, n;\ n=0, 1, 2, \ldots). \tag{9.2}$$

L E M M A 9.1. *$\sigma_n(t\,|\,x_0, t_0)=\sigma_n(t_0+0\,|\,x_0, t_0)$ for all $t>t_0$ if and only if*

$$\psi_n(X, t\,|\,x_0, t_0)=\eta_n(X\,|\,x_0, t_0)\,\varepsilon(t-t_0). \tag{9.3}$$

C O R O L L A R Y. *If $\sigma_n(t_0+0\,|\,x_0, t_0)=1$, then (9.3) is true and $\eta_n(\mathfrak{X}\,|\,x_0, t_0)=1$, and conversely. If $(x_0, t_0)\in K$ then (9.3) is true for all n.*

The "if" part of the lemma is obvious; the "only if" is proved in the same way as Lemma 3.2. The first part of the corollary follows; and from it the second, because owing to the fact that $\{\sigma_n\}$ is a non-increasing sequence, if $\sigma_n(t+0\,|\,x, t)=1$ then $\sigma_j(t+0\,|\,x, t)=1$ for all $j\leqslant n$; hence if $(x, t)\in K$, then $\sigma_n(t+0\,|\,x, t)=1$ for all n. An immediate consequence is:

LEMMA 9.2. *If* $\psi(K^+\,|\,x_0, t_0)=0$, *then for all* $X\in\mathcal{B}_x$ *and all* $t>t_0$

$$\psi_n(X, t\,|\,x_0, t_0)=\int_{t_0}^{t}\int_{\mathfrak{X}}\eta_{n-1}(X\,|\,\xi, \tau)\,\psi(d\xi, d\tau\,|\,x_0, t_0), \quad (n=1, 2, \ldots) \tag{9.4}$$

$$\sigma_\infty(t\,|\,x_0, t_0)=\sigma_n(t\,|\,x_0, t_0)=\sigma(t\,|\,x_0, t_0). \tag{9.5}$$

THEOREM 9.3. (x_0, t_0) *is a trapping phase if and only if* $\eta(K(t_0)\,|\,x_0, t_0)=1$. If $\eta(K(t_0)\,|\,x_0, t_0)=1$, then $\sigma(t_0+0\,|\,x_0, t_0)=1$; hence $(x_0, t_0)\in D$ and by Lemma 9.1

$$\psi(K^+\,|\,x_0, t_0)=\eta(K^+(t_0)\,|\,x_0, t_0)=0. \tag{9.6}$$

Hence by Lemma 9.2 $\sigma_\infty(t_0+0\,|\,x_0, t_0)=1$, and therefore $(x_0, t_0)\in K$. Conversely, if $(x_0, t_0)\in K$, $\varkappa_R(t\,|\,x_0, t_0)=0$ for all $t>t_0$; but $\varkappa_R*\psi=\varkappa_R-\varkappa_0$; hence

$$0=\varkappa_R(t\,|\,x_0, t_0)=\int_{t_0}^{t}\int_{\mathfrak{X}}\varkappa_R(t\,|\,\xi, \tau)\,\psi(d\xi, d\tau\,|\,x_0, t_0), \quad (t\geqslant t_0). \tag{9.7}$$

It follows as in the proof of Lemma 6.2, with $\varkappa_R$, K substituted respectively for $\varkappa_0$, D, that

$$\psi(K^+\,|\,x_0, t_0)=0. \tag{9.8}$$

Hence by Lemma 9.1 $\eta(K^+(t_0)\,|\,x_0, t_0)=0$; but $\eta(\mathfrak{X}\,|\,x_0, t_0)=\sigma(t_0+0\,|\,x_0, t_0)=1$; hence $\eta(K(t_0)\,|\,x_0, t_0)=1$. This completes the proof of the theorem. It follows by induction that:

COROLLARY. *If* $(x_0, t_0)\in K$, *then* $\eta_n(K(t_0)\,|\,x_0, t_0)=1$ *for all* $n\geqslant 1$. Thus a trapping phase is one from which there is no return to non-trapping phases; hence its name.

A measurable set of phases Q will be called *closed* in case $\eta(Q(t)\,|\,x, t)=1$ whenever $(x, t)\in Q$.

LEMMA 9.4. $Q\subset K$.

If $(x, t)\in Q$ implies $\eta(Q(t)\,|\,x, t)=1$, then by induction it implies $\eta_n(Q(t)\,|\,x, t)=1$ for all n; hence $\sigma_n(t+0\,|\,x, t)=1$ for all n, and therefore $\sigma_\infty(t+0\,|\,x, t)=1$ whenever $(x, t)\in Q$: i.e. $Q\subset K$. Combining this lemma with Theorem 9.3:

LEMMA 9.5. *K is the union of one or more closed sets of phases.*

Consider in particular the "degenerate" process where $\psi(D^+ \mid x, t) \equiv 0$ (cf. Lemma 6.2); then $\eta(D(t) \mid x, t) = 1$ whenever $(x, t) \in D$: i.e. D is a closed set, and hence $D = K$: all singular phases are trapping ones; this also follows from the fact that $\sigma = \sigma_\infty$, (cf. (6.10)): i.e. the first jump time is the same as the first singular jump time. In other words, a "degenerate" process of this type has *only* singular transitions, and therefore ψ and the I.E. yield practically no information regarding its evolution. The procedure described in § 8 amounts in this case to specifying the process anew in terms of χ_0 and of $\psi^{(1)}$ instead of ψ: for $\chi_R \equiv \chi_0$ and $\sigma^{(1)} \equiv \sigma$. Hence if $(\xi, \tau) \in K$, $\sigma^{(1)}(t \mid \xi, \tau) = \varepsilon(t - \tau)$ and therefore

$$\psi^{(1)}(X, t \mid \xi, \tau) = \eta(X \mid \xi, \tau)\, \varepsilon(t - \tau).$$

Since $\psi(K^+ \mid x_0, t_0) \equiv 0$, the H.I.E. yields

$$\begin{aligned} \psi^{(1)}(S \mid x_0, t_0) &= \int_{\varkappa} \psi^{(1)}(S \mid \xi, \tau)\, \psi(d\xi, d\tau \mid x_0, t_0) \\ &= \int_{\varkappa} \eta(S(t) \mid \xi, \tau)\, \psi(d\xi, d\tau \mid x_0, t_0); \end{aligned} \tag{9.9}$$

i.e. $\psi^{(1)}$ is specified entirely by $\eta(X \mid \xi, \tau)$, which is an assumed transition probability consequent upon a singular jump into the trapping phase (ξ, τ).

10. Ergodic phase-space distributions

It is natural to try if possible to analyse the behaviour of unstable processes in terms of an ergodic phase-space distribution ω, which is the limit in some sense of the sequence of phase-space distributions $\{\psi_n\}$. Unfortunately the existence theorems for ergodic limits of general Markoff chains are hedged in by many restrictions, so that it does not appear possible at present to consider this problem fully and in its full generality. We must content ourselves here with a tentative investigation.

Suppose that there exists a conditional distribution $\omega(S \mid x, t)$ on $\mathcal{B}_s \times \mathcal{S}$ such that $\psi_n \xrightarrow{e} \omega$ in some mode of "ergodic" convergence (denoted by $\xrightarrow{e}$ or erg. lim.) as $n \to \infty$; suppose that this mode of convergence is such as to ensure that if $\psi_n \xrightarrow{e} \omega$, then:

(1) if $\alpha(x, t)$ is a bounded measurable function on $\mathcal{S}$ then $\alpha * \psi_n \xrightarrow{e} \alpha * \omega$;

(2) if $\beta(S)$ is a distribution on $\mathcal{B}_s$, then $\psi_n * \beta \xrightarrow{e} \omega * \beta$. It then follows by making $n \to \infty$ in the relations $\psi_{n+1} = \psi_n * \psi = \psi * \psi_n$ that

$$\omega = \omega * \psi = \psi * \omega = \omega * \psi_n = \psi_n * \omega = \omega * \omega. \tag{10.1}$$

Hence the ergodic limit ω is unique: for suppose $\psi_n \overset{e}{\to} \omega'$ as well; then

$$\omega = \omega * \psi_n = \omega * \omega' \quad \text{and} \quad \omega' = \psi_n * \omega' = \omega * \omega';$$

therefore $$\omega = \omega'.$$

Since $\psi_n(X, t \mid x_0, t_0) \leqslant \sigma_n(t \mid x_0, t_0)$ for all n, and since σ_n is a non-increasing sequence converging to σ_∞, we also expect that:

(3) $$\omega(\mathfrak{X}, t \mid x_0, t_0) \leqslant \sigma_\infty(t \mid x_0, t_0).$$

ω will be called *maximal* if $\omega(\mathfrak{X}, t \mid x_0, t_0) \equiv \sigma_\infty(t \mid x_0, t_0)$; this is the most interesting case; for the case of "<" may be visualized as arising from singular transitions leading to an "escape" of the system to states "outside" the state space $\mathfrak{X}$, so that ω "no longer tells the whole story".

It follows from (10.1) that

$$\chi_0 * \omega = \chi_0 * \psi_n * \omega = \chi_n * \omega = \operatorname*{erg.\,lim.}_{n\to\infty} \chi_n * \omega = 0 \tag{10.2}$$

since $\chi_n \to 0$; and therefore

$$0 = \sum_0^n \chi_j * \omega = \operatorname*{erg.\,lim.}_{n\to\infty} \sum_0^n \chi_j * \omega = \chi_R * \omega. \tag{10.3}$$

Hence $$\int_{t_0}^{t} \int_{\mathfrak{X}} \varkappa_R(t \mid \xi, \tau)\, \omega(d\xi, d\tau \mid x_0, t_0) = 0. \tag{10.4}$$

It follows from (10.4) as in the proof of Theorem 9.3 that

$$\omega(K^+ \mid x_0, t_0) = 0; \tag{10.5}$$

i.e. *singular transitions lead to trapping phases only.*

Whenever $(x_0, t_0) \in K$, it follows from the corollary of Lemma 9.1 that $\psi_n(S \mid x_0, t_0) = \eta_n(S(t_0) \mid x_0, t_0)$, and hence that

$$\left.\begin{aligned} &\omega(S \mid x_0, t_0) = \pi(S(t_0) \mid x_0, t_0), \\ \text{where} \quad &\pi(X \mid x_0, t_0) = \operatorname*{erg.\,lim.}_{n\to\infty} \eta_n(X \mid x_0, t_0). \end{aligned}\right\} \tag{10.6}$$

Consequently from the relation $\omega = \omega * \omega$

$$\omega(S \mid x_0, t_0) = \int_K \pi(S(t) \mid x, t)\, \omega(dx, dt \mid x_0, t_0). \tag{10.7}$$

If ω is a maximal ergodic distribution, it is a maximal solution of H.I.E. It cannot be substituted directly for $\psi^{(1)}$ in the construction of class A solutions (cf. end of § 8) because $\chi_R \ast \omega = 0$. However, provided σ_∞ satisfies the conditions of Lemma 8.7, there obviously exist an infinity of functions $\phi^{(1)}(X \mid x_0, t_0)$, restrictions to $\mathfrak{X} \times K$ of functions ϕ satisfying the ϕ-conditions (cf. § 4) such that

$$\psi^{(1)}(S \mid x_0, t_0) = \int_K \phi^{(1)}(S(t) \mid x, t)\, \omega(dx, dt \mid x_0, t_0) \tag{10.8}$$

is a maximal solution of H.I.E. and $\chi_R \ast \psi^{(1)} \not\equiv 0$. To each such $\phi^{(1)}$, which may be interpreted as a postulated distribution of "returns" from trapping states, there corresponds a solution $\chi_R^{(1)}$, and more generally sequences $\{\chi_R^{(n)}\}$ of class A solutions. If moreover the process is time-homogeneous, then

$$\chi(X, t \mid x_0, t_0) = \chi_R(X, t \mid x_0, t_0) + \omega(X, t \mid x_0, t_0) \tag{10.9}$$

is a class B solution. For time homogeneity implies that $\sigma_\infty(t \mid x_0, t_0) = \sigma_\infty(x_0, t - t_0)$; hence if $\sigma_\infty(t+0 \mid x, t) = 1$ for one value of $t > t_0$, it is unity for all $t > t_0$, and consequently

$$K = A \times \mathfrak{T}, \quad \text{where } A = \{x \mid \sigma_\infty(x, +0) = 1\}.$$

It follows that ω satisfies conditions (8.6), for:

1) $$\int_X \chi_R(X \mid \xi, t_1)\, \omega(d\xi \mid x_0, t_2) = \int_A \chi_R(X \mid \xi, t_1)\, \omega(d\xi \mid x_0, t_2) = 0, \quad (t_1 \geqslant 0,\ t_2 \geqslant 0), \tag{10.10}$$

because by (10.5) $\omega(A^+ \mid x, t) \equiv 0$ and $\chi_R(X \mid \xi, t_1) = 0$ if $\xi \in A$;

2) $$\int_X \omega(X \mid \xi, t_1)\, \omega(d\xi \mid x_0, t_2) = \int_A \omega(X \mid \xi, t_1)\, (d\xi \mid x_0, t_2)$$
$$= \int_A \pi(X \mid \xi)\, \omega(d\xi \mid x_0, t_2) = \omega(X \mid x_0, t_2) \tag{10.11}$$

because for time-homogeneous processes $\pi(X \mid x_0)$ in (10.6) is independent of t_0, and hence (10.7) becomes

$$\omega(X \mid x_0, t) = \int_0^t \int_A \pi(X \mid \xi)\, \omega(d\xi \mid x_0, d\tau) = \int_A \pi(X \mid \xi)\, \omega(d\xi \mid x_0, t).$$

3) Finally from (8.11)

$$\omega(X \mid x_0, t) = \omega(X \mid x_0, t_2) + \int_X \omega(X \mid \xi, t_1)\, \chi_R(d\xi \mid x_0, t_2).$$

This completes the proof that (10.9) is a class B solution of I.E. satisfying the C.M.P. conditions.

11. Examples

In the examples below the state space $\mathfrak{X}$ is either a set of integers or a real interval. In the first, second and fourth examples, the process is time-homogeneous and it is convenient to introduce Laplace transforms with respect to the time. Let $\chi_\alpha(x \mid x_0, t)$ stand as a generic notation for a cumulative transition probability distribution, $\varkappa_\alpha(x, t)$ for a jump-number distribution, $\psi_\alpha(x \mid x_0, t)$ for a cumulative jump-time and state distribution, $\sigma_\alpha(x, t)$ for a cumulative jump-time distribution. The Laplace transforms are defined as follows:

$$\bar{\chi}_\alpha(x \mid x_0, s) = \int_0^\infty e^{-st} \chi_\alpha(x \mid x_0, t)\, dt;$$

$$\bar{\varkappa}_\alpha(x_0, s) = \bar{\chi}_\alpha(\infty \mid x_0, s) = \int_0^\infty e^{-st} \varkappa_\alpha(x_0, t)\, dt;$$

$$\bar{\psi}_\alpha(x \mid x_0, s) = \int_0^\infty e^{-st} \psi_\alpha(x \mid x_0, dt);$$

$$\bar{\sigma}_\alpha(x_0, s) = \bar{\psi}_\alpha(\infty \mid x_0, s) = \int_0^\infty e^{-st} \sigma_\alpha(x_0, dt).$$

It follows immediately from (5.4), (5.7) and the convolution theorem that

$$\bar{\psi}_n(x \mid x_0, s) = \int_{-\infty}^{\infty} \bar{\psi}_j(x \mid \xi, s)\, \bar{\psi}_{n-j}(d\xi \mid x_0, s), \quad (j = 0, 1, \ldots, n;\ n = 1, 2, \ldots);$$

$$\bar{\chi}_n(x \mid x_0, s) = \int_{-\infty}^{\infty} \bar{\chi}_j(x \mid \xi, s)\, \bar{\psi}_{n-j}(d\xi \mid x_0, s), \quad (j = 0, 1, \ldots, n;\ n = 1, 2, \ldots).$$

The integral equation (2.5) is equivalent to

$$\bar{\chi}(x \mid x_0, s) = \bar{\chi}_0(x \mid x_0, s) + \int_{-\infty}^{\infty} \bar{\chi}(x \mid \xi, s)\, \bar{\psi}(d\xi \mid x_0, s).$$

The relation (6.2), $\sigma_\infty(x, t) = 1 - \varkappa_R(x, t)$ is equivalent to

$$\bar{\sigma}_\infty(x, s) = 1 - s\, \bar{\varkappa}_R(x, s).$$

Example of a q-process

We consider first an example of a q-process (cf. § 5) where $\mathfrak{X}$ is the real line, χ_0 possesses a density, denoted by the bold face symbol $\boldsymbol{\chi}_0(x - x_0, t)$, which is homogeneous in x as well as t: writing $x - x_0 = y$,

$$\boldsymbol{\chi}_0(y, t) = (2\pi t)^{-\frac{1}{2}} \exp\left\{-\frac{y^2}{2t} - qt\right\}$$

where q is a constant; χ_0 is in fact the unique solution of the diffusion equation

$$\frac{\partial \chi_0}{\partial t} = \frac{1}{2} \frac{\partial^2 \chi_0}{\partial y^2} - q \chi_0.$$

The transition probability ϕ given that a jump has occurred has also a density homogeneous in x and independent of t:

$$\varphi(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}.$$

Substituting in (4.10), we find that ψ has the density

$$\psi(y, t) = q \int_{-\infty}^{\infty} \varphi(y-z) \chi_0(z, t)\, dz = [2\pi(t+1)]^{-\frac{1}{2}} q \exp\left\{-\frac{y^2}{2(t+1)} - qt\right\}.$$

Introducing double Laplace-Fourier transforms of χ_0 and ψ,

$$\bar{\bar{\chi}}_0(\theta, s) = \int_0^{\infty} e^{-st} dt \int_{-\infty}^{\infty} e^{i\theta y} \chi_0(y, t)\, dy = (s + q + \tfrac{1}{2}\theta^2)^{-1}$$

$$\bar{\bar{\psi}}(\theta, s) = \int_0^{\infty} e^{-st} dt \int_{-\infty}^{\infty} e^{i\theta y} \psi(y, t)\, dy = (s + q + \tfrac{1}{2}\theta^2)^{-1} q e^{-\theta^2/2},$$

the integral equation (2.5) becomes

$$\bar{\bar{\chi}}(\theta, s)[1 - \bar{\bar{\psi}}(\theta, s)] = \bar{\bar{\chi}}_0(\theta, s).$$

Hence the transform of the regular solution, which in this example is obviously the unique solution, is

$$\bar{\bar{\chi}}(\theta, s) = \bar{\bar{\chi}}_0(\theta, s)[1 - \bar{\bar{\psi}}(\theta, s)]^{-1} = [s + \tfrac{1}{2}\theta^2 + q(1 - e^{\theta^2/2})]^{-1},$$

whose inverse in series form is

$$\chi(y, t) = \sum_{n=0}^{\infty} \frac{(qt)^n}{n!\,[2\pi(n+t)]^{\frac{1}{2}}} \exp\left\{-\frac{y^2}{2(n+t)} - qt\right\}.$$

Example of an unstable process

We give now an example of a time-homogeneous q-step process of the type described at the end of § 4, which is unstable. Here, $\mathfrak{X} = [0, \infty)$,

$$q(x) = \frac{1}{x};\quad \phi(x \mid x_0) = \frac{x \vee x_0}{x_0};\quad \varkappa_0(x, t) = e^{-t/x};$$

where $x \vee x_0 = \min(x, x_0)$. For $x_0 = 0$, we take $\phi(x \mid 0) = \varepsilon(x)$ and $\varkappa_0(0, t) = 1 - \varepsilon(t)$; i.e. $(0, t)$ is a trapping phase for all t. ψ and its Laplace transform $\bar{\psi}$ are

$$\psi(x \mid x_0, t) = (1 - e^{-t/x_0}) \frac{x \vee x_0}{x_0}; \quad \bar{\psi}(x \mid x_0, s) = \frac{x \vee x_0}{x_0(1 + s x_0)}.$$

Hence by induction

$$\bar{\psi}_n(x \mid x_0, s) = \frac{1}{(n-1)!\, x_0(1 + s x_0)} \int_0^{x \vee x_0} \left[\log \frac{x_0(1 + s\xi)}{\xi(1 + s x_0)}\right]^{n-1} d\xi.$$

The transform of the regular solution is then easily seen to be

$$\bar{\chi}_R(x \mid x_0, s) = \frac{x_0}{1 + s x_0} \varepsilon(x - x_0) + \frac{x \vee x_0}{(1 + s x_0)^2},$$

and hence
$$\chi_R(x \mid x_0, t) = \left[\varepsilon(x - x_0) + \frac{t}{x_0} \frac{x \vee x_0}{x_0}\right] e^{-t/x_0}.$$

In this example one sees that the Markoff chain $\{\psi_n\}$ possesses an ergodic limit ω whose transform is

$$\bar{\omega}(x \mid x_0, s) = \lim_{n \to \infty} \bar{\psi}_n(x \mid x_0, s) = \frac{\varepsilon(x)}{(1 + s x_0)^2} = \bar{\sigma}_\infty(x_0, s)\, \varepsilon(x);$$

hence
$$\omega(x \mid x_0, t) = \left[1 - \left(1 + \frac{t}{x_0}\right) e^{-t/x_0}\right] \varepsilon(x);$$

i.e. in accordance with the conclusions of § 10 the total variation of ω is concentrated in the trapping phases, $(0, t)$. It is easily verified that $\chi = \chi_R + \omega$ is a class B solution.

In order to construct a class A solution (cf. § 8), let $\mu(x)$ be an arbitrary cumulative distribution function (with $\mu(\infty) = 1$) representing the probability distribution of "returns" from $x = 0$, and let

$$\bar{\psi}^{(1)}(x \mid x_0, s) = \mu(x)\, \bar{\sigma}_\infty(x_0, s).$$

It is easily shown by induction that

$$\bar{\psi}_n^{(1)}(x \mid x_0, s) = \mu(x)\, \bar{\sigma}_\infty(x_0, s) \left[\int_0^\infty \bar{\sigma}_\infty(\xi, s)\, \mu(d\xi)\right]^{n-1}.$$

Hence

$$\bar{\chi}_R^{(1)}(x \mid x_0, s) = \bar{\chi}_R(x \mid x_0, s) + \bar{\sigma}_\infty(x_0, s) \int_0^\infty \bar{\chi}_R(x \mid \xi, s)\, \mu(d\xi) \left[1 - \int_0^\infty \bar{\sigma}_\infty(\xi, s)\, \mu(d\xi)\right]^{-1}.$$

It is easily verified that $\chi_R^{(1)}$ is normalised to unity: i.e. that $\bar{\sigma}_\infty^{(1)}(x_0, s) = 0$ and $\bar{\varkappa}_R^{(1)}(x_0, s) = 1/s$. If for example we take the "return" from $x = 0$ to be to the state $x = 1$, i.e. take $\mu(x) = \varepsilon(x-1)$, then an explicit expression for $\chi_R^{(1)}$ is easily obtained; this takes a particularly simple form for $x_0 = 1$, namely,

$$\chi_R^{(1)}(x \mid 1, t) = \tfrac{1}{2}\,\varepsilon(x-1)\,(1 + e^{-2t}) + \tfrac{1}{2}\,(x \vee 1)\,(1 - e^{-2t}).$$

The asymptotic distribution when $t \to \infty$ gives a probability $\frac{1}{2}$ that $x = 1$ and a probability $\frac{1}{2}$ that x is uniformly distributed in the interval [0, 1].

Example of a process with barriers

Consider now a q-step process where $\mathfrak{X}$ is the real line, $q(t)$ is independent of x and $\phi(x \mid x_0)$ is independent of t. Let $Q(t, t_0) = \int_{t_0}^{t_0} q(\tau)\, d\tau$

$$\chi_0(x, t \mid x_0, t_0) = e^{-Q(t,t_0)}\,\varepsilon(x - x_0);$$

$$\psi(x, t) \mid x_0, t_0) = [1 - e^{-Q(t,t_0)}]\,\phi(x \mid x_0).$$

If $Q < \infty$ for all t, t_0, then it is easily proved by induction that

$$\psi_n(x, t \mid x_0, t_0) = \frac{1}{(n-1)!} \int_0^{Q(t,t_0)} u^{n-1} e^{-u}\, du\, \phi_n(x \mid x_0),$$

where ϕ_n is defined inductively by

$$\phi_n(x \mid x_0) = \int_{-\infty}^{\infty} \phi_{n-1}(x \mid \xi)\, \phi(d\xi \mid x_0).$$

Hence

$$\chi_n(x, t \mid x_0, t_0) = \frac{Q^n(t, t_0)}{n!}\, e^{-Q(t,t_0)}\, \phi_n(x \mid x_0);$$

$$\chi_R(x, t \mid x_0, t_0) = {}^{-Q(t,t_0)} \sum_0^\infty \frac{Q^n(t, t_0)}{n!}\, \phi_n(x \mid x_0);$$

$$\varkappa_R(t \mid x_0, t_0) = e^{-Q(t,t_0)} \sum_0^\infty \frac{Q^n(t, t_0)}{n!} = 1.$$

Suppose now that there exists a countable set of times $B = \{t_n\}$ such that

$$Q(t, t_0) \begin{cases} < \infty & \text{if } B \cap [t_0, t] = 0 \text{ (the empty set)}, \\ = \infty & \text{otherwise}; \end{cases}$$

take e.g. $q(t) = \sec^2(\pi t/2)$: then $t_n = 2n + 1$, $n = 1, 2, \ldots$. Suppose furthermore that the cumulative probability distribution

$$\pi(x \mid x_0) = \lim_{n\to\infty} Q_n(x \mid x_0)$$

exists. Then it is easily seen that

$$\sigma_\infty(t \mid x_0, t_0) = \begin{cases} 0 & \text{if } B \cap [t_0, t] = 0, \\ 1 & \text{otherwise}; \end{cases}$$

i.e. every $t_n \in B$ is a barrier, and $\sigma_\infty(B^+ \mid x_0, t_0) \equiv 0$. Hence according to Lemma 8.7 there is no non-trivial solution $\chi_R^{(1)}$; every $\chi_R^{(1)} \equiv \chi_R$. On the other hand,

$$\omega(x, t \mid x_0, t_0) = \lim_{n\to\infty} \psi_n(x, t \mid x_0, t_0) = \sigma_\infty(t \mid x_0, t_0)\, \pi(x \mid x_0)$$

and it is easily verified as in (10.9) that

$$\chi(x, t \mid x_0, t_0) = \chi_R(x, t \mid x_0, t_0) + \omega(x, t \mid x_0, t_0)$$

is a class B solution of I.E. satisfying the C.M.P. conditions.

A "pathological" process

This example (due to Kolmogoroff, 1951) may be obtained by making $N \to \infty$ in the following stable q-step process with $N+1$ states $0, 1, \ldots, N$: let n, k denote positive integers

$$q(0) = N; \quad q(n) = q_n > 0; \quad \phi(0 \mid n) = 1; \quad \phi(k \mid n) = 0; \quad \phi(0 \mid 0) = 0; \quad \phi(k \mid 0) = \frac{1}{N}.$$

$$\psi(0 \mid n, t) = 1 - e^{-q_n t}; \quad \psi(k \mid 0, t) = \frac{1}{N}(1 - e^{-Nt}); \quad \psi(k \mid n, t) = \psi(0 \mid 0, t) = 0.$$

The Laplace transform of the regular solution is easily shown to be

$$\bar{\chi}_R(0 \mid 0, s) = \left[s \left(1 + \sum_1^N \frac{1}{q_n + s} \right) \right]^{-1} = \mu(s);$$

$$\bar{\chi}_R(k \mid 0, s) = \frac{\mu(s)}{q_k + s}; \qquad \bar{\chi}_R(0 \mid n, s) = \frac{q_n \mu(s)}{q_n + s};$$

$$\chi_R(k \mid n, s) = \frac{1}{q_k + s} \left[\delta_{nk} + \frac{q_n \mu(s)}{q_n + s} \right].$$

Make $N \to \infty$; if $\Sigma q_n^{-1} < \infty$ then $\bar{\chi}_R$ converges to the Laplace transform of a transition probability satisfying the C.M.P. conditions. The latter, though obtained as the limit of the unique "regular" solution of stable process, is not a regular solution, nor does it belong to the classes of solutions studied in § 8. ψ is not properly defined in the limiting process because $\psi(k \mid 0, t) = 0$ for all k; this can be remedied by adjoining an ideal state ∞ to the state space such that $\varkappa_0(\infty, t)=0$ and $\psi(\infty|0, t)=\psi(0|\infty, t)=1$ for all $t>0$. It then appears that the limiting process is degenerate in the sense of Lemma 6.2, with the whole variation of ψ confined to the trapping phases $(0, t)$ and (∞, t). A much deeper insight into the structure of this example has been obtained by Kendall and Reuter [8] using semi-group theory. Further examples of solutions which do not fall within the ambit of the classes studied in § 8 are given by Kendall [9]; see also Lévy [11] and]12].

Appendix

Proof of Lemma 3.3. It is sufficient to prove the first part of the lemma for $\alpha \geqslant 0$. Since α is measurable, it is the pointwise limit of a non-decreasing sequence of non-negative simple functions $\alpha_n(x, t) = \sum_i a_i^{(n)} I(S_i^{(n)} | x, t)$, where the $a_i^{(n)}$ are real numbers and the collection $\{S_i^{(n)}\}$ is a finite partition of $\mathcal{S}$, and

$$\gamma(x_0, t_0) = \lim_{n\to\infty} \sum_i a_i^{(n)} \beta(S_i^{(n)} \cap S \,|\, x_0, t_0). \tag{A.1}$$

Each finite sum in the right hand side of (A.1) is a measurable function on $\mathcal{S}$; hence γ is a measurable on $\mathcal{S}$ and is obviously bounded by sup α (since $\beta \leqslant 1$). Equation (3.2) then follows by Fubini's theorem. It follows that in the 2nd part of the lemma γ is a bounded measurable function on $\mathcal{S}$ for fixed S, and is obvious that $\gamma \geqslant 0$. Let $\{S_n\}$ be any sequence of disjoint measurable subsets of $\mathcal{S}$, $S = \bigcup_n S_n$; then

$$\begin{aligned}
\gamma(\bigcup_n S_n | x_0, t) &= \int_{S'} \alpha(\bigcup_n S_n | x, t)\, \beta(dx, dt | x_0, t_0) \\
&= \int_{S'} \sum_n \alpha(S_n | x, t)\, \beta(dx, dt | x_0, t_0) \\
&= \sum_n \int_{S'} \alpha(S_n | x, t)\, \beta(dx, dt | x_0, t_0) \\
&= \sum_n \gamma(S_n | x_0, t_0),
\end{aligned} \tag{A.2}$$

where the inversion in the order of summation and integration in 3rd expression is justified by Lebesgue's monotone convergence theorem. Finally

$$\gamma(\mathcal{S} | x_0, t_0) = \int_{S'} \alpha(\mathcal{S} | x, t)\, \beta(dx, dt | x_0, t_0) \leqslant \beta(S' | x_0, t_0) \leqslant 1. \tag{A.3}$$

This completes the proof of the lemma.

Proof of Lemma 3.4. It is sufficient to prove the lemma for $\alpha \geqslant 0$. Express α as the pointwise limit of a non-decreasing sequence of non-negative simple functions, using the same notation as in the proof of Lemma 3.3; then

$$\begin{aligned}
\int_{S_1} \left\{ \int_{S_2} \alpha(x_2, t_2)\, \beta(d x_2, d t_2 \mid x_1, t_1) \right\} \gamma(d x_1, d t_1 \mid x_0, t_0) \\
&= \int_{S_1} \left\{ \lim_{n\to\infty} \sum_i a_i^{(n)}\, \beta(S_i^{(n)} \cap S_2 \mid x_1, t_1) \right\} \gamma(d x_1, d t_1 \mid x_0, t_0) \\
&= \lim_{n\to\infty} \sum_i a_i^{(n)} \int_{S_1} \beta(S_i^{(n)} \cap S_2 \mid x_1, t_1)\, \gamma(d x_1, d t_1 \mid x_0, t_0) \\
&= \int_{S_2} \alpha(x_2, t_2) \left\{ \int_{S_1} \beta(d x_2, d t_2 \mid x_1, t_1)\, \gamma(d x_1, d t_1 \mid x_0, t_0) \right\}, \qquad \text{(A.4)}
\end{aligned}$$

where the passage from the 2nd to the 3rd line is justified by Lebesgue's monotone convergence theorem. This proves the lemma.

Proof of Lemma 3.6. Under the conditions of the lemma, for fixed S $\alpha(S \mid x, t)$ is a measurable function on $\mathfrak{S}$, and for fixed (x, t) a distribution on $\mathcal{B}_s$ (cf. Munroe [13], p. 106); hence α is a conditional distribution on $\mathcal{B}_s \times \mathfrak{S}$. (3.8) follows by Lebesgue's bounded convergence theorem, (3.9) by the generalization of the Helly-Bray theorem for sequences of Lebesgue-Stieltjes integrals to sequences of integrals with respect to general measures (Munroe, loc. cit., p. 173).

Proof of Lemma 3.7. Fix x and t; then $\beta'(S \mid x, t)$ is a distribution on $\mathcal{B}_s$. For if $S \in \mathcal{B}_s$, then $S(t) \in \mathcal{B}_x$ (cf. Halmos [7], p. 141). Clearly $\beta'(S \mid x, t) \geqslant 0$. Let $\{S_n\}$ be any sequence of disjoint sets $\varepsilon\, \mathcal{B}_s$, $S = \bigcup_n S_n$; then $S(t) = \bigcup_n S_n(t)$. Hence since $\beta(X \mid x, t)$ is a distribution on $\mathcal{B}_x$, $\beta'(S \mid x, t) = \beta(S(t) \mid x, t) = \sum_n \beta(S_n(t) \mid x, t) = \sum_n \beta'(S_n \mid x, t)$. Finally, since $\mathfrak{S}(t) = \mathfrak{X}$, $\beta'(\mathfrak{S} \mid x, t) = \beta(\mathfrak{X} \mid x, t) \leqslant 1$; this completes the proof of the assertion. Let now $\mathfrak{M}$ be the class of all measurable sets S such that $\beta'(S \mid x, t)$ is a measurable function on $\mathfrak{S}$. $\mathfrak{M}$ includes all measurable rectangle sets $X \times T$, $X \in \mathcal{B}_x$, $T \in \mathcal{B}_t$, because for each such rectangle

$$\beta'(X \times T \mid x, t) = \beta(X \mid x, t)\, \varepsilon(T \mid t)$$

is the product of a measurable function by the characteristic function of the set $\mathfrak{X} \times T$. $\mathfrak{M}$ is a monotone class: for let $\{S_n\}$ be any monotone sequence of sets in $\mathfrak{M}$ and let $S = \lim_{n\to\infty} S_n$; then $S \in \mathcal{B}_s$, and hence $\beta'(S \mid x, t) = \lim_{n\to\infty} \beta'(S_n \mid x, t)$ is the limit of a convergent sequence of measurable functions, and is therefore measurable; hence $S \in \mathfrak{M}$. But the minimal monotone class containing all measurable rectangle sets is $\mathcal{B}_s$, therefore $\mathfrak{M} \supset \mathcal{B}_s$; but by hypothesis $\mathfrak{M} \subset \mathcal{B}_s$; therefore $\mathfrak{M} \equiv \mathcal{B}_s$. This completes the proof of the first part of the lemma; the proof of the second part is analogous and will be omitted.

Proof of Lemma 3.8. By Lemma 3.7, the right-hand-sides of (3.10) and (3.11) exists; using Fubini's theorem, they are seen to be equal to the left-hand sides.

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