ANALYTIC THEORY OF SINGULAR DIFFERENCE EQUATIONS.

By

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§ 1. Introduction.

In the present paper we study a linear difference system of order n in $y_1(x), \ldots y_n(x)$,

$$y_i(x+1) = \sum_{j=1}^{n} a_{ij}(x)\, y_j(x) \tag{1}$$

$$(|(a_{ij}(x))| \not\equiv 0;\ i = 1, 2, \ldots n),$$

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in which the coefficients $a_{ij}(x)$ will be taken either as known rational functions of x or, more generally, as series convergent for $|x| > r$ which, except for a finite number of ascending powers, are in descending powers of x or of $x^{\frac{1}{p}}$ (p, a positive integer). Still more generally it may be supposed that the coefficients are merely represented asymptotically by such series in certain regions of the complex plane.

The equations (1) will be written in the matrix form as follows

$$Y(x+1) = A(x)\,Y(x), \tag{1 a}$$

$$(Y(x) \equiv (y_{ij}(x));\ A(x) \equiv (a_{ij}(x)));$$

here, for $j = 1, 2, \ldots n$, the elements $y_{1j}(x), \ldots y_{nj}(x)$ in the j-th column of $Y(x)$ form a solution

$$y_1(x) = y_{1j}(x), \ldots y_n(x) = y_{nj}(x)$$

of the equations (1). Such a solution $Y(x)$ of the matrix equation (1 a) will be called a matrix solution in case $|Y(x)| \not\equiv 0$.

In a preceding paper by Birkhoff[1] the well known fact was pointed out, that such a system (1) may be related to a single difference equation of the n-th order

$$L_n(y) \equiv a_0(x)\,y(x+n) + a_1(x)\,y(x+n-1) + \cdots + a_n(x)\,y(x) = 0, \tag{2}$$

$$(a_0(x) \not\equiv 0;\ a_n(x) \not\equiv 0)$$

by means of a linear transformation

$$y(x) = \sum_{j=1}^{n} \lambda_j(x)\,y_j(x), \tag{3}$$

in such a wise that whenever $y_1(x), \ldots y_n(x)$ is a solution of (1), the corresponding $y(x)$ obtained from (3) is a solution of (2), and, conversely, whenever $y(x)$ is a solution of (2), then the n functions $y_1(x), \ldots y_n(x)$ determined by the n equations

[1] Formal Theory of Irregular Linear Difference Equations, these Acta, vol. 54, 1930, pp. 205—246 (cited hereafter as (I)).

$$y(x) = \sum_{j=1}^{n} \lambda_j(x)\, y_j(x),$$

(4) $$y(x+1) = \sum_{j, j_1=1}^{n} \lambda_j(x+1)\, a_{j j_1}(x)\, y_{j_1}(x),$$

$$\cdots\cdots\cdots\cdots$$

$$y(x+n-1) = \sum_{j, j_1, \ldots j_{n-1}}^{n} \lambda_j(x+n-1)\, a_{j j_1}(x+n-2) \ldots a_{j_{n-2} j_{n-1}}(x)\, y_{j_{n-1}}(x)$$

form a solution of (1). Here $a_0(x), \ldots a_n(x)$ are known functions of the same type as the $a_{ij}(x)$ while $\lambda_1(x), \ldots \lambda_n(x)$ are known rational functions of x, arbitrary except that certain special conditions are not to be satisfied. Conversely of course an equation (2) can be related to a system (1), for instance by writing

(5) $$y_1(x+1) = y_2(x),\ y_2(x+1) = y_3(x), \ldots, y_{n-1}(x+1) = y_n(x),$$

$$a_0(x)\, y_n(x+1) = -a_1(x)\, y_n(x) - a_2(x)\, y_{n-1}(x) - \cdots - a_n(x)\, y_1(x)$$

in which case $y = y_1(x)$ will satisfy (2). Since in (2) we have $a_0(x) \not\equiv 0$, without any loss of generality it may be supposed that $a_0(x) = 1$. In much of the text, along with an equation of type (2) there will be occasion to consider the related system

(6) $$Y(x+1) = D(x)\, Y(x),$$

$$D(x) = \begin{pmatrix} 0, & 1, & 0, & \ldots & 0 \\ 0, & 0, & 1, & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -a_n, & -a_{n-1}, & \ldots & & -a_1 \end{pmatrix} = (d_{ij}(x)).$$

If $Y(x) = (y_{ij}(x))$ is a matrix solution of (6) then

(6 a) $$(y_{ij}(x)) = (y_j(x+i-1))$$

and the functions $y_1(x), \ldots y_n(x)$ will constitute a fundamental set of solutions of (2). The converse is also true.

The fundamental result of the paper referred to above is that every system of type (1), or single equation (2), admits precisely n linearly independent formal solutions with elements of the general type

$$e^{Q(x)}s(x),\ Q(x)=\mu x\log x+\gamma x+\delta x^{\frac{p-1}{p}}+\cdots+\nu x^{\frac{1}{p}} \tag{7}$$

where

$$s(x)=x^r\Big[(a+bx^{-\frac{1}{p}}+\cdots)+(a^1+b^1x^{-\frac{1}{p}}+\cdots)\log x+\cdots \\ +(a^m+b^mx^{-\frac{1}{p}}+\cdots)\log{}^m x\Big], \tag{7 a}$$

p is a positive integer, μp is an integer and m is a positive integer or 0. Here p does not need to be the same as the integer, denoted by the same letter, occurring in connection with the coefficients of a system, or single equation. The following definition will be introduced.

Definition 1. *A formal series $s(x)$ which is of the form (7 a) will be termed an s-series.*

An element $e^{Q(x)}s(x)$ may be thought of as representative of $y(x)$ for instance, in which case the corresponding $y_1(x), \ldots y_n(x)$ are given by such s-series with the same exponential factor $e^{Q(x)}$. Two series (7) which differ merely by a periodic factor $e^{2\lambda\pi\sqrt{-1}x}$ (λ, an integer) are considered to be linearly dependent. The series involved may converge but in general will diverge. With an equation of order n, of type (2), there will be associated n functions $Q_j(x)\,(j=1, \ldots n)$.

For purposes of classification the following terminology is found to be convenient. The difference system (1) or single equation (2) will be called *normal* if $p=1$ in all of the formal series, so that each $Q_j(x)$ reduces merely to $\mu_j x\log x+\gamma_j x$; otherwise it will be called *anormal*, since then there enter anormal series with $p>1$. This agreement is in accordance with that used for linear differential equations. Moreover the system (1) or equation (2) will be called *regular* or *irregular*, according as there is only a single value of μ_j or more than one such value. Finally any difference system (1) or equation (2) is called *singular* when it is not both normal and regular.

The earlier methods of Nörlund, Galbrun, Carmichael and Birkhoff were applied primarily to the regular normal case[1]; for a system (1), this case may be looked upon as the 'general' case from a certain point of view.

[1] Cf. N. E. Nörlund, *Differenzen Rechnung*, Berlin (1924). See also Birkhoff's papers *General Theory of Linear Difference Equations*, Trans. Am. Math. Soc., vol. 12 (1911) pp. 243—284 (hereafter cited as (II)); *The Generalized Riemann Problem for Linear Differential Equations and the Allied Problems for Linear Difference and q-Difference Equations*, Proc. Am. Acad. Arts and Sciences, vol. 49 (1913), pp. 521—568 (hereafter cited as (III)).

Furthermore Galbrun[1] has treated a single difference equation of order n with rational coefficients in the special case of the special anormal regular type in which a pair of anormal series enter with $p=2$, so that the two corresponding polynomials $Q(x)$ in $\sqrt{x}$ have the respective forms

$$\gamma(x) + \delta\sqrt{x}, \quad \gamma x - \delta\sqrt{x}.$$

In a recent important paper Adams[2] has shown that to some extent Birkhoff's methods continue to apply in the irregular normal case.

In the present paper the analytic theory of linear difference system (or single equation) is developed so as to apply without restriction upon the form of the formal series. The methods consist, on one hand, of suitable modifications of those of paper II; on the other hand, an important rôle is played by certain new methods involving factorization and group summations. The main result of the paper is embodied in the Fundamental Existence Theorem of section 9. In most of the text preceding section 8 we restrict ourselves to a quadrant Γ,

$$\frac{\pi}{2} \leqq \arg x \leqq \pi + \varepsilon \qquad (|x| > \varrho > 0)$$

the lower boundary of which, h, is a portion of a line parallel to the axis of reals and lying below this axis. In quadrants other than Γ results will hold precisely analogous to those obtained with reference to Γ.

As a matter of notation we shall write

$$Q_{ij}(x) = Q_i(x) - Q_j(x).$$

Moreover, in addition to definition (1), it will be found convenient to set forth the following definitions.

Definiton 2. *A branch extending to infinity and satisfying the equation* $\Re Q'_{ij}(x) = 0$ *will be called a* B' *curve. If* $\Re Q'_{ij}(x) \equiv 0$ *there is no* B' *curve.*

Definition 3. *Let* G *denote a part of* Γ *with the right boundary coincident with that of* Γ. *Let the left boundary of* G *have a limiting direction at infinity;*

[1] H. Galbrun, *Sur certaines solutions exceptionnelles d'une équation linéaire aux différences finies*, Bull. Soc. Math. de France, vol. 49 (1921), pp. 206—241.

[2] C. R. Adams, *On the Irregular Cases of the Linear Ordinary Difference Equations,* Trans. Am. Math. Soc., vol. 30 (1928), pp. 507—54.

if this direction is coincident with that of the negative axis of reals assume this boundary to be of the form $v = h(-u)^e$ $(h > 0;\ 1 > e > 0;\ x = u + \sqrt{-1}\,v)$. *Let* C *denote a curve in* G, *with a limiting direction at infinity.* $V(x)$ *will be said to possess an order* k_0 *along* C *if, as* $|x| \to \infty$ *along* C,

$$\left|e^{V(x) - 2\pi k\sqrt{-1}x}\right| \begin{matrix} \to 0 \\ \to \infty \end{matrix} \begin{cases} k < k_0 \\ k > k_0. \end{cases}$$

A function $V(x)$ *will be said to be proper in* G, *or* $|e^{V(x)}|$ *will be said to be comparable with* $|e^{2\pi\sqrt{-1}x}|$ *in* G, *if along every curve* C, *lying in* G *and of the above description,* $V(x)$ *has an order* k_0 *(in general, dependent on* C*). A set* $Q_j(x)$ $(j = 1, \dots n)$ *will be said to be proper in* G *if all the* $Q_{ij}(x)$ $(i, j = 1, \dots n)$ *are proper in* G. *The region* G *may reduce to a single curve* C.

Definition 4. *An operator* $L_n(y)$ *(or equation* $L_n(y) = 0$*), with coefficients known in a region* G *and of the type, in* G, *which has been assumed with reference to* (1) *and* (2), *will be termed proper in* G *if the equation* $L_n(y) = 0$ *has a fundamental set of solutions with the asymptotic form of the formal series in each of the several regions, separated by* B' *curves, which form* G. *Solutions of this kind will be said to be proper.*

Definition 5. *A function* $p(x)$, *of period unity and analytic in an upper half plane, will be called proper if*

$$p(x) \sim p e^{2\pi\sqrt{-1}Hx}$$

$$(p, \text{ a constant; } H, \text{ an integer})$$

in a region whose left boundary is of the form $v = h(-u)^e$ *and whose right boundary is of the form* $v = hu^e$ $(h > 0,\ e > 0)$.

Definition 6. *An operator* $L_n(y)$ *(or equation* $L_n(y) = 0$*) which is proper in* G *will be called completely proper in* G *if, in* G, *proper fundamental sets of solutions exist which are connected by proper periodic functions.*

Definition 7. *A set*

$$Q_1(x), \dots Q_n(x)$$

has a point of division in G *if this set can be separated into two groups*

$$Q_1(x), \dots Q_\Gamma(x);\ Q_{\Gamma+1}(x), \dots Q_n(x)$$
$$(1 \leqq \Gamma < n)$$

so that for x in G

$$\Re\, Q'_\lambda(x) \geqq \Re\, Q'_{\Gamma+\mu}(x)$$
$$(\lambda = 1, \dots \Gamma;\ \mu = 1, \dots n - \Gamma).$$

Definition 8. *Let G denote a subregion of Γ, the lower boundaries of G and Γ being coincident. An operator $L_n(y)$ (or equation $L_n(y) = 0$) with coefficients of the same kind, in G, as in (1) and (2) will be said to be Q-factorable in G if the set of Q's,*

$$Q_1(x), \dots Q_n(x),$$

belonging to $L_n(y)$, has a point of division in G.

Definition 9. *Let F be a curve extending to infinity and lying in Γ. Let R_F be the portion of Γ between F and the right boundary of Γ. The curve F and the region R_F will be termed proper for the set*

$$Q_1(x), \dots Q_n(x)$$

if this set is proper along F and also in R_F.

§ 2. Some Properties of B' and Proper Curves.

Let us write

(1) $$Q_j(x) = \mu_j\, x \log x + P_j(x) \qquad (j = 1, \dots n).$$

With a system (or equation) of order n we have associated n polynomials, $P_j(x)$ $(j = 1, \dots n)$, in $x^{\frac{1}{p}}$, of degree not greater than p and without constant terms. These polynomials occur in the exponential parts of the formal series (7, (7 a); § 1). In this connection subscripts 1 to n are attached to p, m, r, γ, δ, $\dots \nu$ so as to differentiate between the n exponential factors which accompany the n formal power series (when we consider a single equation) or so as to differentiate between the n columns of the formal matrix (when a system is considered). Unless stated otherwise, we shall take for each of these elements a common value of p and m; this can always be effected by taking these integers sufficiently great. It is obvious of course that if one determination of $x^{\frac{1}{p_j}}$ and of $\log x$ is made in such a formal solution, all of other determinations of $x^{\frac{1}{p_j}}$, namely

$$\omega x^{\frac{1}{p_j}}, \quad \omega^2 x^{\frac{1}{p_j}}, \ldots \omega^{p_j-1} x^{\frac{1}{p_j}}$$

where ω is a primitive p_j-th root of unity, and certain of the other determinations of $\log x$, namely

$$\log x + 2\pi\sqrt{-1}, \quad \log x + 4\pi\sqrt{-1}, \ldots \log x + 2(m_j - 1)\pi\sqrt{-1},$$

yield other related formal solutions. Thus with one such formal solution is associated a group of $m_j p_j$ solutions which are linearly independent.

For the present we suppose a cut made along the positive axis of reals and take the principal determination of $x^{\frac{1}{p}}$ and $\log x$ on the upper side of the cut. In much of the text we deal with 'solutions the left' in such a cut plane. When we 'work from the right' there is a similar procedure, with a cut made along the negative axis of reals.

The B' curves will be seen to be of outstanding importance (Def. 2; § 1). Along such a curve $\Re Q'_{ij}(x) = 0$; conversely, any curve for which $\Re Q'_{ij}(x) = 0$ $(i \neq j)$ is a B' curve, except when $\mu_i = \mu_j$ while $P'_{ij}(x)$ is a pure imaginary constant. In fact, the equation

$$\Re Q'_{ij}(x) = 0 \tag{2}$$

can be written more explicitly in the form

$$\mu_{ij}(\log |x| + 1) + \Re P'_{ij}(x) = 0$$

$$\left(P'_{ij}(x) = \gamma_{ij} + \frac{p-1}{p}\delta_{ij} x^{-\frac{1}{p}} + \cdots;\ \mu_{ij} = \mu_i - \mu_j,\ \gamma_{ij} = \gamma_i - \gamma_j,\ \delta_{ij} = \delta_i - \delta_j\right);$$

this justifies the last statement. There is no B'_{ij} curve[1] when $\mu_i \neq \mu_j$, since for $|x|$ large the first member of the above equation would be arbitrarily large while the second term would remain finite in fact approaching $\Re\gamma_{ij}$. Consequently it follows that for the existence of a B'_{ij} curve it is necessary that

[1] The B' curve along which $\Re Q'_{ij}(x) = 0$ is denoted by B'_{ij}.

$$\mu_{ij} = \Re\gamma_{ij} = 0.$$

Obviously the B' curves are algebraic. Thus when a B'_{ij} curve exists it is given by the equation

$$R_{ij}(x) \equiv \Re\left[P'_{ij}(x) - \gamma_{ij}\right] = 0$$

or

$$\Re\left[\frac{p-1}{p}\delta_{ij}x^{-\frac{1}{p}} + \cdots + \frac{1}{p}\nu_{ij}x^{-\left(\frac{p-1}{p}\right)}\right] = 0. \tag{2 a}$$

If there exists an anormal series (with $p_i > 1$), the p_i determinations of $x^{\frac{1}{p_i}}$ yield p_i associated series solutions with the same value of μ and the same real part of γ; in this case the necessary conditions for the existence of a B'_{ij} curve are satisfied.

If $p_i = p_j = 1$, there exists no B'_{ij} curve since then $P'_{ij}(x)$ is a pure imaginary constant. It is clear then that if a B'_{ij} curve exists it satisfies an equation

$$\Re\left[\frac{s}{p}\eta_{ij}x^{-\left(\frac{p-s}{p}\right)} + \cdots + \frac{1}{p}\nu_{ij}x^{-\left(\frac{p-1}{p}\right)}\right] = 0 \tag{2 b}$$

$$(\eta_{ij} = |\eta_{ij}|\,e^{\sqrt{-1}\,\bar{\eta}_{ij}};\quad \eta_{ij} \neq 0;\quad 1 \leqq s \leqq p-1).$$

The leading term in (2 b) is

$$\frac{s}{p}|\eta_{ij}|\,|x|^{-\left(\frac{p-s}{p}\right)}\cos\left(\bar{\eta}_{ij} - \left(\frac{p-s}{p}\right)\alpha\right) \qquad (\alpha = \arg x);$$

consequently there will be several B'_{ij} curves with limiting directions α defined by the equation

$$\cos\left(\bar{\eta}_{ij} - \left(\frac{p-s}{p}\right)\alpha\right) = 0. \tag{2 c}$$

It is of interest to determine in what cases there are no B' curves. From the preceding it is clear that whe must then have for all i and j either $\mu_{ij} \neq 0$ or $\mu_{ij} = 0$, $\Re\gamma_{ij} \neq 0$, or else $\mu_{ij} = 0$, $\Re\gamma_{ij} = 0$, $P'_{ij} - \gamma_{ij} \equiv 0$. Suppose then that we consider those formal series for which μ_i has a given value and at the same time $\Re\gamma_i$ has a given value. For the corresponding formal solutions we shall have

$$P'_{ij}(x) - \gamma_{ij} \equiv P(x).$$

Now if $P_i(x)$ has various determinations, then μ_i and $\Re\gamma_i$ are the same for all of these, so that these determinations fall into the same group, and $P'_{ij} - \gamma_{ij} \not\equiv 0$ in all cases. Consequently $P'_i(x) - \gamma_i$ must be identically zero, so that $P_i(x) \equiv \gamma_i x$.

The case in which there are no B' curves is accordingly the general normal case, regular or irregular.

The B' curves are important for the analytic theory since they limit the scope of validity of iterations and summations.

In the cases hitherto treated with success there have been no B' curves.

Generally speaking the B' curves, or any other curves depending on the $Q(x)$'s are not to be regarded as fixed up to a translation. The region Γ consists of several consecutive regions separated by B' curves. These regions will be denoted as

$$(1),\ (2), \dots (m), \dots$$

where (1) is the region having for its lower boundary that of Γ. Unless there is only one such region, so that (1) is Γ, the upper boundary of (1) will be denoted by B'; in general, the right boundary of (m) will be denoted by B^m. The number of these regions is finite, the right boundary of the last one of them being coincident with that of Γ (i.e., with the positive half of the axis of imaginaries or a line parallel to that axis). Moreover, these regions may be considered overlapping in the sense that any such two consecutive regions may be considered as having a strip of, say, unit width in common. *In the sections leading up to § 9 it will be assumed that B' curves are simple in the sense that as we pass across such a curve precisely two of the functions $\Re\, Q'(x)$ are interchanged in order.* How to meet the situation when the above is not the case will be apparent from the text.

The following lemma will be now proved.

Lemma 1. *Let B^m be a B'_{ij} curve with the limiting direction of the negative axis of reals. Assume that it is not coincident with the negative axis of reals. Then*

$$Q_{ij}(x) = \sqrt{-1}\left(\sigma_{ij}x + a_{ij}(-x)^{\frac{s_{ij}}{p}} + \cdots\right) + \left(b_{ij}(-x)^{\frac{r_{ij}}{p}} + \cdots\right) \tag{3}$$

$$(\sigma, a, \dots b, \dots, \textit{real};\ a,\ b \neq 0;\ p > s_{ij} > r_{ij} \geqq 1)$$

and the equation of B^m will be of the form

$$v = c(-u)^{\frac{d_{ij}}{p}} + \cdots \tag{3 a}$$

$$(x = u + \sqrt{-1}\, v; \quad d_{ij} = p - s_{ij} + r_{ij}; \quad c > 0).$$

Moreover, in the region between a curve $F_{r_{ij}}$,

$$v = \gamma(-u)^{\frac{r_{ij}}{p}} \qquad (\gamma > 0) \tag{3 b}$$

(lying, of course, below B^m) and the positive axis of imaginaries (as well as further to the right) the function $Q_{ij}(x)$ is proper.

Proof. Writing $Q_{ij}(x)$ in the form (3) we note that a term in $x^{\frac{r_{ij}}{p}}$ with $p > r_{ij} \geqq 1$ actually enters, since otherwise the negative axis of reals is evidently the only possible corresponding B' curve; this possibility is excluded by hypothesis. Hence $b_{ij} \neq 0$. The equation of the curve will be of the form

$$\Re\, Q'_{ij}(x) = \Re\left[\sqrt{-1}\left(-\frac{s_{ij}\, a_{ij}(-x)^{\frac{s_{ij}}{p}}}{p(-x)} + \cdots\right) + \right.$$
$$\left. + \left(-\frac{r_{ij}\, b_{ij}(-x)^{\frac{r_{ij}}{p}}}{p(-x)} + \cdots\right)\right] = 0.$$

If $s_{ij} < r_{ij}$ the dominant term in the parenthesis is clearly the second one of those displayed. The expression in the parenthesis would be nearly real for $|x|$ large. Hence there could be no B' curve of the specified type in this case. If $s_{ij} = r_{ij}$ the parenthesis has a dominant term

$$-\left(\sqrt{-1}\,\frac{s_{ij}\, a_{ij}}{p} + \frac{r_{ij}\, b_{ij}}{p}\right)\frac{(-x)^{\frac{s_{ij}}{p}}}{(-x)}$$

and this is also impossible since $b_{ij} \neq 0$.

There remains the case $s_{ij} > r_{ij}$ with $a_{ij} \neq 0$, when the dominant term is the first explicitly written. Here there is actually a B' curve with horizontal direction to the left. Now we may write the above equation in the form

$$\Im h(x) = 0, \quad h(x) \equiv \frac{1}{(-x)^{\frac{p-s_{ij}}{p}}} \left[\left(-\frac{s_{ij}\, a_{ij}}{p} + \cdots \right) + \sqrt{-1} \left(\frac{\frac{r_{ij}}{p} b_{ij}}{(-x)^{\frac{s_{ij}-r_{ij}}{p}}} + \cdots \right) \right].$$

Along the curve, for $|x|$ large, $h(x)$ will be arbitrarily near to a small real value. Furthermore by suitably ordering i and j we may make a_{ij} negative so that $h(x)$ will approach a positive value. Introducing a new variable $\bar{x}$,

$$\frac{1}{(-\bar{x})^{\frac{p-s_{ij}}{p}}} = \frac{1}{(-x)^{\frac{p-s_{ij}}{p}}} \left[-\frac{s_{ij}\, a_{ij}}{p} + \cdots \right],$$

$h(x)$ becomes

$$\frac{1}{(-\bar{x})^{\frac{p-s_{ij}}{p}}} \left[1 + \sqrt{-1} \left(d_1 (-\bar{x})^{-\frac{s_{ij}-r_{ij}}{p}} + \cdots \right) \right] \qquad (d_1 \neq 0).$$

In the last parenthesis the power series has real cofficients. By inversion it follows that

$$-\bar{x} = h(x)^{\frac{-p}{p-s_{ij}}} \left[1 + \sqrt{-1}\, d_2\, h(x)^{\frac{s_{ij}-r_{ij}}{p-s_{ij}}} + \cdots \right] \qquad (d_2 \neq 0).$$

Hence, writing $\bar{x} = \bar{u} + \sqrt{-1}\, \bar{v}$, we have

$$-\bar{u} = h(x)^{\frac{-p}{p-s_{ij}}} (1 + \cdots), \quad -\bar{v} = d_2 h(x)^{-\frac{p-s_{ij}+r_{ij}}{p-s_{ij}}} + \cdots$$

so that the B' curve in the $\bar{x}$ plane will have the form

$$\bar{v} = d_3 (-\bar{u})^{\frac{d_{ij}}{p}} + \cdots (d_3 \neq 0;\ p > d_{ij} = p - s_{ij} + r_{ij} \geqq 1).$$

Consequently the equation in the x plane will be of the form (3 a).

We shall prove now that the function $Q_{ij}(x)$ is proper in a region $R_{F_{r_{ij}}}$ between a curve $F_{r_{ij}}$, given by (3 b), and the positive axis of imaginaries (as well as further to the right until a line, in the first quadrant, is reached making an arbitrarily small angle with the positive axis of reals). In other words, we wish to show that along any particular curve, lying in $R_{F_{r_{ij}}}$ and having a limiting direction at infinity, a definite finite order k_0 (depending on the curve) for $e^{Q_{ij}(x)}$ as compared with $e^{2\pi\sqrt{-1}x}$ exists in the sense that, as $|x|$ becomes infinite along this curve, we have

(4) $$\lim e^{Q_{ij}(x)-2k\pi\sqrt{-1}x} = \begin{cases} \infty & (k>k_0) \\ 0 & (k<k_0). \end{cases}$$

(If we work below the real axis, $e^{2\pi\sqrt{-1}x}$ is replaced by $e^{-2\pi\sqrt{-1}x}$.) It can be easily verified that, since in $Q_{ij}(x)$ the constant μ_{ij} is zero, $Q_{ij}(x)$ will be proper in the whole plane excepting possibly in two arbitrarily small sectors containing the positive and negative axes of reals, respectively. In other words, $Q_{ij}(x)$ will certainly be proper in any region in which there can be no curves having limiting direction at infinity, coinciding with that of the real axis. For simplicity of demonstration the subscripts i, j in the following will be omitted. It is sufficient to consider curves F_δ lying in R_{F_Γ} and given by equations of the form

(5) $$v = h(-u)^{\frac{\delta}{p}} \quad (h>0;\ p>\delta\geqq\Gamma);$$

here δ is not necessarily an integer.

For x on F_δ we shall have

$$\pi-\alpha = \operatorname{tg}^{-1}\left(\frac{v}{-u}\right) = \frac{h}{(-u)^{\frac{p-\delta}{p}}} + \cdots$$

so that, along F_δ,

(5 a) $$\Re\left(\sqrt{-1}\,(-x)^{\frac{s}{p}}\right) = |x|^{\frac{s}{p}} \sin\frac{s}{p}(\pi-\alpha) = \frac{\dfrac{s\,h\,|x|^{\frac{s}{p}}}{p}}{(-u)^{\frac{p-\delta}{p}}} + \cdots$$

and

(5 b) $$\Re(-x)^{\frac{\Gamma}{p}} = |x|^{\frac{\Gamma}{p}} \cos\frac{\Gamma}{p}(\pi-\alpha) = |x|^{\frac{\Gamma}{p}} + \cdots.$$

Using (5), (5 a), (5 b) and the fact that, along F_δ, $|x| = -u+\cdots$ we have for x on F_δ

(5 c) $$\Re\left[Q(x) - 2\pi k\sqrt{-1}\,x\right] = (2k\pi - o)\,h(-u)^{\frac{\delta}{p}} + \left[\frac{sha}{p}(-u)^{\frac{\delta+s-p}{p}} + \cdots\right] + \left[b(-u)^{\frac{\Gamma}{p}} + \cdots\right];$$

here the terms displayed in the parentheses are correspondingly the leading ones. When $\delta = \Gamma$ the leading term in the second member of (5 c) will be $[(2k\pi - \sigma)h + b](-u)^{\frac{\Gamma}{p}}$; on the other hand, when $\delta > \Gamma$ the leading term will be $(2k\pi - \sigma)h(-u)^{\frac{\delta}{p}}$. In the first case the order k_0 will satisfy the equation

$$(2k_0\pi - \sigma)h + b = 0$$

and in the second case it will satisfy the equation

$$2k_0\pi - \sigma = 0.$$

This completes the proof of the lemma.[1]

The following lemma will be proved.

Lemma 2. *If in the region* $(1) + \cdots + (m)$ *the set*

$$Q_1(x), \ldots Q_\Gamma(x)$$

has no point of division then this set has a proper curve F in (m) or further to the left (Cf. Def. 7; § 1).

Proof. This is obviously true, in any case when the limiting direction of the upper boundary of $(1) + \cdots + (m)$ is not that of the negative axis of reals. Accordingly we assume that the upper boundaries of the regions $(1), \ldots (m)$,

$$B^1, B^2, \ldots B^m,$$

each have limiting directions coinciding with that of the negative axis of reals.

By hypothesis, if the set of functions $Q(x)$ is separated into two groups

$$(6) \qquad Q_1(x), \ldots Q_s(x); \quad Q_{s+1}(x), \ldots Q_\Gamma(x)$$

there exists at least one member of the first group and at least one member of the second group which interchange order in $(1) + \cdots + (m)$. This statement will hold true for $s = 1, 2, \ldots \Gamma - 1$. Consider (6) with $s = 1$. Let $Q_{k_1^1}(x)$ be the member with the least subscript which interchanges order with $Q_1(x)$. This would necessitate existence, in $(1) + \cdots + (m)$, of a B'_{1, k_1^1} curve. Obviously there

[1] It is sufficiently obvious that existence of an order k_0 along F_Γ insures the desired result.

could be no $B'_{ij} (j = 2, \dots k_1^1 - 1)$ curves in $(1) + \cdots + (m)$, but necessarily curves

$$B'_{2, k_1^1},\ B'_{3, k_1^1},\ \dots\ B'_{k_1^1 - 1, k_1^1}$$

exist in $(1) + \cdots + (m)$. On applying Lemma (1) it follows then that the functions $Q_{1, k_1^1}(x)$, $Q_{2, k_1^1}(x)$, $\dots$ $Q_{k_1^1 - 1, k_1^1}(x)$ are proper on and above a curve F in (m). Noting that

$$Q_{ij}(x) = Q_{i1}(x) + Q_{1j}(x) \qquad (1 \leqq i, j \leqq k_1^1)$$

we conclude that the functions $Q_{ij}(x) (1 \leqq i, j \leqq k_1^1)$ are all proper on and above F. If there are any other $Q(x)$'s changing order with $Q_1(x)$ let

$$Q_{k_2^1}(x) \ (k_2^1 > k_1^1)$$

be the member, lying nearest to $Q_{k_1^1}(x)$, which has this property. It follows that there is a B'_{1, k_2^1} curve in $(1) + \cdots + (m)$, and that, in $(1) + \cdots + (m)$, there can be no curves

$$B'_{1, k_1^1 + j} \ \ (j = 1, \dots k_2^1 - k_1^1 - 1).$$

Hence necessarily the following curves would exist in $(1) + \cdots + (m)$

$$B'_{k_1^1 + 1, k_2^1}, \dots B'_{k_2^1 - 1, k_2} \text{ and } B'_{1, k_2^1}.$$

By Lemma (1) the functions

$$Q_{k_1^1 + 1, k_2^1}, \dots Q_{k_2^1 - 1, k_2^1} \text{ and } Q_{1, k_2^1}(x)$$

are proper on and above a curve F in (m) (F will now stand for the upper one of the two F-curves so far considered; we proceed in this fashion in each of the consecutive steps). Now

$$Q_{ij}(x) = Q_{i, k_2^1}(x) + Q_{k_2^1, j}(x)$$

$$(k_1^1 + 1 \leqq i, j \leqq k_2^1)$$

so that the $Q_{ij}(x) (k_1^1 + 1 \leqq i, j \leqq k_2^1)$ are proper on and above F. We note, further, that

$$Q_{ij}(x) = Q_{i, k_1^1 + \sigma}(x) = Q_{i, k_1^1}(x) + Q_{k_1^1, 1}(x) +$$

$$Q_{1, k_2^1}(x) + Q_{k_2^1, k_1^1 + \sigma}(x) \ \ (1 \leqq i \leqq k_1^1;\ k_1^1 + 1 \leqq j \leqq k_2^1;\ 1 \leqq \sigma \leqq k_2^1 - k_1^1).$$

Here the terms of the last member are proper on and above F so that the above $Q_{ij}(x)$ have the same property. In conjunction with the preceding it follows that the

$$Q_{ij}(x) \quad (1 \leqq i,\ j \leqq k_2^1)$$

are proper above F.

Suppose that $Q_{k_1^1}(x),\ Q_{k_2^1}(x), \ldots Q_{k_{j_1}^1}(x)\,(j_1 \geqq 1)$

$$(2 = k_1^1 < k_2^1 \cdots < k_{j_1}^1)$$

are all the $Q(x)$'s which change order with $Q_1(x)$ in $(1) + \cdots + (m)$. By the reasoning of the kind just employed we can demonstrate that the $Q_{ij}(x)\,(1 \leqq i,$ $j \leqq k_{j_1}^1)$ are proper above a curve F lying in (m).

Consider (6) with $s = k_{j_1}^1$. Let $Q_{\delta_2}(x)\,(\delta_2 > 1)$ be the $Q(x)$ of the first group which has the least subscript and changes order, in $(1) + \cdots + (m)$, with at least one of the set of $Q(x)$'s of the second group. Such a subscript $\delta_2\,(1 < \delta_2 \leqq k_{j_1}^1)$ certainly exists. Let all the $Q(x)$'s of the second group, having this property, be

$$Q_{k_1^2}(x),\ Q_{k_2^2}(x), \ldots Q_{k_{j_2}^2}(x)\,(j_2 \geqq 1)$$
$$(k_{j_1}^1 + 1 \leqq k_1^2 < k_2^2 < \cdots < k_{j_2}^2).$$

By the reasoning already employed and using the proved fact that the

$$Q_{ij}(x) \qquad (1 \leqq i,\ j \leqq k_{j_1}^1)$$

are proper above F, we conclude that the functions

$$Q_{ij}(x) \qquad (1 \leqq i,\ j \leqq k_{j_2}^2)$$

are proper above a curve F in (m). Unless $k_{j_2}^2 = \Gamma$, when the desired result is achieved, we consider (6) again, with $s = k_{j_2}^2$ and continue the process as specified above (we have $\delta_3 > \delta_2 > \delta_1 = 1$). The proof of the lemma can be completed by induction and *is seen to be applicable also when the B' curves are not 'simple'*.

Another lemma will be essential for the purposes at hand.

Lemma 3. *If the set*

$$Q_1(x), \ldots Q_n(x) \tag{7}$$

has no point of division in $(1) + \cdots + (m+1)$ but in $(1) + \cdots + (m)$ there is a point of division, then this set has a proper curve F in (m).

Proof. Suppose that the point of division in (1) + ⋯ + (m) is between the Γ-th and $\Gamma + 1$-st members of the above set. Since the set (7) has no point of division in $(1) + \cdots + (m + 1)$ it follows that B^m, the right boundary of (m), is a $B'_{r,\varrho}$ curve with, say, $r \leqq \Gamma$ and, necessarily, $\varrho \geqq \Gamma + 1$. In fact, if we had $r, \varrho \leqq \Gamma$ or $r, \varrho \geqq \Gamma + 1$ the set (7) would have a point of division (between the Γ-th and $\Gamma + 1$-st members) in $(1) + \cdots + (m + 1)$. Now the set

(7 a) $$Q_1(x), \; \ldots \; Q_\Gamma(x)$$

has no point of division in $(1) + \cdots + (m)$, the same being true for the set

(7 b) $$Q_{\Gamma+1}(x), \; \ldots \; Q_n(x).$$

The truth of this statement follows from the fact that B^m is $B'_{r,\varrho}$ $(r \leqq \Gamma;$ $\varrho \geqq \Gamma + 1)$ so that a point of division, in $(1) + \cdots + (m)$, of either one of the sets (7 a), (7 b) would imply a corresponding point of division for the set (7) in $(1) + \cdots + (m + 1)$; the latter situation, however, had been excluded by hypothesis. Hence, by Lemma 2, the sets (7 a) and (7 b) are each proper to the right of a curve F lying in (m). In other words, we have the functions

(8) $$Q_{ij}(x) \qquad (1 \leqq i, j \leqq \Gamma)$$

proper to the right of F (in (m)), and the functions

(8 a) $$Q_{ij}(x) \qquad (\Gamma + 1 \leqq i, j \leqq n)$$

also proper to the right of this curve.

Consider

(8 b) $$Q_{ij}(x) = Q_{i,\Gamma+\sigma}(x)$$
$$(1 \leqq i \leqq \Gamma; \quad \Gamma + 1 \leqq j \leqq n; \quad 1 \leqq \sigma \leqq n - \Gamma).$$

Any one of the functions (8 b) could be written in the form

$$Q_{ij}(x) = Q_{i,\Gamma+\sigma}(x) = Q_{i,r}(x) + Q_{r,\varrho}(x) + Q_{\varrho,\Gamma+\sigma}(x)$$
$$(i, r \leqq \Gamma; \; \varrho, \Gamma + \sigma \geqq \Gamma + 1).$$

Now $Q_{i,r}(x)$ is a function of (8) and $Q_{\varrho,\Gamma+\sigma}(x)$ is a function of (8 a); these two functions are proper to the right of F, in (m). On the other hand $Q_{r,\varrho}(x)$ corresponds to B^m, the right boundary of (m). Thus on applying, if necessary, Lemma 1 we conclude that $Q_{r,\varrho}(x)$ is proper to the right of a curve F lying in (m). Consequently the functions given by (8 b) have the same property.

Consider

$$Q_{ij}(x) \qquad (1 \leqq i, j \leqq n). \tag{9}$$

Any particular $Q_{ij}(x)$ of the set (9) belongs to one of the sets (8), (8 a), (8 b)[1]; hence the functions (9) are all proper to the right of a curve F, in (m). This completes the proof of the lemma. *By a similar, though slightly more complicated reasoning, we show that the Lemma is true also when B' curves which are not 'simple' are admitted.*

Proper curves will be seen to be important since, as will be shown later, along paths lying in proper regions bounded by such curves, certain summations are possible. Such curves (and regions) are also essential in demonstrating that the periodic functions connecting certain sets of solutions are proper (Cf. Def. 5; § 1).

§ 3. Lemmas on Iteration.

Consider the quadrant Γ and the consecutive regions

$$(1),\ (2), \ldots (m), \ldots$$

which are separated by B' curves, B^1, B^2, $\ldots B^m$, $\ldots$ and constitute Γ (see § 2). On the other hand, there is a quadrant Ω of a similar kind lying below the negative axis of reals and having for its upper boundary, (h^1), a portion of a line $v = c > 0$. The consecutive regions of Ω separated by B' curves will be denoted by

$$[1],\ [2], \ldots [m], \ldots;$$

here [1] will be the region having (h^1) for its upper boundary. In some of the following sections we propose to envisage a process of iteration from the left (or equally from the right of course). Such a process will be first applied to region of type (1) or of type (1) + [2].

As seen from § 2 the upper boundary of (1) extends indefinitely upwards while the lower boundary of [1] will extend indefinitely downwards. The negative axis of reals may be a B' curve or it may not. In the latter case a region of type (1) + [1] is suitable for iteration. Regions like (1) or (1) + [2] will be

[1] $Q_{ij}(x)$ and $Q_{ji}(x) = -Q_{ij}(x)$ are both considered as belonging to the same set.

said to be of type K. More generally, K will denote a region either having the negative axis of reals for a part of its boundary or containing this axis; such a region will be allowed to extend to the right but always excepting a neighbourhood of the positive axis[1] bounded, say, by curves of the form $v = \pm hu^e (h, e > 0)$.

The following lemma will be proved.

Lemma 4. *Suppose that in a region K (that is, a region of type K) we have*

$$(1) \qquad \Re Q_1'(x) \geqq \Re Q_j'(x) \qquad (j = 2, \ldots n)$$

the set $Q_j(x)$ $(j = 1, \ldots n)$ being associated with a difference system (1) *or* (1 a); (§ 1). *Let the matrix of formal solutions of this system be denoted by*

$$(1\text{ a}) \qquad S(x) = (e^{Q_j(x)} s_{ij}(x))$$

(here the series $s_{ij}(x)$ are s-series (Def. 1; § 1)) *and let $T(x)$ be a matrix,*

$$(1\text{ b}) \qquad T(x) = (e^{Q_j(x)} t_{ij}(x)),$$

in which $t_{ij}(x) (i, j = 1, \ldots n)$ denotes $s_{ij}(x)$ with the power series terminated after k terms (k being sufficiently great). Form the matrix

$$(2) \qquad Y^r(x) = (y_{ij}^r(x)) = A(x-1) \ldots A(x-r)\, T(x-r).$$

The following holds true. For x in K the limits

$$(2\text{ a}) \qquad \lim_{r \to \infty} y_{i1}^r(x) = y_{i1}(x) \qquad (i = 1, \ldots n),$$

exist, are independent of k, are analytic and are the elements of a solution of the system $Y(x+1) = A(x)\, Y(x)$. Moreover, in K,

$$(3) \qquad y_{i1}(x) \sim e^{Q_1(x)} s_{i1}(x) \qquad (i = 1, \ldots n).$$ [2]

Proof. The matrix $Y^r(x)$ can be expressed as follows

$$(4) \qquad Y^r(x) = T(x)\, \overline{Y}^r(x) \quad (\overline{Y}^r(x) = (\bar{y}_{ij}^r(x)),$$

$$\overline{Y}^r(x) = \prod_{i=1}^{r} T^{-1}(x-i+1)\; A(x-i)\; T(x-i).$$

---

[1] That is, when working from the left.

[2] Relations like (3) are to be construed as denoting asymptotic relationship with respect to the power series factors.

Now define a matrix $B(x) = (b_{ij}(x))$ by the relation

$$B(x) = T(x + 1)T^{-1}(x).$$

On the other hand, we have formally

$$A(x) = (a_{ij}(x)) = S(x + 1)S^{-1}(x);$$

here the $a_{ij}(x)$ are in K of the form specified in the beginning of § 1. By a direct computation[1] we show that

$$(5) \qquad A(x) - B(x) = \frac{1}{x^{k_1}} H(x) \quad (H(x) = (h_{ij}(x))$$

where $|h_{ij}(x)| \leqq h$, for x in K, while $k_1 = \frac{k}{p} - d_1$ $(d_1 \geqq 0;\ k_1 \to \infty$ as $k \to \infty)$. Now $|A(x)| \not\equiv 0$ so that $|B(x)| \not\equiv 0$; thus, writing

$$(5\text{ a}) \qquad A(x) = B(x)[I + N(x)]$$

$$(I = (\delta_{ij}) = \text{identity matrix}),$$

we have

$$N(x) = \frac{1}{x^{k_1}} B^{-1}(x) H(x) \quad (B^{-1}(x) = (\bar{b}_{ij}(x)).$$

The $\bar{b}_{ij}(x)$, if not bounded (for $|x| > p > 0$), are infinite at $x = \infty$ to finite order. Hence

$$(5\text{ b}) \qquad N(x) = \frac{1}{x^{k_2}} C(x) \quad (C(x) = (c_{ij}(x)))$$

$\left(k_2 = \frac{k}{p} - d_2;\ d_2 \geqq 0;\ k_2 \to \infty \text{ as } k \to \infty\right)$; here, for x in K, $|c_{ij}(x)| \leqq c$. Consider now the product

$$T^{-1}(x + 1)A(x)T(x) = T^{-1}(x)B^{-1}(x)A(x)T(x)$$

$$= T^{-1}(x)\left(I + \frac{1}{x^{k_2}} C(x)\right)T(x) = I + \frac{1}{x^{k_2}} T^{-1}(x)C(x)T(x).$$

Writing

$$T^{-1}(x) = (e^{-Q_i(x)}\bar{t}_{ij}(x))$$

$$T^{-1}(x)C(x)T(x) = (e^{Q_j i(x)}\Gamma_{ij}(x))$$

[1] Compare with the analogous procedure in (II).

where

$$\Gamma_{ij}(x) = \sum_{\lambda_1, \lambda_2=1}^{n} \bar{t}_{i\lambda_1}(x) c_{\lambda_1\lambda_2}(x) t_{\lambda_2 j}(x).$$

The $\Gamma_{ij}(x)$, if not bounded, are infinite at $x = \infty$ (in K) to finite order. Consequently

$$T^{-1}(x+1)A(x)T(x) = I + \frac{1}{x^b}H(x), \tag{6}$$

$$H(x) = (H_{ij}(x)) = (e^{Q_{ji}(x)} h_{ij}(x))$$

$(|h_{ij}(x)| \leqq h$ in K; $b = \frac{k}{p} - d$; $d \geqq 0$; $b \to \infty$ as $k \to \infty)$.

By (4) and (6)

$$\bar{Y}^r(x) = \left(I + \frac{1}{(x-1)^b}H(x-1)\right)\dots\left(I + \frac{1}{(x-r)^b}H(x-r)\right) \tag{7}$$

$$= I + \sum_{k_1}^{r} \frac{H(x-k_1)}{(x-k_1)^b} + \sum_{k_1<k_2}^{r} \frac{H(x-k_1)H(x-k_2)}{(x-k_1)^b(x-k_2)^b} + \cdots$$

$$+ \sum_{k_1<\dots<k_s}^{r} \frac{H(x-k_1)\dots H(x-k_s)}{(x-k_1)^b\dots(x-k_s)^b} + \cdots$$

so that

$$\bar{y}^r_{i1}(x) = \delta_{i1} + \sum_{k_1}^{r} \frac{H_{i1}(x-k_1)}{(x-k_1)^b} + \cdots \tag{7 a}$$

$$+ \sum_{k_1<\dots<k_s}^{r} \frac{1}{(x-k_1)^b\dots(x-k_s)^b} \sum_{\lambda_1\dots\lambda_{s-1}=1}^{n} H_{i\lambda_1}(x-k_1)\dots H_{\lambda_{s-1}1}(x-k_s) + \cdots$$

$$= \delta_{i1} + \sum_{k_1}^{r} e^{Q_{1i}(x-k_1)} \frac{h_{i1}(x-k_1)}{(x-k_1)^b} + \cdots$$

$$+ \sum_{k_1<\dots<k_s}^{r} \frac{1}{(x-k_1)^b\dots(x-k_s)^b} \cdot$$

$$\cdot \sum_{\lambda_1\dots\lambda_{s-1}=1}^{n} e^{G^{k_1\dots k_s}_{\lambda_1\dots\lambda_{s-1}}(x)} h_{i\lambda_1}(x-k_1) h_{\lambda_1\lambda_2}(x-k_2)\dots h_{\lambda_{s-1}1}(x-k_s) + \cdots$$

Take k sufficiently great so that $b > 2$; then the series (7 a) will converge uniformly to a function analytic in K (for $i = 1, \ldots n$) provided that the functions $G^{k_1 \ldots k_s}_{\lambda_1 \ldots \lambda_{s-1}}(x)$ have a non-positive or limited positive real part. Now such a function is expressible as follows

$$\begin{aligned} G^{k_1 \ldots k_s}_{\lambda_1 \ldots \lambda_{s-1}}(x) &= Q_{\lambda_1 i}(x-k_1) + Q_{\lambda_2 \lambda_1}(x-k_2) + \cdots \\ &\quad + Q_{\lambda_{s-1} \lambda_{s-2}}(x-k_{s-1}) + Q_{1 \lambda_{s-1}}(x-k_s) \\ &= Q_{1i}(x) + (Q_{i1}(x) - Q_{i1}(x-k_1)) + (Q_{\lambda_1 1}(x-k_1) - Q_{\lambda_1 1}(x-k_2)) \\ &\quad + \cdots + (Q_{\lambda_{s-1} 1}(x-k_{s-1}) - Q_{\lambda_{s-1} 1}(x-k_s)) \\ &= Q_{1i}(x) + \int\limits_{x-k_1}^{x} Q'_{i1}(x)\,dx + \int\limits_{x-k_2}^{x-k_1} Q'_{\lambda_1 1}(x)\,dx + \cdots + \int\limits_{x-k_s}^{x-k_{s-1}} Q'_{\lambda_{s-1} 1}(x)\,dx . \end{aligned}$$

If $\Im x_1 = \Im x$, $\Re x_1 < \Re x$, then with path of integration along the straight line goining x_1 and x we have

$$\Re \int\limits_{x_1}^{x} Q'(x)\,dx = \int\limits_{x_1}^{x} \Re\, Q'(x)\,dx \leqq 0$$

inasmuch as $\Re\, Q'(x) = \Re\, Q'_{s,1}(x) \leqq 0$. The latter inequality holds, in K, for $s = 1, \ldots n$. Hence, in K,

$$\Re\, G^{k_1 \ldots k_s}_{\lambda_1 \ldots \lambda_{s-1}}(x) \leqq \Re\, Q_{1i}(x). \tag{7 b}$$

Using the fact that $|h_{ij}(x)| \leqq h$ (in K) we conclude that the limits

$$\lim_{r \to \infty} \bar{y}^r_{i1}(x) = \bar{y}_{i1}(x)$$

exist. In fact

$$\begin{aligned} |\bar{y}_{i1}(x) - \delta_{i1}| &\leqq \frac{1}{n} e^{\Re Q_{1i}(x)} \left(\sum_{k_1} \frac{nh}{|x-k_1|^b} + \cdots + \sum_{k_1 \ldots k_s} \frac{(nh)^s}{|x-k_1|^b \ldots |x-k_s|^b} + \cdots \right) \\ &= \frac{1}{n} e^{\Re Q_{1i}(x)} \left[\left(\left(1 + \frac{nh}{|x-1|^b} \right) \left(1 + \frac{nh}{|x-2|^b} \right) \cdots \right) - 1 \right]. \end{aligned}$$

Moreover, using formulas (12), (13) of (II; p. 248), we have, for x in K,

$$\bar{y}_{i1}(x) = \delta_{i1} + \frac{e^{Q_{1i}(x)} m_i(x)}{x^{b-1}}$$

$$(|m_i(x)| \leqq m;\ u \leqq a;\ a, \text{ any fixed number}),$$

and

$$\bar{y}_{i1}(x) = \delta_{i1} + \frac{e^{Q_{1i}(x)} m_i^1(x)}{v^{b-1}}$$

$$(|m_i^1(x)| \leqq m^1;\ \Im x = v;\ u \geqq a;\ |v| \geqq d)$$

where c is a sufficiently great positive number. Using (4) we get, for x in K,

$$y_{i1}(x) = \lim\ y_{i1}^r(x) = \sum_{\lambda=1}^{n} e^{Q_\lambda(x)} (t_{i\lambda}(x) (\lim\ \bar{y}_{\lambda 1}^r(x))$$

$$= \sum_{\lambda=1}^{n} e^{Q_\lambda(x)} t_{i\lambda}(x) \left(\delta_{\lambda 1} + \frac{e^{Q_{1\lambda}(x)}\ m_\lambda(x)}{x^{b-1}}\right), \qquad (u \leqq a);$$

$$y_{i1}(x) = \sum_{\lambda=1}^{n} e^{Q_\lambda(x)} t_{i\lambda}(x) \left(\delta_{\lambda 1} + \frac{e^{Q_{1\lambda}(x)}\ m_\lambda^1(x)}{v^{b-1}}\right), \qquad (u \geqq a).$$

Thus we have for x in K and for $i = 1, \ldots n$

(8)
$$y_{i1}(x) = e^{Q_1(x)} \left(t_{i1}(x) + \frac{\eta_i(x)}{x^{b-1}}\right)$$

$$(|\eta_i(x)| \leqq \eta;\quad u \leqq a),$$

$$y_{i1}(x) = e^{Q_1(x)} \left(t_{i1}(x) + \frac{\eta_i^1(x)}{v^{b-1}}\right)$$

$$(|\eta_i^1(x)| \leqq \eta^1;\quad u \geqq a).$$

Noting that if the functions $\eta_i(x)$, $\eta_i^1(x)$ are not bounded they are infinite to finite order at infinity (in the two regions, respectively), we have the two formulas, valid in K,

(8 a)
$$y_{i1}(x) = e^{Q_1(x)} \left(t_{i1}(x) + \frac{\sigma_i(x)}{x^c}\right)$$

$$(|\sigma_i(x)| \leqq \sigma;\quad u \leqq a),$$

(8 b)
$$y_{i1}(x) = e^{Q_1(x)} \left(t_{i1}(x) + \left(\frac{x}{v}\right)^\beta \frac{\sigma_i^1(x)}{x^c}\right)$$

$$(|\sigma_i^1(x)| \leqq \sigma^1;\quad u \geqq a).$$

In the above $c = \frac{k}{p} - d^1$, $d^1 \geqq 0$ and $c \to \infty$ as $k \to \infty$ and β is the maximum order to which the $\eta_i(x)$, $\eta_i^1(x)$ may be infinite at infinity. Now, as specified previously, the region K is not allowed to contain a neighbourhood of the positive axis of reals bounded by curves of the form

$$v = \pm\, h u^e (h,\ e > 0).$$

Consequently for x in K, in a right half plane,

$$\left|\frac{x}{v}\right| = \sqrt{1 + \frac{u^2}{v^2}} \leqq \sqrt{1 + \frac{u^2}{h^2 u^{2e}}} = \frac{u^{1-e}}{h}(1 + \cdots).$$

Thus $\left|\frac{x}{v}\right|$ may become infinite along a path to infinity but not faster than $|x|^{1-e}$ where $1 - e < 1$. It follows then that a relation of the form (8 a) will hold throughout K with the constant c possibly smaller near the right boundary of K. However, for the whole region K, the relation (8 a) will hold with $c \to \infty$ as $k \to \infty$. That is, by taking h sufficiently great c can be made arbitrarily great.

The $y_{i1}(x)$ are the constituent elements of a solution of the system $Y(x+1) = A(x)\,Y(x)$. Moreover, the limits are independent of k. This can be demonstrated by the reasoning of the kind employed for an analogous purpose in (II). This completes the proof of the lemma.

The above lemma is concerning determinant limits of first order. We shall now consider determinant limits of higher order. Determinant limits of various orders have been previously used by Birkhoff in the paper (II). In this connection the following facts, needed for the purposes at hand, will be stated. With the system (1 a) (§ 1) there is associated a difference system of order C_k^n $(k = 2, \ldots n)$,

(9) $$Y_k(x + 1) = A_k(x)\, Y_k(x)$$

where

(9 a) $$A_k(x) = (a_{i_1 \ldots i_k;\, j_1 \ldots j_k}(x))$$

$$(i_1, \ldots i_k,\ j_1, \ldots j_k = 1, \ldots n;\ i_1 < i_2 < \cdots < i_k;\ j_1 < j_2 < \cdots < j_k).$$

The multiple subscripts in the above and in what follows are to be construed in the sense made apparent by the relationship

(10)
$$h_{i_1\ldots i_k;\, j_1\ldots j_k} = \begin{vmatrix} h_{i_1j_1} & h_{i_1j_2} & \ldots & h_{i_1j_k} \\ h_{i_2j_1} & h_{i_2j_2} & \ldots & h_{i_2j_k} \\ \ldots & \ldots & \ldots & \ldots \\ h_{i_kj_1} & h_{i_kj_2} & \ldots & h_{i_kj_k} \end{vmatrix},$$

while in the matrix (of order C_k^n)

(10 a)
$$(h_{i_1\ldots i_k;\, j_1\ldots j_k})$$

the set of subscripts $(i_1 \ldots i_k)$ refers to a row and the set of subscripts $(j_1 \ldots j_k)$ refers to a column of the matrix. The difference system (9) possesses a formal matrix solution

(11)
$$S_k(x) = (e^{Q_{j_1}(x)+\cdots+Q_{j_k}(x)} s_{i_1\ldots i_k;\, j_1\ldots j_k}(x)).$$

The formal series $s_{i_1\ldots i_k;\, j_1\ldots j_k}(x)$ are s-series (Def. 1; § 1) and linearly independent. If $(y_{ij}(x))$ is a matrix solution of $Y(x+1) = A(x)\,Y(x)$ then

$$Y_k(x) = (y_{i_1\ldots i_k;\, j_1\ldots j_k}(x))$$

will constitute a matrix solution of (9).

With the above in view we shall state the following lemma.

Lemma 5. *Suppose that in a region K the coefficients of a system* (1 a) (§ 1) *are known. Assume, moreover, that in K, for all $j_1, \ldots j_k \leqq n$ with $j_1 < \cdots < j_k$,*

(12)
$$\Re[Q_1'(x) + \cdots + Q_k'(x)] \geqq \Re[Q_{j_1}'(x) + \cdots + Q_{j_k}'(x)].$$

The functions $Q_1(x), \ldots Q_k(x)$ are to be considered as associated with the first, second, ... and k-th columns of $S(x)$, respectively. Form the determinants

$$y^r_{i_1\ldots i_k;\, 1\ldots k}(x)$$

by means of the elements $y^r_{ij}(x)$ of the matrix $Y^r(x)$, defined by (2).

The following is true. For x in K the determinant limits

(12 a)
$$\lim_{r\to\infty} y^r_{i_1\ldots i_k;\, 1\ldots k}(x) = y_{i_1\ldots i_k;\, 1\ldots k}(x)$$
$$(i_1, \ldots i_k = 1, \ldots n)$$

exist, are independent of the number of terms retained in the power series factors of the $t_{ij}(x)$ *and they are the constituent elements of a solution of the system* $Y_k(x+1) = A_k(x)\,Y_k(x)$ ((9), (9 a)). *Moreover, in* K,

$$y_{i_1 \ldots i_k;\, 1 \ldots k}(x) \sim e^{Q_1(x) + \cdots + Q_k(x)} s_{i_1 \ldots i_k;\, 1 \ldots k}(x) \qquad (12\,\text{b})$$

$$(i_1, \ldots i_k = 1, \ldots n).$$

Proof. By Lemma 4 this lemma is true for $k = 1$. When $k = 2$, as can be seen from (II; pp. 253—254), there is the following situation. With reference to the system (9), formed for $k = 2$, consider the product

$$Y_2^r(x) = (y_{2;\, ij}^r(x)) = A_2(x-1) \ldots A_2(x-r)\, T_2(x-r) \qquad (13)$$

$$= (y_{i_1 i_2;\, j_1 j_2}^r(x)) \qquad (i, j = 1, \ldots C_2^n;\ i_1, \ldots j_2 = 1, \ldots n)$$

where $T_2(x)$ is $S_2(x)$ with the power series factors terminated after a sufficiently great number of terms. Let the columns be so ordered that the function $Q(x)$ of the first column of $S_2(x)$ is $Q_1(x) + Q_2(x)$. With (12) assumed in K for $k = 2$, by Lemma 4 it would follow that the limits of the elements in the first column of (13) exists in K. Moreover, these limits will satisfy all other properties specified, in the lemma, for the determinant limits (of order two). Now by the reasoning precisely of the kind employed in (II; p. 254) it follows that the elements in the first column[1] of $Y_2^r(x)\,(r = 1, 2, \ldots)$ are correspondingly identical with the determinants

$$y_{i_1 i_2;\, 1\, 2}^r(x) = \begin{vmatrix} y_{i_1 1}^r(x), & y_{i_1 2}^r(x) \\ y_{i_2 1}^r(x), & y_{i_2 2}^r(x) \end{vmatrix}.$$

The cases $k = 3, \ldots n$ can be treated in a similar way.

We shall consider now a region R bounded on the left by a curve with a limiting direction at infinity, and extending indefinitely upwards (or downwards) while to the right such a region will be allowed to extend at most up to a curve of the form

$$v = \pm\, h u^e \qquad (h, e > 0)$$

(In general both boundaries of R will be B' curves). Iterations for regions of this type will be specified by the following lemma.

[1] The column specified by the pair of subscripts $(j_1 j_2) = (1\ 2)$.

Lemma 6. *Let V denote a strip of unit width situated immediately to the right of the left boundary of R (a region specified above). Suppose that we know in V a set of $s\,(n \geqq s \geqq 1)$ solutions of the system (1 a) (§ 1),*

$$y_{ij}(x) \qquad (i = 1, \ldots n;\ j = 1, \ldots s)$$

which satisfy in V the asymptotic relations

$$y_{ij}(x) \sim e^{Q_j(x)} s_{ij}(x)$$

and are analytic in V. Moreover, assume that in R

(14) $$\Re\, Q_1'(x) \geqq \Re\, Q_j'(x) \qquad (j = 2, \ldots n)$$

and that in the first column of $S(x)$ (and of $T(x)$) we have $Q(x) = Q_1(x)$.

Letting $T^1(x)$ denote $T(x)$, with the first s columns correspondingly replaced by the $y_{ij}(x)$ $(i = 1, \ldots n;\ j = 1, \ldots s)$, and defining the matrix $Y^{r_x}(x)$ by the product

(15) $$Y^{r_x}(x) = (y^{r_x}_{ij}(x)) = A(x-1)\ldots A(x-r_x)\,T^1(x-r_x)$$
$$(x - r_x \text{ in } V;\ A(x) \text{ known in } R)$$

the following can be asserted.

The $y^{r_x}_{ij}(x)$ $(i = 1, \ldots n;\ j = 1, \ldots s)$ are s solutions in R, and constitute analytic extensions to R of the s solutions originally assumed as known in V. The asymptotic form of the elements of the first one of these solutions will be

(15 a) $$y^{r_x}_{i1}(x) = y_{i1}(x) \sim e^{Q_1(x)} s_{i1}(x)$$
$$(i = 1, \ldots n;\ x \text{ in } R).$$

Proof. It is observed that

$$Y^{r_x}(x) = T(x)\,\overline{Y}^{r_x}(x)$$

where

$$\overline{Y}^{r_x}(x) = \left[\prod_{i=1}^{r_x-1} T^{-1}(x-i+1)A(x-i)\,T(x-i)\right]\cdot$$
$$\cdot\,(T^{-1}(x-r_x+1)A(x-r_x)\,T^1(x-r_x)).$$

Here the expression for $\overline{Y}^{r_x}(x)$ differs from that given before for $\overline{Y}^{r}(x)$ in the last factor, and that just in the power series factors, by arbitrarily great powers of

$x^{-\frac{1}{p}}$. This is due to the fact that, as a consequence of the conditions of the lemma,

$$T^1(x) \sim S(x).$$

Moreover, for every fixed x we have r_x finite, so that there is no necessity for passing to the limit. Making direct use of the matrix equation (1 a) (§ 1) it is immediately obvious that the first s columns of $Y^{r_x}(x)$ are analytic extensions of the solutions whose asymptotic forms have been assumed in V. Using (14) and applying the reasoning of the kind employed in proving Lemma 4 we derive the relations (15 a) thereby establishing the lemma.

For determinant limits we have, by application of Lemma 6, the following result.

Lemma 7. *Let R and V have the meaning indicated in Lemma 6. Suppose that we know in V a set of s ($C_k^n \geqq s \geqq 1$) solutions of the system* (9),

(16) $$y_{i_1 \ldots i_k;\, j_1 \ldots j_k}(x)$$
$$(i_1, \ldots i_k = 1, \ldots n;\ s \text{ sets } (j_1 \ldots j_k))$$

which satisfy in V the asymptotic relations

$$y_{i_1 \ldots i_k;\, j_1 \ldots j_k}(x) \sim e^{Q_{j_1}(x) + \cdots + Q_{j_k}(x)} s_{i_1 \ldots i_k;\, j_1 \ldots j_k}(x)$$

and are analytic in V. Assume that in R

(16 a) $$\Re[Q_1'(x) + \cdots + Q_k'(x)] \geqq \Re[Q_{j_1}'(x) + \cdots + Q_{j_k}'(x)]$$
$$(j_1 < \cdots < j_k = 1, \ldots n)$$

and that in the first column of $S_k(x)$ (11) (and of $T_k(x)$) we have $Q(x) = Q_1(x) + \cdots + Q_k(x)$.

Let $T_k^1(x)$ denote $T_k(x)$, with the first s columns (corresponding to the s sets $(j_1^1 \ldots j_k^1)$ in (16)) replaced by the elements of (16), respectively. Define the matrix $Y_k^{r_x}(x)$ by the product

(17) $$Y_k^{r_x}(x) = A_k(x-1) \ldots A_k(x - r_x) T_k^1(x - r_x)$$
$$= (y^{r_x}_{i_1 \ldots i_k;\, j_1 \ldots j_k}(x))$$
$$(x - r_x \text{ in } V;\ A(x) \text{ known in } R).$$

The functions $y^{rx}_{i_1\ldots i_k;\, j_1\ldots j_k}(x)$ $(i_1, \ldots i_k = 1, \ldots n$; *the s sets* $(j_1 \ldots j_k)$ *of* (16)) *will be constituent elements of s solutions, in* R, *of the system* (9) *and will represent analytic extensions to* R *of solutions* (16) *originally assumed as known in* V. *The elements of the first one of these solutions will have in* R *the asymptotic form*

(17 a)
$$y^{rx}_{i_1\ldots i_k;\, 1\ldots k}(x) = y_{i_1\ldots i_k;\, 1\ldots k}(x)$$
$$\sim e^{Q_1(x) + \cdots + Q_k(x)} s_{i_1\ldots i_k;\, 1\ldots k}(x)$$
$$(i_1 < \cdots < i_k = 1, \ldots n).$$

Theorems entirely analogous to those of this section may be formulated when we work from the right instead of from the left. In this case a cut is made along the negative axis of reals to fix the determination of $S(x)$ and we consider the symbolic product

$$Y^r_*(x) = A^{-1}(x)A^{-1}(x+1)\ldots A^{-1}(x+r-1)\,T(x+r)$$

instead of $Y^r(x)$. In this case we exclude the neighborhood of the negative axis of reals bounded by curves of the form

$$v = \pm\, h(-u)^e \qquad (h, e > 0).$$

§ 4. A Lemma on Summation.

We shall now establish a modification of the method of contour summation used in (II).

Let R denote a region the left boundary of which is either h (the lower boundary of Γ (§ 1)) or a curve, extending indefinitely upwards, with a limiting direction at infinity. Let the right boundary of R be a curve, extending indefinitely upwards, with a limiting direction at infinity. This latter boundary, if with the limiting direction of the axis of reals will be assumed to be a curve of the form $v = hu^e$ $(h, e > 0)$. The left boundary of R, if extending upwards and with its limiting direction coincident with that of the negative axis, will be of the form $v = h(-u)^e + \cdots$ $(h > 0;\ 1 > e > 0)$.

The following lemma will be proved.

Lemma 8. *Assume that the function*

$$H(x) = e^{Q(x)} h(x) \tag{1}$$

is analytic in R while $Q(x) = \mu x \log x + \gamma x + \cdots + \nu x^{\frac{1}{p}}$ *is proper on and in the neighborhood of the right boundary of R and*

$$h(x) \sim H(x) \qquad (in\ R) \tag{1 a}$$

where $H(x)$ *is a formal s-series (Def.* 1; § 1). *Furthermore, suppose that*

$$\Re\, Q'(x) \leqq 0 \qquad (in\ R). \tag{1 b}$$

The equation

$$y(x+1) - y(x) = e^{Q(x)} h(x) \tag{2}$$

possesses a solution $y(x)$, *analytic in a region* R' *interior to R by a distance* $\varepsilon (> 0)$, *for which an asymptotic relation,*

$$y(x) \sim e^{Q(x)} s(x), \tag{2 a}$$

where $s(x)$ *is a formal s-series, is maintained in the above region.*

Proof. The formal equation

$$y(x+1) - y(x) = e^{Q(x)} H(x)$$

is formally satisfied by $y(x) = e^{Q(x)} s(x)$ where $s(x)$ is an s-series. This follows from a Lemma proved by Birkhoff in (I; p. 218). Let $t(x)$ denote $s(x)$ with the power series factors terminated after m terms (with m sufficiently great). Substitute in (2)

$$y(x) = e^{Q(x)}\left(t(x) + \frac{z(x)}{x^k}\right) \qquad \left(k = \frac{m}{p}\right). \tag{3}$$

The new variable $z(x)$ will satisfy the equation

$$q(x+1) z(x+1) - q(x) z(x) = \frac{e^{Q(x)} \beta(x)}{x^{k'}} \tag{3 a}$$

$$\left(q(x) = \frac{e^{Q(x)}}{x^k};\ k' \to \infty \ \text{ as } \ k \to \infty\right).$$

Here

$$\frac{e^{Q(x)}\beta(x)}{x^{k'}} = e^{Q(x)}h(x) - \Delta e^{Q(x)}t(x)$$

and $\beta(x)$ is analytic and bounded[1] ($|\beta(x)| \leqq \beta$) in R. To demonstrate the truth of the lemma we need first to show that (3 a) has a solution $z(x)$ analytic in R' and, if not bounded in R', infinite at $x = \infty$ to a finite order $\bar{k}$ which is such that $k - \bar{k}$ approaches infinity with k.

The equation (3 a) with the second member replaced by zero is satisfied by

$$z^0(x) = x^k e^{-Q(x)}.$$

Hence

(3 b) $$z(x) = x^k e^{-Q(x)} \underset{t=x}{\mathfrak{S}} \frac{e^{Q(t)}\beta(t)}{t^{k'}},$$

(3 c) $$\underset{t=x+1}{\mathfrak{S}} \varphi(t) - \underset{t=x}{\mathfrak{S}} \varphi(t) = \varphi(x),$$

will be a solution of (3 a) provided the operation $\underset{t=x}{\mathfrak{S}}$ is suitably specified.

With x in R' let L denote a contour lying interior to R and defined as follows. When $\Re\, Q(x) = 0$ along the negative axis of reals while the lower boundary of R is h (the lower boundary of Γ) then L is to consist of h and of a path L^* near the right boundary of R. In all other cases L is to consist of a path near the right boundary of R. With x not nearly congruent to L (that is, if x' represents the point on L for which $\Im x' = \Im x$ we have $\Re(x - x')$ not an integer) let $x + k_x$ ($k_x \geqq 0$) be the last one of the sequence of points $x, x+1, \ldots$ lying to the left of L. Let l_x denote a loop which contains the points $x, x+1, \ldots, x + k_x$ and passes between $x - 1$ and x and between $x + k_x$ and L. Now, by hypothesis, $Q(x)$ is proper along L (and also at least within a limited distance from L). Hence a least integer λ can be found so that

(4) $$\varrho_\lambda(x) = 2\pi\lambda v + \Re\, Q(x) \to +\infty \qquad (\Im x = v)$$

as $|x| \to \infty$ upwards from the axis of reals along L. If $\mu \neq 0$, L is near the

[1] $\beta(x)$ could be considered to be a function asymptotic in R to a formal s-series whose power series factors begin with low powers of $x^{-\frac{1}{p}}$.

imaginary axis of reals.[1] This integer λ will be unchanged if L is shifted a finite distance in the direction of the axis of reals.

Write, for x in R',

$$\mathop{\mathfrak{S}}_{t=x} \frac{e^{Q(t)}\beta(t)}{t^{k'}} = \int_{L_x} \frac{e^{2\pi\sqrt{-1}\lambda(x-t)+Q(t)}\beta(t)\,dt}{(1-e^{2\pi\sqrt{-1}(x-t)})\,t^{k'}}. \tag{5}$$

Here λ will be supposed to have the value specified above. The path L_x is to consist of L, described upwards (if $L = h + L^*$, then h is described from infinity to the neighborhood of the origin and L^* is described upwards), and of l_x, described in the clockwise direction. When x approaches a position of congruency to L we shift L through a suitable distance. The integral (5) will converge since $\varrho_{\lambda-1}(x)$ remains bounded along L; moreover, it will represent a sum formula in the sense of (3 c), and the function of x given by the second member of (5) will be analytic in R'. It remains to show that this function is such that $z(x)$, as defined by (3 b), has the desired properties.

Let x' be the point on L for which $I_{x'} = I_x$. Denote the portion of L up to that point by L_1 and from that point up by L_2. If $L = h + L^*$, let L_1^* denote the part of L^* up to x' and let L_2^* denote the part of L^* up from x'. For t on L and for x in R' we have

$$|1 - e^{\pm 2\pi\sqrt{-1}(x-t)}| > d > 0. \tag{5 a}$$

The inequality

$$\Re Q(x) \geqq \Re Q(x_1) \tag{6}$$

$$(\Im x = \Im x_1;\ \Re x \leqq \Re x_1;\ x, x_1 \text{ in } R)$$

will be also needed. It is seen to hold, in virtue of (1 b), since we have

$$\Re Q = -\int_x^{x_1} \Re Q'(x)\,dx.$$

With these preliminaries in view consider the integral along l_x,

[1] This is due to the fact that along such a path and, in general, along a path extending upwards with its limiting direction coinciding with that of the positive axis of imaginaries we have $R(x \log x)$ behaving as a constant multiple of $|x|$ (i. e. r).

$$(7)\qquad \int\limits_{l_x} = -\frac{e^{Q(x)}\beta(x)}{x^{k'}} - \frac{e^{Q(x+1)}\beta(x+1)}{(x+1)^{k'}} - \cdots - \frac{e^{Q(x+k_x)}\beta(x+k_x)}{(x+k_x)^{k'}}.$$

In virtue of (6), for x in R',

$$(7\,\mathrm{a})\qquad \left|\int\limits_{l_x}\right| \leq e^{\Re Q(x)}\beta\left(\frac{1}{|x|^{k'}} + \frac{1}{|x+1|^{k'}} + \cdots + \frac{1}{|x+k_x|^{k'}}\right).$$

Now, substituting $z = -x + 1$ in an inequality of Birkhoff[1],

$$\sum_{i=1}^{\infty}\left|\frac{1}{(z-i)^{k'}}\right| < \frac{\pi}{2\,|z|^{k'-1}} \qquad (\Re z < 0),$$

we find that

$$\sum_{i=0}^{\infty}\frac{1}{|x+i|^{k'}} < \frac{1}{2\,|x-1|^{k'-1}} \qquad (\Re x > 1)$$

so that, if R extends to the right of the imaginary axis,

$$(7\,\mathrm{b})\qquad \frac{1}{|x|^{k'}} + \cdots + \frac{1}{|x+k_x|^{k'}} < \frac{h_1}{|x|^{k'-1}}$$

($\Re x = u > 0$; h_1 independent of k').

Let x ($|x| > \varrho > 0$; $u \leqq 0$) and $x + k_x$ be above curves

$$(8)\qquad v = h_1(-u)^{e_1} \qquad (h_1 > 0;\ 1 > e_1 > 0),$$

$$(8\,\mathrm{a})\qquad v = h_2 u^{e_2} \qquad (h_2 > 0;\ 1 > e_2 > 0).$$

Substituting $z = -x + 1$ in the inequality (13) of (II),

$$\sum_{i=1}^{\infty}\left|\frac{1}{(z-i)^{k'}}\right| < \frac{2}{|v|^{k'-1}}\left[\frac{1}{|v|} + \frac{\pi}{2}\right] \qquad (\Re z > 0;\ \Im z = v),$$

it is found that

$$(8\,\mathrm{b})\qquad \frac{1}{|x|^{k'}} + \cdots + \frac{1}{|x+k_x|^{k'}} < \frac{h_1'}{v^{k'-1}} \qquad (u \leqq 0;\ \Im x = v).$$

[1] See formula 12 in (II).

Let $\bar{u}\,(\bar{u}<0)$ be the value of u for the point on the curve (8) whose ordinate is $\Im x = v$. Then, from (8 b), it follows that

$$\frac{h_1'}{v^{k'-1}} < \frac{h_1''(k')}{|\bar{u}|^{e_1(k'-1)}} .$$

Now for a suitable l, independent of x $(u \leqq 0)$,

$$|\bar{u}| \geqq l\,|x|,$$

whenever x is above the curves (8), (8 a). Whence

$$\frac{1}{|x|^{k'}} + \cdots + \frac{1}{|x+k_x|^{k'}} < \frac{h_2''(k')}{|x|^{e_1(k-1)}}$$
$$(u \leqq 0;\quad x \text{ above } (8)).$$

Thus the inequality

$$\frac{1}{|x|^{k'}} + \cdots + \frac{1}{|x+k_x|^{k'}} < \frac{h'}{|x|^{k_2}} \tag{9}$$

holds, whenever $x\,(|x| > \varrho > 0)$ lies above curves of form (8), (8 a). Here k_2 can be made arbitrarily great by taking k' sufficiently great and h' is independent of x. Hence for x in k', above curves of type (8), (8 a), we have

$$\left|\int_{l_x}\right| \leqq \frac{\beta h' e^{\Re Q(x)}}{|x|^{k_2}} . \tag{10}$$

Whenever R contains the negative axis of reals the inequality (10) will continue to hold in the whole region R' provided that along the negative axis of reals, for $|x|$ sufficiently great, $\Re Q(x) \neq 0$. In fact, from (7) it follows that

$$\left|\int_{l_x}\right| \leqq e^{\Re Q(x)} \beta \left(\frac{1}{|x|^{k'}} + \frac{e^{\Re(Q(x+1)-Q(x))}}{|x+1|^{k'}} + \cdots \right.$$
$$\left. + \frac{e^{\Re(Q(x+k_x)-Q(x))}}{(x+k_x)^{k'}} \right); \tag{10 a}$$

on the other hand, we have

$$e^{\Re(Q(x')-Q(x))} \tag{10 b}$$
$$(\Im x = \Im x';\quad \Re x < \Re x';\quad x,\, x' \text{ in } R)$$

diminishing very rapidly as $x'-x$ increases. So it is clear that the sum of terms in (10a), involving factors of the form (10b), is negligible to the extent that (10) would hold throughout R'.

The only case when the inequality (10) is not asserted is when R contains the negative axis of reals, while x is in R' below a curve of the form (8) and $e^{\Re Q(x)}$ remains bounded as x moves to the left along a line parallel to the negative axis. In the sequel it will be seen that it is precisely in this case that it is not necessary to consider the integral

$$\int_{l_x}$$

[In particular, see Case II below].

There are two cases to be considered.

Case I. *Along the negative axis of reals, for* $|x|$ *sufficiently great,* $\Re Q(x) \neq 0$.

In this case along any line in R, parallel to the axis of reals, $\Re Q(x)$ increases not slower than a positive fractional power of $|x|$, as x moves along such a line to the left. This is an immediate consequence of the nature of the function $Q(x)$ and of the inequality (1 b) (which insures (6)).

Now, taking into consideration (5 a) and the integrand in (5),

$$\left|\int_{L_1}\right| < \frac{\beta\,|e^{2\pi\sqrt{-1}\,\lambda x}|}{d}\int_{L_1}\frac{e^{\varrho_\lambda(t)}\,|dt|}{|t|^{k'}}.$$

By (4) the integrand in the above increases exponentially along L_1 and attains its maximum at x', the upper end point of L_1. The integral will be of the order of magnitude of the value of this integrand at x'. Thus

$$\left|\int_{L_1}\right| < \frac{\beta_1\,e^{\Re Q(x')}}{|x'|^{k_3}} = \frac{\beta_1\,e^{\Re Q(x)}}{|x|^{k_3}}\left(\left|\frac{x}{x'}\right|^{k_3} e^{\Re(Q(x')-Q(x))}\right)$$

$$(k_3 = k' - d_3;\ d_3 \geqq 0;\ k_3 \to \infty \text{ as } k' \to \infty;\ \Im x = \Im x').$$

In the case at hand, the function

$$g(x) = \left|\frac{x}{x'}\right|^{k_3} e^{\Re(Q(x')-Q(x))}$$

approaches zero very rapidly as x moves to the left from x' along a line through

x' parallel to the axis of reals. When x remains in a sufficiently close neighborhood of x' and does not part from x' too rapidly as $v (= \Im x = \Im x')$ increase, the fact can be used that $\Re(Q(x') - Q(x)) \leqq 0$ $(\Re x \leqq \Re x')$; we shall then have $g(x)$ either bounded or infinite at infinity to an order $\bar{k}_3$ such that $k_3 - \bar{k}_3 \to \infty$ as $k_3 \to \infty$. More precisely, this will be the case for x in any region bounded on the left by a curve of the form

$$v = h(-u)^e \quad (h,\ e > 0)$$

where e can be taken arbitrarily small. If there is occasion to consider a region below such a curve, for x in such a region (with e sufficiently small) we shall have $e^{\Re(Q(x') - Q(x))}$ approaching zero exponentially (i. e., as $e^{-\Gamma |x|^\gamma}$ $(\Gamma, \gamma > 0)$) as $|x| \to \infty$ along any path to infinity in that region. It is clear then that

$$\left| \int_{L_1} \right| < \frac{\beta_1 e^{\Re Q(x)}}{|x|^{k_4}} \tag{11}$$

$$(k_4 \to \infty \text{ as } k' \to \infty;\ x \text{ in } R').$$

The integral along L_2 will be written in the form

$$\left| \int_{L_2} \right| = \left| \int_{L_2} \frac{e^{2\pi\sqrt{-1}(\lambda-1)(x-t) + Q(t)} \beta(t)\, dt}{(e^{-2\pi\sqrt{-1}(x-t)} - 1)\, t^{k'}} \right|.$$

For x in R'

$$\left| \int_{L_2} \right| < \frac{\beta \left| e^{2\pi\sqrt{-1}(\lambda-1)x} \right|}{d} \int_{L_2} \frac{e^{\varrho_{\lambda-1}(t)} |dt|}{|t|^{k'}}.$$

As t moves along L_2 from x' upwards we have $\varrho_{\lambda-1}(t)$ bounded. Therefore the maximum of the integrand, last written, occurs at x'. The reasoning of the type used in deriving (11) will show that

$$\left| \int_{L_2} \right| < \frac{\beta_2 e^{\Re Q(x)}}{|x|^{k_5}} \tag{11 a}$$

$$(k_5 \to \infty \text{ as } k' \to \infty;\ x \text{ in } R')$$

so that

$$\left| \int_{L} \right| < \frac{\beta' e^{\Re Q(x)}}{|x|^{k''}} \tag{12}$$

$$(k'' \to \infty \text{ as } k' \to \infty;\ x \text{ in } R').$$

Case II. *Along the negative axis of reals* $\Re\, Q(x) = 0$.

If the left boundary of R is not h this boundary will be of the form

$$v = h(-u)^e + \cdots \quad (h,\, e > 0).$$

An inequality like (12) will continue to hold in R'. This can be shown by the reasoning used to derive (12). If the lower boundary of R is h the contour L will consist of

$$L = h + L^* = h + L_1^* + L_2^*.$$

The contour L_x, in (5), will then be deformed into a loop, described in the counter clockwise direction and extending to infinity, containing the points $x - 1$, $x - 2$, ... and not containing the points x, $x + 1$, ... The formula (5) will yield the following

$$(13) \qquad \underset{t=x}{\mathfrak{S}} \frac{e^{Q(t)}\beta(t)}{t^{k'}} = \frac{e^{Q(x-1)}\beta(x-1)}{(x-1)^{k'}} + \frac{e^{Q(x-2)}\beta(x-2)}{(x-2)^{k'}} + \cdots,$$

inasmuch as convergence may be asserted.

Now

$$(13\text{ a}) \qquad \Re\, Q(x) = \mu\,|x|\log|x|\cos\alpha + (\gamma' u - \gamma'' v)$$
$$+ |\eta|\,|x|^{\frac{s}{p}}\cos\left(\bar{\eta} + \frac{s}{p}\alpha\right) + \cdots$$
$$(\gamma = \gamma' + \sqrt{-1}\,\gamma'';\ \eta = |\eta|\,e^{\sqrt{-1}\,\bar{\eta}}, \ldots;\ x = |x|\,e^{\sqrt{-1}\,\alpha};\ p > s \geqq 1).$$

Necessarily $\mu = \gamma' = 0$ and, whenever a coefficient η in $Q(x)$ is not zero,

$$(13\text{ b}) \qquad \cos\left(\bar{\eta} + \frac{s}{p}\pi\right) = 0.$$

Hence $Q(x) = \sqrt{-1}\,\gamma'' x + \eta x^{\frac{s}{p}} + \cdots (p > s \geqq 1)$ while

$$(13\text{ c}) \qquad \Re\, Q(x) = -\gamma'' v \pm |\eta|\,|x|^{\frac{s}{p}} \sin\frac{s}{p}(\pi - \alpha) + \cdots$$
$$(p > s \geqq 1);$$

here we may have $|\eta| = 0$. This relation is derived by noting that, in virtue of (13 b),

$$\cos\left(\bar{\eta} + \frac{s}{p}\alpha\right) = \pm \sin\frac{s}{p}(\pi - \alpha).$$

If $\Re\, Q(x) = -\gamma'' v$ then

$$\Re\,[Q(x-i) - Q(x)] \equiv 0 \qquad (i = 1, 2, \ldots). \tag{13 d}$$

If in (13 c) $|\eta| \neq 0$, we define a curve F_H in Γ, by an equation

$$r = h(-u)^H \left(h > 0;\ H = 1 - \frac{s}{p}\right). \tag{14}$$

For x in R below F_H we have

$$\pi - \alpha = \operatorname{tg}^{-1}\frac{v}{-u} = \frac{v}{-u} + \cdots$$

and

$$\sin\frac{s}{p}(\pi - \alpha) = \frac{s}{p}\left(\frac{v}{-u}\right) + \cdots$$

so that

$$\left|\sin\frac{s}{p}(\pi - \alpha)\right| \leqq \frac{hs}{p}\frac{(-u)^H}{(-u)} + \cdots. \tag{14 a}$$

Thus, below F_H,

$$|\eta|\,|x|^{\frac{s}{p}} \sin\frac{s}{p}(\pi - \alpha) \leqq \frac{|\eta|\,hs}{p}\frac{|x|^{\frac{s}{p}}|x|^H}{|x|} + \cdots = \frac{|\eta|\,hs}{p} + \cdots \tag{14 b}$$

since $|x| = -u + \cdots$.

Similarly, if x is in R below F_H,

$$|\eta|\,|x-i|^{\frac{s}{p}} \sin\frac{s}{p}(\pi - \alpha_i) \leqq \frac{|\eta|\,hs}{p} + \cdots \tag{14 c}$$

$$(x - i = |x - i|\,e^{\sqrt{-1}\,\alpha_i};\ i = 1, 2, \ldots).$$

Noting that

$$\Re\,[Q(x-i) - Q(x)] = \left[\pm\,|\eta|\,|x-i|^{\frac{s}{p}} \sin\frac{s}{p}(\pi - \alpha_i) + \cdots\right] - \left[\pm\,|\eta|\,|x|^{\frac{s}{p}} \sin\frac{s}{p}(\pi - \alpha) + \cdots\right],$$

we have by (14 b) and (14 c)

(15) $$\Re\,[Q(x-i)-Q(x)] \leqq 2\frac{|\eta|\,hs}{p}+\cdots<q$$

(q independent of x, i; x in R below F_H; $i=1, 2, \ldots$).

Hence, whether $\Re\,Q(x)=-\gamma'' v$ or $|\eta|$ in (13 c) is not zero, the inequalities (15) are seen to hold at least for x in R below F_H. Thus, from (13) it follows that

$$\left|\underset{t=x}{\mathfrak{S}}\frac{e^{Q(t)}\,\beta(t)}{t^{k'}}\right|<\beta\,e^{q}e^{\Re\,Q(x)}\sum_{i=1}^{\infty}\frac{1}{|x-i|^{k'}}$$

(x in R', below F_H).

Further, by formula (12) of (II),

(15 a) $$\left|\underset{t=x}{\mathfrak{S}}\frac{e^{Q(t)}(t)}{t^{k'}}\right|<\frac{\pi\beta\,e^{q}}{2}\frac{e^{\Re\,Q(x)}}{x^{k'-1}}$$

(x in R', below F_H).

If R extends above F_H the expression (13) does not appear useful for purposes of demonstration, whenever x is above F_H. In this case we use the relation

$$\underset{t=x}{\mathfrak{S}}=\int_{l_x}+\int_{L_1}+\int_{L_2}$$

The first of the last three integrals satisfies inequality (10). As to the second one, we have the integrand (as displayed in the second member of (5)) bounded along h (while x has a fixed value in R' (on or above F_H)). We have

$$\left|\int_{L_1}\right|=\frac{\beta\,|e^{2\pi\sqrt{-1}\lambda x}|}{d}\int_{L_1}\frac{e^{\varrho_\lambda(t)}\,|dt|}{|t|^{k'}}.$$

Here $\varrho_\lambda(t)$ is bounded along h and increasing exponentially along the remaining part of L_1, i.e., along L_1^*. Hence

$$\left|\int_{L_1}\right|<\frac{\beta_3\,e^{\Re\,Q(x^1)}}{|x^1|^{k_5}}=\frac{\beta_3\,e^{\Re\,Q(x)}}{|x|^{k_5}}\left(\left|\frac{x}{x^1}\right|^{k_5}e^{\Re\,(Q(x^1)-Q(x))}\right)$$

$$\leqq\frac{\beta_3\,e^{R\,Q(x)}}{|x|^{k_5}}\left|\frac{x}{x^1}\right|^{k_5}$$

($k_5\to\infty$, as $k'\to\infty$; $\Im x=\Im x^1=v$).

With x restricted as stated, we have $\left|\frac{x}{x^1}\right|$ behaving in the most unfavorable way when x is on F_H; we have then

$$(16) \qquad \left|\frac{x}{x^1}\right| \leqq \frac{(-u)}{v} + \cdots = \frac{(-u)}{h(-u)^H} + \cdots = \frac{1}{h}(-u)^{\frac{s}{p}} + \cdots < \bar{h}\,|x|^{\frac{s}{p}}.$$

Thus, for x in R' on and to the right of F_H,

$$\left|\int_{L_1}\right| < \frac{\beta_4\, e^{\Re\, Q(x)}}{|x|^{k_6}} \qquad (k_6 = k_5 H \to \infty \text{ as } k' \to \infty).$$

A similar inequality is obtained for $\int_{L_2}$, valid in the same region. In proving this inequality we again make use of (16). Hence

$$(17) \qquad \left|\int_{L_x}\right| < \frac{\beta_1'\, e^{\Re\, Q(x)}}{|x|^{k_1''}}$$

$(k_1'' \to \infty$ as $k' \to \infty$; x in R' on and above $F_H)$.

But in virtue of (15 a) an inequality like (17) is seen to hold throughout R'. This completes the examination of Case II.

The result just mentioned, together with (10) and (12), enables us to assert that an inequality like (17) holds, for x in R', in any case. It follows therefore that $z(x)$, as given by (3 b), satisfies in R' an inequality

$$|z(x)| < \beta_1'\, |x|^{\bar{k}} \qquad (\bar{k} = k - k_1'').$$

Now $k - \bar{k} = k_1''$ and approaches infinity as k approaches infinity (see (3 a)). In (3) attach subscript to $y(x)$, $t(x)$, $z(x)$. It is clear then that (2) holds for $y_k(x)$ to $m(k)$ termes $(m(k) \to \infty$ as $k \to \infty)$.

It remains to show that $(y_\sigma(x) - y_k(x))\, e^{-Q(x)}\ (\equiv g_{\sigma k}(x);\ \sigma > k) \sim 0$ in R'. If h is part of L, $g_{\sigma k}(x) \equiv 0$; otherwise, $|g_{\sigma k}(x)| \leqq h_{\sigma k} e^{-\varrho_\lambda(x)}$ (in R'). Application of (4) and (6) completes the proof.

§ 5. Construction of Proper Solutions to the Right of a Proper Curve.

The following theorem will be proved.

Theorem I. *Assume that the coefficients of an equation* $L_n(y) = 0$ (2; § 1) *are known* (*Cf.* § 1) *in a subregion of* Γ (§ 1),

$$G = (1) + (2) + \cdots + (m) + \cdots + (\eta).$$

Let the corresponding functions $Q(x)$ *be*

(1) $$Q_1(x), \ldots Q_n(x).$$

Suppose that F, *a proper curve* (*Def.* 9; § 1) *for the set* (1), *is the left boundary of* (m) $(2 \leqq m \leqq \eta)$ *or lies to the left of it. Assume that in a strip* V, *of unit width and with its left boundary coincident with the left boundary of* (m), *there exists a proper fundamental set of solutions* (*Def.* 4; § 1) *satisfying the equation* $L_n(y) = 0$. *It will necessarily follow that* $L_n(y)$ *is completely proper* (*Def.* 6; § 1) *in* $(m) + \cdots + (\eta)$. *If* F, *a proper curve for the set* (1), *exists in the region* (1) *then the above assumption concerning existence of solutions in* V *may be omitted and it will necessarily follow that* $L_n(y)$ *is completely proper in* $(m) + \cdots + (\eta)$ $(m = 1)$.

Proof. As stated previously the regions (1), (2), ... (η) are separated by B' curves,

(2) $$B^1, B^2, \ldots B^{\eta-1}.$$

In any region (s) of this set of regions the $Q_j'(x)$ $(j = 1, \ldots n)$ maintain a certain ordering. We shall write

(3) $$\Re\, {}_sQ_1'(x) \geqq \Re\, {}_sQ_2'(x) \geqq \cdots \geqq \Re\, {}_sQ_n'(x). \quad (x \text{ in } (s)).$$

In connection with this ordering the subscript s will be attached, from the left, to some other symbols; thus, ${}_sS(x)$ will denote the formal matrix of a difference system corresponding to $L_n(y) = 0$, with ${}_sQ_j(x)$ entering in the j-th column. The set $[{}_sQ_1(x), {}_sQ_2(x), \ldots {}_sQ_n(x)]$ is merely a permutation of the set $[Q_1(x), Q_2(x), \ldots Q_n(x)]$.

It is sufficient to prove the theorem for the system $Y(x + 1) = D(x)\, Y(x)$, related to the given equation (2; § 1) and given by (6; § 1). This follows from the relationship (6 a; § 1) between solutions of the system and the single equation. It is clear that the $d_{ij}(x)$ are known and of the same character as the $a_j(x)$ $(j = 1, \ldots n)$, the coefficients of $L_n(y)$. The process of construction of solutions, about to be given, is of course equally applicable to any system $Y(x+1)=Z(x)\,Y(x)$ (1 a; § 1).

By iteration (3) we construct, in (m), determinant limits of orders $1, 2, \ldots n$ corresponding to the ${}_mQ(x)$'s

$$ {}_mQ_1(x),\ {}_mQ_1(x) + {}_mQ_2(x),\ \ldots,\ {}_mQ_1(x) + \cdots + {}_mQ_n(x), $$

respectively. When $m=1$ this process will be carried on by »iteration from the infinite left» (Lemmas 4 and 5; § 3). When $m>1$ the process will be carried on by iteration from the strip V, specified in the theorem. In the latter case use will be made of the existence of solutions in V, as stated in the theorem; in this connection Lemmas 6 and 7 (§ 3) are to be used. Generally speaking, application of Lemmas 4, 5, 6 and 7 is possible in virtue of the inequalities (3) being valid in (m) (for $s=m$).

In agreement with the notation of § 3 let these determinant limits be denoted, for $k=1, \ldots n$, by

$$ {}_my_{i_1 \ldots i_k;\, 1 \ldots k}(x) \qquad (i_1 < \cdots < i_k;\ i_1, \ldots i_k = 1, \ldots n). \tag{4} $$

These functions are analytic in (m) and satisfy the asymptotic relations

$$ {}_my_{i_1 \ldots i_k;\, 1 \ldots k}(x) \sim e^{{}_mQ_1(x) + {}_mQ_2(x) + \cdots + {}_mQ_k(x)}\, {}_ms_{i_1 \ldots i_k;\, 1 \ldots k}(x) \tag{4 a} $$
$$ (i_1 < \cdots < i_k;\ i_1, \ldots i_k = 1, \ldots n;\ x \text{ in } (m)). $$

For $k=1$ the functions (4) are elements of a solution of the system $Y(x+1)=D(x)\,Y(x)$; write

$$ {}_mz_{i1}(x) = {}_my_{i;\,1}(x) \qquad (i=1, \ldots n). \tag{4 b} $$

This solution is proper in (m). Assume that, for $k-1 \geqq 1$, there exist $k-1$ solutions,

$$ {}_mz_{ij}(x) \qquad (i=1, \ldots n;\ j=1, \ldots k-1). \tag{4 c} $$

which are analytic in (m), satisfy the relations

$$ {}_mz_{i_1 \ldots i_s;\, 1 \ldots s}(x) = {}_my_{i_1 \ldots i_s;\, 1 \ldots s}(x) \tag{4 d} $$
$$ (i_1 < \cdots < i_s = 1, \ldots n;\ s = 1, \ldots k-1;\ x \text{ in } (m)) $$

and are such that, in (m)

(4 e) $$ {}_m z_{ij}(x) \sim e^{mQ_j(x)}\, {}_m s_{ij}(x) \qquad (i = 1, \ldots n;\ j = 1, \ldots k-1). $$

Existence of a matrix of solutions, proper in (m), will be demonstrated by induction if we show that there exists a solution ${}_m z_{ik}(x)$ $(i = 1, \ldots n)$ analytic in (m) and such that (4 d), (4 e) will hold for $s = k, j = k$. Analogous to a similar construction, in II, such a solution can be found in terms of certain determinant limits and the solutions (4 c). For this purpose use will be made of the following formulas, found in (II). (The notation used in this paper is different from that of (II)).

We have

(5) $$ {}_m z_{1k}(x) = \sum_{j=1}^{k-1} {}_m z_{1j}(x) \mathop{\mathfrak{S}}_{t=x} {}_m V_{kj}(t), $$

(5 a) $$ {}_m V_{kj}(t) = \frac{{}_m\theta^{(k)}(t)\, {}_m m_{j,\,k-1}(t)}{{}_m\theta^{(k-1)}(t)\, {}_m\theta^{(k-1)}(t+1)}, $$

(5 b) $$ {}_m m_{j,\,k-1}(t) = (-1)^{k-1+j} \begin{vmatrix} {}_m z_{11}(t+1) \ldots {}_m z_{1,\,j-1}(t+1), & {}_m z_{1,\,j+1}(t+1) \ldots \\ {}_m z_{11}(t+2) \ldots \quad \ldots & \ldots \quad \ldots \\ \ldots \quad \ldots \quad \ldots & \ldots \quad \ldots \\ {}_m z_{11}(t+k-2) \quad \ldots & \ldots \quad \ldots \end{vmatrix}, $$

(5 c) $$ {}_m\theta^{(k)}(t) = \sum_{i_1 \ldots i_{k-1}=1}^{n} \left\{ d_{1 i_1}(t) \left[\sum_{i_1=1}^{n} d_{1 i_1}(t+1)\, d_{i_1 i_2}(t) \right] + \cdots \right\} \cdot {}_m y_{1 i_1 \ldots i_{k-1};\, 1 \ldots k}(t). $$

Moreover, for x in (m),

(5 d) $$ {}_m\theta^{(k)}(x) \sim \begin{vmatrix} e^{mQ_1(x)}\, {}_m s_1(x), & e^{mQ_2(x)}\, {}_m s_2(x), \ldots & e^{mQ_k(x)}\, {}_m s_k(x) \\ e^{mQ_1(x+1)}\, {}_m s_1(x+1), & \ldots \quad \ldots & \ldots \\ \ldots & \ldots \quad \ldots & \ldots \\ e^{mQ_1(x+k-1)}\, {}_m s_1(x+k-1), & \ldots \quad \ldots & \ldots \end{vmatrix}, $$

while

(5 e) $$ {}_m\theta^{(k)}(x) = \begin{vmatrix} {}_m z_{11}(x), & {}_m z_{12}(x), \ldots & {}_m z_{1k}(x) \\ \ldots & \ldots \quad \ldots & \ldots \\ {}_m z_{11}(x+k-1), & \ldots \quad \ldots & \ldots \end{vmatrix}. $$

The symbol $\underset{t=x}{\mathfrak{S}}$ stands for summation and is to be suitably determined. It is seen that the ${}_m m_{j,k-1}(t)$ $(j = 1, \ldots k-1)$ and ${}_m\theta^{(k)}(t)$ are known and analytic in (m).

Using (5 b) and the known asymptotic forms (4 e) the asymptotic form of ${}_m m_{j,k-1}(x)$ will be seen to be

$$(6) \qquad {}_m m_{j,k-1}(x) \sim e^{{}_m Q_1(x) + \cdots + {}_m Q_{j-1}(x) + {}_m Q_{j+1}(x) + \cdots + {}_m Q_{k-1}(x)} {}_m \mu_{j,k-1}(x)$$
$$({}_m\mu_{j,k-1}(x), \text{ an } s\text{-series}; \ x \text{ in } (m)).$$

On the other hand, making use of (5 d) and taking account of the way several formal series with logarithms in the s-series factors are related (See (I); in particular, (6″) on p. 213), we conclude that

$$(6\,\mathrm{a}) \qquad {}_m\theta^{(k)}(x) \sim e^{{}_m Q_1(x) + \cdots + {}_m Q_k(x)} {}_m\varphi^{(k)}(x) \qquad (x \text{ in } (m))$$

where ${}_m\varphi^{(k)}(x)$ is an s-series without logarithms.[1] The series ${}_m\varphi^{(k)}(x)$ $(k = 1, \ldots n)$ cannot be identically zero since the formal series are assumed to be linearly independent. Consequently, by (5 a), (6) and (6 a),

$$(6\,\mathrm{b}) \qquad {}_m V_{kj}(x) \sim e^{{}_m Q_{kj}(x)} {}_m v_{kj}(x)$$
$$(j = 1, \ldots k-1; \ x \text{ in } (m)).$$

Here the series ${}_m v_{kj}(x)$ are all s-series; moreover, by ((3); $s = m$),

$$\Re\, {}_m Q'_{kj}(x) \leqq 0 \qquad (j = 1, \ldots k-1; \ x \text{ in } (m)).$$

It is easily seen that Lemma 8 (§ 4) is applicable for evaluation of any of the expressions

$$(6\,\mathrm{c}) \qquad \underset{t=x}{\mathfrak{S}} {}_m V_{kj}(t) \qquad (j = 1, \ldots k-1)$$

occurring in (5). In that lemma we only need to take $R = (m)$, $Q(x) = {}_m Q_{kj}(x)$, $h(x) = {}_m v_{kj}(x)$. Thus, by the methods of § 4 we evaluate (6 c) as a function analytic in a region $(m)'$, slightly interior to (m), and satisfying in $(m)'$ an asymptotic relation

[1] We can show this by a reasoning, applied to the second member of (5 d), similar to that in (I; p. 215).

$$\mathop{\mathfrak{S}}_{t=x} {}_mV_{kj}(t) \sim e^{mQ_{kj}(x)}\,{}_mu_{kj}(x) \tag{6 d}$$

where ${}_mu_{kj}(x)$ is an s-series. Since the boundaries of (m) can be translated, it may be considered that (6 d) holds in (m). Substituting (4 e) and (4 d) in (5) it is seen that ${}_mz_{1k}(x)$ is analytic in (m) and

$$ {}_mz_{1k}(x) \sim e^{mQ_k(x)}\,{}_m\sigma_{1k}(x) \qquad (x \text{ in } (m)), \tag{7}$$

where ${}_m\sigma_{1k}(x)$ is a formal s-series.

The remaining elements ${}_mz_{ik}(x)$ $(i = 2, \ldots n)$ of the solution may be determined as follows. We have

$$ {}_mz_{1k}(x+q) = \sum_{\lambda_1 \ldots \lambda_q = 1}^{n} d_{1\lambda_1}(x+q-1)\,d_{\lambda_1\lambda_2}(x+q-2)\ldots d_{\lambda_{q-1}\lambda_q}(x)\,{}_mz_{\lambda_q k}(x)$$
$$(q = 1, \ldots n).$$

By the reasoning of (II; p. 259) these equations have a non-vanishing determinant so that

$$ {}_mz_{ik}(x) = \delta_{i1}(x)\,{}_mz_{1k}(x+1) + \cdots + \delta_{in}(x)\,{}_mz_{1k}(x+n) \tag{8}$$
$$(i = 1, \ldots n).$$

Here the $\delta_{ij}(x)$ $(i, j = 1, \ldots n)$ are known in (m) and are of the same nature as the $d_{ij}(x)$. Thus, by (7) and (8), the elements ${}_mz_{ik}(x)$ $(i = 1, \ldots n)$ are analytic in (m) and satisfy, in (m), the asymptotic relations

$$ {}_mz_{ik}(x) \sim e^{mQ_k(x)}\,{}_m\sigma_{ik}(x) \qquad (i = 1, \ldots n) \tag{9}$$

where the series ${}_m\sigma_{ik}(x)$ are s-series. Necessarily the relations (4 d) will be satisfied for $s = 1, \ldots k$. The function ${}_mz_{1k}(x)$ is such that (5 e) holds; thus, using tha asymptotic relationships (5 d), (4 e) and (9), we conclude that the ${}_m\sigma_{ik}(x)$ $(i = 1, \ldots n)$ in (9) can be replaced by the ${}_ms_{ik}(x)$ $(i = 1, \ldots n)$, respectively. Thus a solution ${}_mz_{ik}(x)$ $(i = 1, \ldots n)$, possessing all the desired properties, has been constructed, This proves existence of a matrix solution proper in (m). The above indicates also the actual process of construction in any given case which satisfies the specified hypotheses. It is essential to note that in applying Lemma 8 we have, according to the hypotheses of the Theorem and as required by the Lemma, the function $Q(x) = {}_mQ_{kj}(x)$ proper along the portion of the path of integration near the right boundary of (m). If in the various summations involved in (5) additive periodic functions are admitted the kth solution, ${}_mz_{ik}(x)$ $(i = 1, \ldots k)$, will be modified by addition of linear expressions (with periodic coefficients) in the elements of the preceding $k - 1$ solutions. Unless stated otherwise such periodic functions will not be introduced, the summations in (5) being specified by § 4. The n solutions constituting the n columns of the matrix

$$ (10) \qquad {}_mZ(x) = ({}_mz_{ij}(x)) \qquad (i, j = 1, \ldots n) $$

may be spoken of as 'associated with determinant limits'.

The regions (1), (2), ... (η) may be considered as having strips $V_{j,j+1}$ (between (j) and $(j + 1)$; $j = 1, \ldots \eta - 1$), of unit width, in common. In the case at hand, there exists a proper matrix solution in $V_{m,m+1}$ (if $m + 1 \leqq \eta$). By the process indicated for the construction of ${}_mZ(x)$ we now obtain a matrix solution,

$$ (10\text{ a}) \qquad {}_{m+1}Z(x) = ({}_{m+1}z_{ij}(x)), $$

proper in $(m + 1)$, the constituent solutions (columns) being associated with the determinant limits, known in terms of the ${}_mz_{ij}(x)$ in $(m + 1)$. By a finite number of steps proper matrix solutions, ${}_rZ(x) = ({}_rz_{ij}(x))$ $(r = m, \ldots n)$, are constructed in (m), $(m + 1)$, ... (η); these solutions will be associated with determinant limits.

It remains to demonstrate that the periodic functions connecting these solutions are proper (Def. 5; § 1). Let $Z^r(x) = (z^r_{ij}(x))$ denote ${}_rZ(x)$ with the columns so rearranged that

$$ (11) \qquad Z^r(x) \sim S(x) = (e^{Q_j(x)} s_{ij}(x)) \qquad (x \text{ in } (r);\ r \geqq m). $$

Write

$$ (11\text{ a}) \qquad Z^r(x) = Z^{r+1}(x) P^{r,r+1}(x), \quad P^{r,r+1}(x) = (p^{r,r+1}_{ij}(x)). $$

Let $\Im x_r = \Im x$, $x - x_r =$ integer and restrict x_r to lie in the strip $V_{r,r+1}$ (when $\Im x \geqq \varrho > 0$). We have then for the matrix $P^{r,r+1}(x)$ of periodic functions the relation

$$ (11\text{ b}) \qquad P^{r,r+1}(x) = P^{r,r+1}(x_r) = Z^{r+1^{-1}}(x_r) Z^r(x_r). $$

For $x(=x_r)$ in $V_{r,r+1}$ the following asymptotic relation will hold in virtue of (11)

$$(12)\qquad P^{r,\,r+1}(x) \sim S^{-1}(x)\,S(x) = (e^{Q_{ji}(x)}\,\delta_{ij})$$

$$((\delta_{ij}) = I).$$

In other words, for $\Im x \geqq \varrho > 0$,

$$(12\,a)\qquad p_{ij}^{r,\,r+1}(x) = e^{Q_{ij}(x_r)}\left(\delta_{ij} + \frac{b_{ij}(x_r)}{x_r^k}\right) \qquad (i, j = 1, \ldots n);$$

in the above k can be made arbitrarily great and the $|b_{ij}(x_r)|$, for a fixed k, are bounded. The strips $V_{r,\,r+1}$ ($r=m, \ldots \eta-1$) extend indefinitely upwards (i. e., when $|x|$ approaches infinity in $V_{r,\,r+1}$, $v = \Im x \to +\infty$). These strips are to the right of a proper curve F and they are in a proper region R_F (Def. 9; § 1). The term 'proper' refers, in this connection, to the set

$$Q_1(x), \ldots Q_n(x).$$

By (11 b) the $p_{ij}^{r,\,r+1}(x)$ are analytic for $v \geqq \varrho > 0$. By (12 a) and in virtue of the fact that the $Q_{ij}(x_r)$ are proper in $V_{r,\,r+1}$ (Def. 3; § 1) it follows that

$$(13)\qquad q_{ij}^{r,\,r+1}(z) = p_{ij}^{r,\,r+1}\left(\frac{\log z}{2\pi\sqrt{-1}}\right) = z^{H_{ij}^{r,\,r+1}}\left(p_{ij;\,0}^{r,\,r+1} + p_{ij;\,1}^{r,\,r+1} z + \cdots\right)$$

$$(i, j = 1, \ldots n;\ r = m, \ldots \eta-1;\ H_{ij}^{r,\,r+1},\ \text{an integer});$$

here the power series converge within a sufficiently small circle with $z=0$ for center and, unless $p_{ij}^{r,\,r+1}(x) \equiv 0$, it may be supposed that $p_{ij;\,0}^{r,\,r+1} \neq 0$. Now $|z| = |e^{2\pi\sqrt{-1}x}| = e^{-2\pi v}$; thus, it is clear that

$$(13\,a)\qquad p_{ij}^{r,\,r+1}(x) \sim p_{ij;\,0}^{r,\,r+1} e^{2\pi\sqrt{-1}H_{ij}^{r,\,r+1}x}$$

$$(i, j = 1, \ldots n;\ r = m, \ldots \eta-1)$$

in every region of the kind indicated in (Def. 5; § 1). Hence, in accordance with this definition, these periodic functions are proper. In view of the relationship between solutions of the single equation $L_n(y)=0$, and those of the system, $L_n(y)$ is seen to be completely proper in $(m) + \cdots + (\eta)$.

§ 6. A Lemma on Factorization.

The following lemma will be indispensible as a preliminary to establishing the fundamental result.

Lemma 9. *Let coefficients of*

$$L_n(y) \equiv y(x+n) + a_1(x)\, y(x+n-1) + \cdots + a_n(x)\, y(x) = 0 \tag{1}$$

be known (and be of the kind specified in the beginning of § 1) *in* $(1) + \cdots + (m)$, *a subregion of* Γ. *If the equation is Q-factorable in* $(1)+\cdots+(m)$ (*Def.* 8; § 1), *a point of division being between the* Γ*-th and* $\Gamma+1$*-st terms (not belonging to the same logarithmic group*[1] *of the sequence*

$$Q_1(x), \ldots Q_n(x) \qquad\qquad (1 \leqq \Gamma < n),$$

it necessarily follows that the equation is factorable,

$$L_n(y) \equiv L_{n-\Gamma}\, L_\Gamma(y) = 0, \tag{1 a}$$

so that the coefficients in the operators $L_{n-\Gamma}(z)$, $L_\Gamma(y)$ *are of the same kind as in* (1). *With the* $e^{Q_j(x)} s_j(x)$ $(j=1,\ldots n)$ *denoting a linearly independent set of formal solutions of* (1), *the factorization* (1 a) *can be so effected that the series*

$$e^{Q_1(x)} s_1(x), \ldots e^{Q_\Gamma(x)} s_\Gamma(x) \tag{1 b}$$

are formal solutions of $L_\Gamma(y) = 0$.

Proof. In connection with the system $Y(x+1) = D(x)\, Y(x)$ (6; § 1), associated with (1), functions $y^r_{ij}(x)$ are defined by the product

$$\begin{aligned} Y^r(x) &= (y^r_{ij}(x)) \\ &= D(x-1) \ldots D(x-r)\, T(x-r) \end{aligned} \tag{2}$$

where $T(x)$ denotes $S(x)\,[\equiv (e^{Q_j(x)} s_{ij}(x)) = (e^{Q_j(x+i-1)} s_j(x+i-1))]$ with the s-series factors in the involved elements terminated after, say, k terms (k being sufficiently great). In accordance with the notation of § 3 we write

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[1] Cf. 6″ (p. 213; I).

(2 a) $$y^r_{i_1 \ldots i_\Gamma;\, 1 \ldots \Gamma}(x) = \begin{vmatrix} y^r_{i_1 1}(x), & \ldots & y^r_{i_1 \Gamma}(x) \\ \cdot & \cdot & \cdot \\ y^r_{i_\Gamma 1}(x), & \ldots & y^r_{i_\Gamma \Gamma}(x) \end{vmatrix}$$

where

$$i_1 < i_2 < \cdots < i_\Gamma \text{ and } i_1, i_2, \ldots i_\Gamma = 1, \ldots n.$$

Since, by hypothesis,

$$\Re[Q'_1(x) + \cdots + Q'_\Gamma(x)] \geqq \Re[Q'_{j_1}(x) + \cdots + Q'_{j_\Gamma}(x)]$$

$$(j_1 < \cdots < j_\Gamma = 1, \ldots n;\ x \text{ in } (1) + \cdots + (m))$$

in virtue of Lemma 5 (§ 3) the limits

(2 b) $$\lim_{r \to \infty} y^r_{i_1 \ldots i_\Gamma;\, 1 \ldots \Gamma}(x) = y_{i_1 \ldots i_\Gamma;\, 1 \ldots \Gamma}(x)$$

$$(i_1 < \cdots < i_\Gamma = 1, \ldots n)$$

will exist in $(1) + \cdots + (m)$ and will be analytic in this region; moreover, the asymptotic relations

(2 c) $$y_{i_1 \ldots i_\Gamma;\, 1 \ldots \Gamma}(x) \sim e^{Q_1(x) + \cdots + Q_\Gamma(x)} s_{i_1 \ldots i_\Gamma;\, 1 \ldots \Gamma}(x)$$

$$(i_1 < \cdots < i_\Gamma = 1, \ldots n)$$

will hold in $(1) + \cdots + (m)$. Form the operator

(3) $$L'^r_\Gamma(y) \equiv (-1)^\Gamma \begin{vmatrix} y(x) & , & y^r_{11}(x) & \ldots & y^r_{1\Gamma}(x) \\ y(x+1), & & y^r_{21}(x) & \ldots & y^r_{2\Gamma}(x) \\ \cdot & & \cdot & \cdot & \cdot \\ y(x+\Gamma), & & y^r_{\Gamma+1,1}(x) & \ldots & y^r_{\Gamma+1,\Gamma}(x) \end{vmatrix}$$

$$= b'^r_0(x)\, y(x+\Gamma) + \cdots + b'^r_{\Gamma-s}(x)\, y(x+s) + \cdots + b'^r_\Gamma(x)\, y(x);$$

here

(3 a) $$b'^r_{\Gamma-s}(x) = (-1)^{\Gamma-s} y^r_{1 \ldots s,\, s+2 \ldots \Gamma+1;\, 1 \ldots \Gamma}(x).$$

From the way asymptotic relations (2 c) were derived in § 3 it follows that

(3 b) $$y^r_{1 \ldots s,\, s+2 \ldots \Gamma+1;\, 1 \ldots \Gamma}(x) \sim e^{Q_1(x) + \cdots + Q_\Gamma(x)} s_{1 \ldots s,\, s+2 \ldots \Gamma+1;\, 1 \ldots \Gamma}(x)$$

$$(s = 0, \ldots \Gamma;\ x \text{ in } (1) + \cdots + (m);\ r = 1, 2, \ldots).$$

Hence an equation $L'^r_\Gamma(y) = 0$ will possess in $(1) + \cdots + (m)$, formal solutions

$$e^{Q_1(x)}s_1(x),\ e^{Q_2(x)}s_2(x),\ \ldots\ e^{Q_\Gamma(x)}s_\Gamma(x).$$

In virtue of (2 c) and of the fact just stated, if $L'_\Gamma(y)$ denotes $\lim L'^{\tau}_\Gamma(y)$, the equation

$$(4)\qquad L'_\Gamma(y) \equiv b'_0(x)\,y(x+\Gamma) + \cdots + b'_\Gamma(x)\,y(x) = 0$$

$$(b'_{\Gamma-s}(x) = (-1)^{\Gamma-s}y_{1\ldots s,s+2\ldots\Gamma+1;1\ldots\Gamma}(x))$$

will possess the same formal solutions. Here

$$(4\text{ a})\qquad b'_{\Gamma-s}(x) \sim e^{Q_1(x)+\cdots+Q_\Gamma(x)}s_{1\ldots s,s+2\ldots\Gamma+1;1\ldots\Gamma}(x)$$

$$(s=0,\ 1,\ \ldots\ \Gamma;\ x \text{ in } (1)+\cdots+(m)).$$

In particular, the s-series in the second member of (4 a) cannot be identically zero for $s=0$ and $s=\Gamma$; this is a consequence of linear independence of the formal series. Hence

$$b'_0(x) \not\equiv 0,\quad b'_\Gamma(x) \not\equiv 0.$$

Thus the equation (4) is actually of order Γ. Dividing out the coefficient $b'_0(x)$ we write (4) in the form

$$(5)\qquad L_\Gamma(y) \equiv y(x+\Gamma) + \cdots + b_{\Gamma-s}(x)\,y(x+s) + \cdots + b_\Gamma(x)\,y(x) = 0$$

$$(b_{\Gamma-s}(x) = b'_{\Gamma-s}(x)/b'_0(x);\ s=0,\ \ldots\ \Gamma-1).$$

The coefficients in (5) are analytic in $(1)+\cdots+(m)$; moreover,

$$(5\text{ a})\qquad b_{\Gamma-s}(x) \sim (-1)^{\Gamma-s}\frac{s_{1\ldots s,s+2\ldots\Gamma+1;1\ldots\Gamma}(x)}{s_{1\ldots\Gamma;1\ldots\Gamma}(x)} = \beta_{\Gamma-s}(x)$$

$$(s=0,\ \ldots\ \Gamma-1;\ x \text{ in } (1)+\cdots+(m)).$$

Now, the formal series $s_{1\ldots s,s+2\ldots\Gamma+1;1\ldots\Gamma}(x)$ $(s=0,\ 1,\ \ldots\ \Gamma)$ will contain no logarithms since the columns in the formal determinants

$$s_{1\ldots s,s+2\ldots\Gamma+1;1\ldots\Gamma}(x)$$

can be so combined as to get rid of these logarithms (Cf. I; in particular, pp. 213, 215). It is clear, moreover, that there will be only rational powers of x present in the formal series $\beta_{\Gamma-s}(x)$ since the constants r occurring in (7 a; § 1) differ by rational fractions in the consecutive formal series (7 a; § 1) in any group (I; p. 213) of such series (containing logarithms). Thus the operator $L_\Gamma(y)$ has all the properties required by the lemma. The factorization (1 a) follows immediately. The coefficients in the operator $L_{n-\Gamma}(z)$,

(6) $$L_{n-\Gamma}(z) \equiv z(x+n-\Gamma)+c_1(x)\,z(x+n-\Gamma-1)+\cdots+c_{n-\Gamma}(x)\,z(x),$$

will be analytic in $(1)+\cdots+(m)$ and will be of the required character in $(1)+\cdots+(m)$. The lemma is therefore proved.

The equation $L_{n-\Gamma}(z)=0$ will be formally satisfied, in $(1)+\cdots+(m)$, by the series

(6 a) $$e^{Q_{\Gamma+\mu}(x)}\,\sigma_{\Gamma+\mu}(x)\,(=L_\Gamma(e^{Q_{\Gamma+\mu}(x)}\,s_{\Gamma+\mu}(x))\,[1]$$
$$(\mu=1,\ldots n-\Gamma).$$

On taking account of the established nature of the $b_{\Gamma-s}(x)$ $(s=0, 1, \ldots \Gamma-1)$, it is seen that the series $\sigma_{\Gamma+\mu}(x)$ are s-series.

§ 7. On Products of Completely Proper Operators.

The following theorem will be now proved.

Theorem II. *Suppose that the set*

(1) $$Q_1(x), \ldots Q_n(x),$$

belonging to an equation

(1 a) $$L_n(y)=0,$$

has a point of division in $(1)+\cdots+(m)$. *Here, as before,* $(1)+\cdots+(m)$ *is a subregion of* Γ *the constituent regions* $(1), \ldots (m)$ *being separated by* B' *curves. Assume that corresponding to this point of division we have*

$$\Re\, Q'_\lambda(x) > \Re\, Q'_{\Gamma+\mu}(x)$$
$$(\lambda=1,\ldots \Gamma;\ \mu=1,\ldots n-\Gamma;\ x \text{ in } (1)+\cdots+(m)),$$

where an equality sign is admitted on the boundary of $(1)+\cdots+(m)$. *With the coefficients in* (1 a) *of the right kind* (*Cf.* § 1) *in* $(1)+\cdots+(m)$, *let*

[1] Whenever a formal series is formed by writing $L_\Gamma(e^{Q(x)}\,s(x))$ ($s(x)$ an s-series) the coefficients in L_Γ, if not representable by convergent series, are replaced by the formal series to which these coefficients are asymptotic.

(1 b) $$L_n(y) \equiv L_{n-\Gamma} L_\Gamma(y) \qquad (1 \leqq \Gamma < n)$$

be the corresponding factorization, as specified in Lemma 9 (§ 6). Suppose, moreover, that in (m) (or further to the left) there is a curve F which is proper with respect to the set (1).

It will necessarily follow that, if the operators $L_{n-\Gamma}(z)$, $L_\Gamma(y)$ are completely proper (Def. 6; § 1) in (1) + ⋯ + (m), the product $L_n(y)$ will be completely proper in (1) + ⋯ + (m).

Proof. As an immediate consequence of the hypotheses of the theorem the following is true.

The equation

(2) $$L_\Gamma(y) \equiv y(x+\Gamma) + b_1(x)\, y(x+\Gamma-1) + \cdots + b_\Gamma(x)\, y(x) = 0$$

possesses, in (1) + ⋯ + (m), Γ linearly independent formal series solutions

(2 a) $$e^{Q_j(x)} s_j(x) \qquad (j = 1, \ldots \Gamma).$$

The related system of order Γ

(2 b) $$Y_\Gamma(x+1) = D_\Gamma(x)\, Y_\Gamma(x),$$

$$D_\Gamma(x) = \begin{pmatrix} 0, & 1, & \ldots & 0 \\ 0, & 0, & 1, \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ -b_\Gamma(x), & & \cdots & -b_1(x) \end{pmatrix}$$

possesses, in (1) + ⋯ + (m), a formal matrix solution

(2 c) $$S_\Gamma(x) = (e^{Q_j(x)} s_{ij}(x)) = (e^{Q_j(x+i-1)} s_j(x+i-1))$$
$$(i, j = 1, \ldots \Gamma)$$

(the $s_{ij}(x)$, s-series). This system is satisfied by a matrix solution

(2 d) $$Y^\sigma(x) = (y^\sigma_{ij}(x)) = (y^\sigma_j(x+i-1)) \qquad (i, j = 1, \ldots \Gamma)$$

consisting of elements analytic in (σ) and of the asymptotic form

(2 e) $$Y^\sigma(x) \sim S_\Gamma(x) \qquad (x \text{ in } (\sigma);\ \sigma = 1, \ldots m).$$

Of course, the elements in the first row of $Y^\sigma(x)$ form a fundamental set of solutions of (2), and conversely. By hypothesis, such matrices $Y^s(x)$ $(s=1,\dots m)$ exist so that the matrices $R^s(x)$, of periodic functions, defined by the relations

$$(3) \qquad Y^s(x) = Y^{s+1}(x)\, R^s(x) \qquad (R^s(x)=(r^s_{ij}(x))),$$

are asymptotically representable as follows

$$(3\,\mathrm{a}) \qquad R^s(x) \sim \left(e^{2\pi\sqrt{-1}\, r^s_{ij} x}\, r^{s*}_{ij}\right)$$

$$(\text{the } r^s_{ij}, \text{ integers; } s=1,\dots m-1;\ \Im x \geqq \varrho > 0)$$

for x in any region as in (Def. 5; § 1). Here a constant r^{s*}_{ij} is not zero unless the corresponding function $r^s_{ij}(x)$ is.

Letting $V_{s,s+1}$ have the same meaning as in § 5 and reasoning as at the end of that section we conclude that, for $\Im x \geqq \varrho > 0$,

$$(3\,\mathrm{b}) \qquad (r^s_{ij}(x)) = (e^{Q_{ji}(x_s)}\,\varrho^s_{ij}(x_s)),\ (\varrho^s_{ij}(x_s)) = \left(\delta_{ij} + \frac{\alpha^s_{ij}(x)}{x^k}\right)$$

$$s=1,\dots m-1;\ \Im x = \Im x_s;\ \Re(x-x_s), \text{ integer; } x_s \text{ in } V_{s,s+1}).$$

Here k can be made arbitrarily great and the $\alpha^s_{ij}(x)$ are bounded in $V_{s,s+1}$.[1] B^s, the right boundary of (s), while a B' curve for the set (1), may be not a B' curve for the set $Q_j(x)$ $(j=1,\dots \Gamma)$. In such a case $(r^s_{ij}(x))$ can be taken as $I(=(\delta_{ij}))$.

Analogous facts can be stated concerning the equation

$$(4) \qquad L_{n-\Gamma}(z) \equiv z(x+n-\Gamma) + c_1(x)\, z(x+n+\Gamma-1) + \cdots + c_{n-\Gamma}(x)\, z(x) = 0.$$

In $(1)+\cdots+(m)$ it possesses $n-\Gamma$ formal solutions

$$(4\,\mathrm{a}) \qquad e^{Q_{\Gamma+\mu}(x)}\,\sigma_{\Gamma+\mu}(x) = L_\Gamma\left(e^{Q_{\Gamma+\mu}(x)}\, s_{\Gamma+\mu}(x)\right)$$

$$(\mu=1,\dots n-\Gamma; \text{ the } \sigma_{\Gamma+\mu}(x), \text{ formal } s\text{-series}),$$

the $e^{Q_j(x)}\,s_j(x)$ $(j=1,\dots n)$ constituting a linearly independent set of formal solutions of (1 a). The related system of order $n-\Gamma$

[1] Relations (3 b) would continue to hold if the operator $L_\Gamma(y)$ were merely proper (Def. 4; § 1).

(4 b)
$$Z_{n-\Gamma}(x+1) = D_{n-\Gamma}(x)\, Z_{n-\Gamma}(x),$$

$$D_{n-\Gamma}(x) = \begin{pmatrix} 0, 1 & \dots & 0 \\ 0, 0, 1 & \dots & 0 \\ \dots & \dots & \dots \\ -c_{n-\Gamma}(x), & \dots & -c_1(x) \end{pmatrix}$$

will possess, in $(1)+\cdots+(m)$, a formal matrix solution

(4 c)
$$S_{n-\Gamma}(x) = (e^{Q_{\Gamma+j}(x)}\sigma_{\Gamma+i,\,\Gamma+j}(x)) = (e^{Q_{\Gamma+j}(x+i-1)}\sigma_{\Gamma+j}(x+i-1))$$
$$(i, j = 1, \dots n-\Gamma)$$

(the $\sigma_{\Gamma+i,\,\Gamma+j}(x)$, s-series). There exists a matrix solution of (4 b)

(4 d)
$$Z^{\sigma}(x) = (z^{\sigma}_{\Gamma+i,\,\Gamma+j}(x)) = (z^{\sigma}_{\Gamma+j}(x+i-1))$$
$$(i, j = 1, \dots n-\Gamma)$$

with elements analytic in (σ) and of the asymptotic form

(4 e)
$$Z^{\sigma}(x) \sim S_{n-\Gamma}(x) \qquad (x \text{ in } (\sigma);\ \sigma = 1, \dots m).$$

Matrix solutions, like these, exist with the additional property that the matrices $P^s(x)$, of periodic functions defined by the relations

(5)
$$Z^s(x) = Z^{s+1}(x)\, P^s(x) \qquad (P^s(x) = (p^s_{\Gamma+i,\,\Gamma+j}(x)))$$

are of the asymptotic form

(5 a)
$$P^s(x) \sim \left(e^{2\pi\sqrt{-1}\,p^s_{\Gamma+i,\,\Gamma+j}\,x}\, p^{s*}_{\Gamma+i,\,\Gamma+j}\right)$$
$$(\text{the } p^s_{\Gamma+i,\,\Gamma+j}, \text{ integers};\ s = 1, \dots m-1;\ \Im x \geqq \varrho > 0)$$

for x in any region as in (Def. 5; § 1). Unless a function $p^s_{ij}(x)$ is identically zero the corresponding constant p^{s*}_{ij} is not zero. Moreover, for $\Im x \geqq \varrho > 0$,

(5 b)
$$(p^s_{\Gamma+i,\,\Gamma+j}(x)) = (e^{Q_{\Gamma+j,\,\Gamma+i}(x_s)}\pi^s_{\Gamma+i,\,\Gamma+j}(x_s)),$$
$$(\pi^s_{\Gamma+i,\,\Gamma+j}(x_s)) = \left(\delta_{ij} + \frac{\beta^s_{ij}(x)}{x^k}\right)$$
$$(s = 1, \dots m-1;\ \Im x = \Im x_s;\ \Re(x-x_s), \text{ integer};\ x_s \text{ in } V_{s,\,s+1}).$$

In the above k can be made arbitrarily great and the $\beta^s_{ij}(x)$ are bounded in $V_{s,\,s+1}$

A process of group summation will be now applied for the purpose of solving the equation

$$L_\Gamma(y) = z^s_{\Gamma+\mu}(x) \qquad (1 \leqq \mu \leqq n-\Gamma;\ 1 \leqq s \leqq m). \tag{6}$$

Let $Z^{s*}_{\Gamma+\mu}$ denote the matrix of order Γ

$$Z^{s*}_{\Gamma+\mu}(x) = \begin{pmatrix} 0 & \dots & 0 \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ 0 & \dots & 0 \\ z^s_{\Gamma+\mu}(x) & \dots & z^s_{\Gamma+\mu}(x) \end{pmatrix} \tag{6 a}$$

In each of the several rows the elements of the matrix

$$Y^s(x) \mathop{\mathbf{S}}_{t=x} Y^{s^{-1}}(t+1)\, Z^s_{\Gamma+\mu}(t) \tag{6 b}$$

$$\left(\mathop{\mathbf{S}}_{t=x+1} \varphi(t) - \mathop{\mathbf{S}}_{t=x} \varphi(t) = \varphi(x)\right)$$

are the same. An element of the first row will be a solution of (6); denote it by $y^s_{\Gamma+\mu}(x)$. This function will be also a solution of (1 a). Writing $Y^{s^{-1}}(x) = (\bar{y}^s_{ij}(x))$ $(i, j = 1, \dots \Gamma)$ we have

$$y^s_{\Gamma+\mu}(x) = \sum_{\lambda=1}^{\Gamma} y^s_{1\lambda}(x) \mathop{\mathbf{S}}_{t=x} \bar{y}^s_{\lambda\Gamma}(t+1)\, z^s_{\Gamma+\mu}(t). \tag{7}$$

Here the summations are to be suitably determined. Consider a summand $\bar{y}^s_{\lambda\Gamma}(t+1)\, z^s_{\Gamma+\mu}(t)$. By (3)

$$Y^s(x) = Y^e(x)\, R(x) \qquad (s < e \leqq m) \tag{7 a}$$

where

$$R(x) = (r_{ij}(x)) = R^{e-1}(x)\, R^{e-2}(x) \dots R^s(x)$$

and

$$r_{ij}(x) \sim e^{2\pi\sqrt{-1}\, r_{ij} x}\, r^*_{ij} \tag{7 b}$$

$$(i, j = 1, \dots \Gamma;\ \text{the } r_{ij},\ \text{integers};\ \Im x \geqq \varrho > 0).$$

Thus, noting that

(7 c)
$$(\bar{y}^s_{ij}(x)) = R^{-1}(x)\ Y^{e-1}(x)$$

$$(R^{-1}(x) = (\bar{r}_{ij}(x)) \sim (e^{2\pi\sqrt{-1}\,\bar{r}_{ij}x}\,\bar{r}^*_{ij});\quad Y^{e-1}(x) = (\bar{y}^e_{ij})),$$

we have

(7 d)
$$\bar{y}^s_{\lambda\Gamma}(t+1) = \sum_{\sigma=1}^{\Gamma} \bar{r}_{\lambda\sigma}(t)\,\bar{y}^e_{\sigma\Gamma}(t+1) \qquad (1 \leqq \lambda \leqq \Gamma).$$

On the other hand, by (5),

(8)
$$Z^s(x) = Z^e(x)\,P(x) \qquad (s < e \leqq m)$$

where

$$P(x) = (p_{\Gamma+i,\,\Gamma+j}(x)) = P^{e-1}(x)\,P^{e-2}(x)\ldots P^s(x)$$

and

(8 a)
$$p_{\Gamma+i,\,\Gamma+j}(x) \sim e^{2\pi\sqrt{-1}\,p_{\Gamma+i,\,\Gamma+j}x}\,p^*_{\Gamma+i,\,\Gamma+j}$$

$(i, j = 1, \ldots n-\Gamma$; the $p_{\Gamma+i,\,\Gamma+j}$, integers)

in any region extending indefinitely upwards, as in (Def. 5; § 1). Hence

(8 b)
$$z^s_{\Gamma+\mu}(t) = \sum_{w=1}^{n-\Gamma} z^e_{\Gamma+w}(t)\,p_{\Gamma+w,\,\Gamma+\mu}(t).$$

With the periodic functions in (7 a) and (8) formed for $e=m$ and using relations (2 e), (4 e) we conclude that, for x in (m),

(9)
$$\bar{y}^s_{\lambda\Gamma}(x+1)\,z^s_{\Gamma+\mu}(x) =$$
$$\sum_{\sigma=1}^{\Gamma}\sum_{w=1}^{n-\Gamma} z^e_{\Gamma+w}(x)\,\bar{y}^e_{\sigma\Gamma}(x+1)\,\bar{r}_{\lambda\sigma}(x)\,p_{\Gamma+w,\,\Gamma+\mu}(x)$$
$$\sim \sum_{\sigma=1}^{\Gamma}\sum_{w=1}^{n-\Gamma} e^{Q_{\Gamma+w,\,\sigma}(x)+2\pi\sqrt{-1}\left(\bar{r}_{\lambda\sigma}+p_{\Gamma+w,\,\Gamma+\mu}\right)x}\ \bar{r}_{\lambda\sigma}\,p^*_{\Gamma+w,\,\Gamma+\mu}\,\sigma_{\Gamma+w}(x)\,\bar{s}_{\sigma\Gamma}(x)$$
$$(\lambda = 1, \ldots \Gamma).$$

In the above the $\bar{s}_{\sigma\Gamma}(x)$ are s-series defined by the relation

$$S_\Gamma^{-1}(x) = (e^{Q_j(x)}\,s_{ij}(x))^{-1} = (e^{-Q_i(x)}\,\bar{s}_{ij}(x))$$
$$(i, j = 1, \ldots \Gamma).$$

This is a consequence of the fact, pointed out before that $|(s_{ij}(x))|$ $(i, j = 1, \ldots \Gamma)$

has no logarithms present. In a strip $V_{m,\,m+1}$ of, say, unit width and situated in (m) near the right boundary of (m) the real part of some exponent,

$$(9\,\mathrm{a})\qquad Q_{\Gamma+w,\,\sigma}(x)+2\,\pi\sqrt{-1}\,(\bar r_{\lambda\sigma}+p_{\Gamma+w,\,\Gamma+\mu})\,x\,,$$

corresponding to non zero constants $\bar r^*_{\lambda\sigma}$, $p^*_{\Gamma+w,\,\Gamma+\mu}$ is equal or greater than the real parts of all other exponents (corresponding to non zero constants $\bar r^*$, p^*). Let $\sigma=\sigma'$, $w=w'$ be subscripts for which this occurs. Then, for x in $V_{m,\,m+1}$,

$$(10)\qquad \bar y^s_{\lambda\Gamma}(x+1)\,z^s_{\Gamma+\mu}(x)\sim$$
$$e^{Q_{\Gamma+w',\,\sigma'}(x)+2\pi\sqrt{-1}\left(\bar r_{\lambda\sigma'}+p_{\Gamma+w',\,\Gamma+\mu}\right)x}\,(\bar r^*_{\lambda\sigma'}\,p^*_{\Gamma+w',\,\Gamma+\mu}\,\sigma_{\Gamma+w'}(x)\,\bar s_{\sigma'\Gamma}(x)+g^s_{\lambda\mu}(x)).$$

Now, by hypothesis, to the left of $V_{m,\,m+1}$ there is a curve F which is proper with respect to the set (1). Consequently along every path, extending to infinity and lying in $V_{m,\,m+1}$ the function $|e^{Q_{\Gamma+w',\,\sigma}(x)}|$ has a definite order with respect to $|e^{2\pi\sqrt{-1}\,x}|$; in fact, *the exponential factor in the second member of* (10) *will have the same property*. The expression $g^s_{\lambda\mu}(x)$ stands for a sum of a finite number of products of the form

$$(10\,\mathrm{a})\qquad p(x)\,s(x)$$

where $|p(x)|=1$ and $s(x)$ is a formal s-series. Such terms may be present only if there are more than one exponent (9 a) with the same real part (which is greater than the real parts of all other exponents).

Suppose again that $s<e\leqq m$. Consider (7 a), but in place of (7 b) use (3 b). We have

$$(11)\qquad (r^q_{ij}(x))^{-1}=(\bar r^q_{ij}(x))=(e^{Q_{ji}(x_q)}\,\bar\varrho^q_{ij}(x_q))\,,$$
$$(\bar\varrho^q_{ij}(x))=\left(\delta_{ij}+\frac{\bar\alpha^q_{ij}(x)}{x^{k_1}}\right)$$

where the $\bar\alpha^q_{ij}(x)$ are bounded in $V_{q,\,q+1}$ and k_1 can be made arbitrarily great. Thus

$$(11\,\mathrm{a})\qquad (\bar y^s_{ij}(x+1))=(\bar r_{ij}(x))\,(\bar y^e_{ij}(x+1))$$
$$=(\bar r^s_{ij}(x))\,(\bar r^{s+1}_{ij}(x))\ldots(\bar r^{e-1}_{ij}(x))\,(\bar y^e_{ij}(x+1))$$

$$= \Bigg(\sum_{\lambda_1 \ldots \lambda_{e-s}=1}^{\Gamma} e^{Q_{\lambda i}(x_s) + Q_{\lambda_2 \lambda_1}(x_{s+1}) + \cdots + Q_{\lambda_{e-s}\lambda_{e-s-1}}(x_{e-1})} \cdot$$

$$\cdot \bar{\varrho}^{s}_{i\lambda_1}(x_s)\, \bar{\varrho}^{s+1}_{\lambda_1 \lambda_2}(x_{s+1}) \ldots \bar{\varrho}^{e-1}_{\lambda_{e-s-1}\lambda_{e-s}}(x_{e-1})\, \bar{y}^{e}_{\lambda_{e-s}, j}(x+1) \Bigg) \cdot$$

Now, by (2 e), for x in (e)

$$(\bar{y}^e_{ij}(x)) \sim S_\Gamma^{-1}(x) = (e^{-Q_i(x)}\, \bar{s}_{ij}(x)) \qquad (i, j = 1, \ldots \Gamma)$$

where the $\bar{s}_{ij}(x)$ are s-series. If we write

$$\text{(11 b)} \qquad (\bar{y}^e_{ij}(x+1)) \sim (e^{-Q_i(x)}\, \bar{s}'_{ij}(x)) \qquad (x \text{ in } (e)),$$

it is apparent that the $\bar{s}'_{ij}(x)$ are s-series. The relations (11 a), (11 b) give the asymptotic form of the $\bar{y}^s_{ij}(x+1)$, for x in (e), in such a way that the exponential factors are explicitly given in terms of the $Q_j(x)$ $(j = 1, \ldots \Gamma)$.

Similarly, by (8) and (5 b),

$$\text{(12)} \qquad (z^s_{\Gamma+i,\, \Gamma+j}(x)) = (z^e_{\Gamma+i,\, \Gamma+j}(x))\,(p_{\Gamma+i,\, \Gamma+j}(x))$$

$$= (z^e_{\Gamma+i,\, \Gamma+j}(x))\,(p^{e-1}_{\Gamma+i,\, \Gamma+j}(x)) \ldots (p^s_{\Gamma+i,\, \Gamma+j}(x))$$

$$= \Bigg(\sum_{\sigma_1 \ldots \sigma_{e-s}=1}^{n-\Gamma} e^{Q_{\Gamma+\sigma_2,\, \Gamma+\sigma_1}(x_{e-1}) + Q_{\Gamma+\sigma_3,\, \Gamma+\sigma_2}(x_{e-2}) + \cdots + Q_{\Gamma+j,\, \Gamma+\sigma_{e-s}}(x_s)} \cdot$$

$$\cdot z^e_{\Gamma+i,\, \Gamma+\sigma_1}(x)\, \pi^{e-1}_{\Gamma+\sigma_1,\, \Gamma+\sigma_2}(x_{e-1})\, \pi^{e-2}_{\Gamma+\sigma_2,\, \Gamma+\sigma_3}(x_{e-2}) \ldots \pi^s_{\Gamma+\sigma_{e-s},\, \Gamma+j}(x_s) \Bigg)$$

$$(i, j = 1, \ldots n - \Gamma).$$

For x in (e) the $z^e_{\Gamma+i,\, \Gamma+j}(x)$ satisfy the asymptotic relations (4 e) (with $\sigma = e$). Thus we have an expression for the asymptotic form, in (e), of the $z^s_{\Gamma+i,\, \Gamma+j}(x)$ in which the exponential factors are in terms of the $Q_{\Gamma+j}(x)$ $(j = 1, \ldots n - \Gamma)$.

Consequently, for x in (e) $(s < e)$, the following asymptotic relationship will hold, with $1 \leqq \lambda \leqq \Gamma$,

$$\text{(13)} \qquad \bar{y}^s_{\lambda \Gamma}(x+1)\, z^s_{\Gamma+\mu}(x) \sim$$

$$\sum_{\lambda_1 \ldots \lambda_{e-s}=1}^{\Gamma} \; \sum_{\sigma_1 \ldots \sigma_{e-s}=1}^{n-\Gamma} e^{H^{\sigma_1 \ldots \sigma_{e-s};\, \mu}_{\lambda_1 \ldots \lambda_{e-s};\, \lambda}(x)}\, \bar{s}'_{\lambda_{e-s},\, \Gamma}(x)\, \sigma_{\Gamma+\sigma_1}(x).$$

Here

(13 a) $$H^{\sigma_1 \dots \sigma_{e-s}; \mu}_{\lambda_1 \dots \lambda_{e-s}; \lambda}(x) = Q_{\Gamma+\sigma_1, \lambda_{e-s}}(x) + [Q^*_{\lambda_1, \lambda}(x_s) + Q^*_{\lambda_2, \lambda_1}(x_{s+1}) + \cdots$$
$$+ Q^*_{\lambda_{e-s}, \lambda_{e-s-1}}(x_{e-1})] + [Q^*_{\Gamma+\sigma_2, \Gamma+\sigma_1}(x_{e-1}) + Q^*_{\Gamma+\sigma_3, \Gamma+\sigma_2}(x_{e-2}) + \cdots + Q^*_{\Gamma+\mu, \Gamma+\sigma_{e-s}}(x_s)]$$

where

(13 b) $$Q^*_{ji}(x_q) = Q_{ji}(x) + \log \bar{\varrho}^q_{ij}(x)$$
$$(i, j = 1, \dots \Gamma;\ s \leqq q)$$

and

(13 c) $$Q^*_{\Gamma+j, \Gamma+i}(x_q) = Q_{\Gamma+j, \Gamma+i}(x_q) + \log \pi^q_{\Gamma+i, \Gamma+j}(x_q)$$
$$(i, j = 1, \dots n - \Gamma).$$

The expressions Q^* involved in (13 b), (13 c) are logarithms of corresponding periodic functions. We take suitable determinations of the logarithms. Whenever a periodic function is identically zero the corresponding term in (13), or in any similar sum, will be zero. The summation signs in (13) will be considered as extended only over those terms for which the periodic functions are not zero. Only superscripts and subscripts corresponding to terms actually present will be considered.

In the sum (13), for any given x in (e), there is a set of subscripts and superscripts

(13 d) $$(\lambda_1 \dots \lambda_{e-s}) = (\lambda'_1 \dots \lambda'_{e-s}),\ (\sigma_1 \dots \sigma_{e-s}) = (\sigma'_1 \dots \sigma'_{e-s})$$

such that

(13 e) $$\Re\, H^{\sigma'_1 \dots \sigma'_{e-s}; \mu}_{\lambda'_1 \dots \lambda'_{e-s}; \lambda}(x) \geqq \Re\, H^{\sigma_1 \dots \sigma_{e-s}; \mu}_{\lambda_1 \dots \lambda_{e-s}; \lambda}(x)$$
$$[(\lambda_1 \dots \lambda_{e-s};\ \sigma_1 \dots \sigma_{e-s}) \neq (\lambda'_1 \dots \lambda'_{e-s};\ \sigma'_1 \dots \sigma'_{e-s});\ \lambda_1 \dots \lambda_{e-s} = 1 \dots \Gamma;$$
$$\sigma_1 \dots \sigma_{e-s} = 1 \dots n - \Gamma].$$

Accordingly, for x in (e) $(s < e)$,

(14) $$\bar{y}^s_{\lambda\Gamma}(x+1)\, z^s_{\Gamma+\mu}(x) \sim$$
$$e^{H^{\sigma'_1 \dots \sigma'_{e-s}; \mu}_{\lambda'_1 \dots \lambda'_{e-s}; \lambda}(x)} (\bar{s}_{\lambda'_{e-s}, \Gamma}(x)\, \sigma_{\Gamma+\sigma'_1}(x) + \cdots)$$
$$(1 \leqq \lambda'_{e-s} \leqq \Gamma;\ 1 \leqq \sigma'_1 \leqq n - \Gamma)$$

where ... stands for a sum of finite number of products of the form (10 a), which may be present only if there are more than one set of subscripts-superscripts (13 d) for which (13 e) holds. In the sequel the inequalities satisfied by λ'_{e-s} and σ'_1 will be found essential.

In (s) the asymptotic form of $\bar{y}^s_{\lambda\Gamma}(x+1)z^s_{\Gamma+\mu}(x)$ will be (Cf. (2 e), (4 e))

$$\bar{y}^s_{\lambda\Gamma}(x+1)z^s_{\Gamma+\mu}(x) \sim e^{Q_{\Gamma+\mu,\lambda}(x)}\,\bar{s}'_{\lambda\Gamma}(x)\,\sigma_{\Gamma+\mu}(x) \tag{15}$$
$$(\mu \geqq 1;\ \lambda \leqq \Gamma).$$

Since by hypothesis, for x in $(1)+\cdots+(m)$,

$$\Re\, Q'_\lambda(x) > \Re\, Q'_{\Gamma+\mu}(x) \qquad (1 \leqq \lambda \leqq \Gamma;\ 1 \leqq \mu \leqq n-\Gamma)$$

(except possibly along the boundary) so that

$$\Re\, Q'_{\Gamma+\mu,\lambda}(x) < 0,$$

it follows that

$$\Re\, Q_{\Gamma+\mu,\lambda}(x) > \Re\, Q_{\Gamma+\mu,\lambda}(x') \tag{15 a}$$
$$(\Im x = \Im x';\ \Re x < \Re x')$$

whenever x and x' are in $(1)+\cdots+(m)$.

Define the function $L_{\Gamma+\mu,\lambda}(x)$ as follows

$$L_{\Gamma+\mu,\lambda}(x) = \begin{cases} Q_{\Gamma+\mu,\lambda}(x) & (x \text{ in } (s)), \\ H^{\sigma'_1\ldots\sigma'_{e-s};\,\mu}_{\lambda'_1\ldots\lambda'_{e-s};\,\lambda}(x) & (x \text{ in } (e);\ s < e \leqq m). \end{cases} \tag{16}$$

By (15 a)

$$\Re\, L_{\Gamma+\mu,\lambda}(x) > \Re\, L_{\Gamma+\mu,\lambda}(x') \tag{16 a}$$
$$(\Im x = \Im x';\ \Re x < \Re x';\ x,\ x' \text{ in } (s)).$$

It will be proved now that, more generally,

$$\Re\, L_{\Gamma+\mu,\lambda}(x) > \Re\, L_{\Gamma+\mu,\lambda}(x') + \xi \tag{17}$$
$$(\Im x = \Im x';\ \Re x < \Re x')$$

provided that x is in (s) while x' is in $(s)+\cdots+(m)$, say, in (e) $(s \leqq e \leqq m)$; while ξ is a real magnitude negligible in a sense to be specified below, which for x' in (s) can be taken as zero. Assume for a moment that (17) holds. In virtue of (16), (13 a), (13 b) and (13 c) the inequality (17) can be written in the form

$$(17\text{ a})\quad \Re\, Q_{\Gamma+\mu,\lambda}(x) > \Re\{Q_{\Gamma+\sigma'_1,\lambda'_{e-s}}(x') + [Q_{\lambda'\lambda}(x_s) + Q_{\lambda'_2\lambda'_1}(x_{s+1}) + \cdots + Q_{\lambda'_{e-s-1}\lambda'_{e-s-2}}(x_{e-2}) + Q_{\lambda'_{e-s}\lambda'_{e-s-1}}(x_{e-1})] + [Q_{\Gamma+\sigma'_2,\Gamma+\sigma'_1}(x_{e-1}) + Q_{\Gamma+\sigma'_3,\Gamma+\sigma'_2}(x_{e-2}) + \cdots + Q_{\Gamma+\sigma'_{e-s},\Gamma+\sigma'_{e-s-1}}(x_{s+1}) + Q_{\Gamma+\mu,\Gamma+\sigma'_{e-s}}(x_s)]\} + \xi_1 + \xi$$

where

$$(17\text{ b})\quad \xi_1 = \log|\bar\varrho^{s}_{\lambda\lambda'_1}(x_s)\,\bar\varrho^{s+1}_{\lambda'_1\lambda'_2}(x_{s+1})\ldots\bar\varrho^{e-1}_{\lambda'_{e-s-1}\lambda'_{e-s}}(x_{e-1})| + \log|\pi^{e-1}_{\Gamma+\sigma'_1,\Gamma+\sigma'_2}(x_{e-1})\ldots\pi^{s+1}_{\Gamma+\sigma'_{e-s-1},\Gamma+\sigma'_{e-s}}(x_{s+1})|.$$

Since (16 a) holds it is necessary to consider only the case $1 < e$. The points $x_s \ldots x_{e-1}$, while correspondingly in the strips $V_{s,s+1}, \ldots V_{e-1,e}$, depend on x' ($x' - x_s, \ldots x' - x_{e-1}$ are integers). By (11) and (5 b) the real function ξ_1 may approach $-\infty$ as $|x'| \to \infty$; but, in any case,

$$(17\text{ c})\quad \xi_1 \leqq \left|\frac{b(x')}{x'^k}\right|$$

where $|b(x')|$ is bounded and k can be made arbitrarily great. Let $\xi = -\xi_1$. If (17 a) is demonstrated, with $\xi = -\xi_1$, the inequality (17) will have been demonstrated (with $\xi = -\xi_1$). Regrouping terms in (17 a), with the inequality sign displayed tentatively,

$$(17\text{ d})\quad \Re\, Q_{\Gamma+\mu,\lambda}(x) > \Re\{Q_{\Gamma+\sigma'_1,\lambda'_{e-s}}(x') + [Q_{\Gamma+\sigma'_2,\lambda'_{e-s-1}}(x_{e-1}) - Q_{\Gamma+\sigma'_1,\lambda'_{e-s}}(x_{e-1})] + [Q_{\Gamma+\sigma'_3,\lambda'_{e-s-2}}(x_{e-2}) - Q_{\Gamma+\sigma'_2,\lambda'_{e-s-1}}(x_{e-2})] + \cdots + [Q_{\Gamma+\sigma'_{e-s},\lambda'_1}(x_{s+1}) - Q_{\Gamma+\sigma'_{e-s-1},\lambda'_2}(x_{s+1})] + [Q_{\Gamma+\mu,\lambda}(x_s) - Q_{\Gamma+\sigma'_{e-s},\lambda'_1}(x_s)]\}$$

or

$$(17\text{ e})\quad \Re[Q_{\Gamma+\mu,\lambda}(x) - Q_{\Gamma+\mu,\lambda}(x_s)] > \Re\{[Q_{\Gamma+\sigma'_1,\lambda'_{e-s}}(x') - Q_{\Gamma+\sigma'_1,\lambda'_{e-s}}(x_{e-1})] + [Q_{\Gamma+\sigma'_2,\lambda'_{e-s-1}}(x_{e-1}) - Q_{\Gamma+\sigma'_2,\lambda'_{e-s-1}}(x_{e-2})] + [Q_{\Gamma+\sigma'_3,\lambda'_{e-s-2}}(x_{e-2}) - Q_{\Gamma+\sigma'_3,\lambda'_{e-s-2}}(x_{e-3})] + \cdots + [Q_{\Gamma+\sigma'_{e-s},\lambda'_1}(x_{s+1}) - Q_{\Gamma+\sigma'_{e-s},\lambda'_1}(x_s)]\}.$$

Now

$$(17\text{ f})\quad \Im x = \Im x_s = \Im x_{s+1} = \cdots = \Im x_{e-1} = \Im x', \qquad \Re x < \Re x_s < \Re x_{s+1} < \cdots < \Re x_{e-1} < \Re x'.$$

The functions $Q(\zeta)$, with double subscripts, occurring in (17 e) have increasing real parts as ζ moves to the left parallelly to the axis of reals and within the region $(1) + \cdots + (m)$. This follows from the fact that the first subscript, in such a function, is greater than Γ and the second subscript is equal or less than Γ (see 15 a). Taking account of (17 f) we conclude that the left member in (17 e) is positive while the real parts of the differences occurring in the square brackets in the second member of (17 e) are negative. Hence (17 e) holds. The steps by means of which (17 e) was derived are reversible. Thus (17) is demonstrated with $\xi = -\xi_1$ ($\xi = 0$ in (s); (17 b), (17 c)).

In view of the preceding we are led to consider a summand

$$H(x) \left(= \bar{y}^s_{\lambda\Gamma}(x+1) z^s_{\Gamma+\mu}(x)\right) = e^{L(x)} h(x) \tag{18}$$

where $L(x)$ $(= L_{\Gamma+\mu,\lambda}(x))$ *satisfies* (16) *and* (17) *while*

$$h(x) \sim H(x) \qquad (x \text{ in } (s) + \cdots + (m)). \tag{18 a}$$

Here the formal expression $H(x)$ *is a formal s-series* ($H_0(x) = \bar{s}'_{\lambda\Gamma}(x)\sigma_{\Gamma+\mu}(x)$; (15)) *for* x *in* (s); *in* (e) $(s < e \leqq m)$ $H(x)$ *is a sum of a finite number of formal expressions of the form* (10 a) ($H(x) = \bar{s}'_{\lambda'_{e-}\,\Gamma}(x)\sigma_{\Gamma+\sigma'_1}(x) + \cdots$; *see* (14)). ***Furthermore,*** $H(x)$ *is analytic in* $(s) + \cdots + (m)$.

In this connection, as well as throughout, when we say that a function is asymptotic to a formal expression it is meant that the function is representable by this expression with the power series factors terminated after a sufficiently great number of terms, while in place of the discarded terms expressions are introduced of the form $\frac{b(x)}{x^k}$ ($|b(x)|$, for k fixed, bounded; k arbitrarily great).

In order to obtain an evaluation of

$$y(x) = \mathop{\mathfrak{S}}_{t=x} e^{L(t)} h(t),$$

that is, a solution of

$$y(x+1) - y(x) = e^{L(x)} h(x) \tag{19}$$

consider first the formal equation

$$y(x+1) - y(x) = e^{Q(x)} H_0(x) \;\left(= e^{Q_{\Gamma+\mu,\lambda}(x)} \bar{s}'_{\lambda\Gamma}(x) \sigma_{\Gamma+\mu}(x)\right). \tag{19 a}$$

By a Lemma of Birkhoff, previously quoted, such an equation certainly will possess a formal solution

$$y(x) = e^{Q(x)} s(x), \tag{19 b}$$

where $s(x)$ is an s-series. Let $t(x)$ denote $s(x)$ with the power series factors terminated after m' terms, m' being sufficiently great. Substitute

$$y(x) = e^{Q(x)}\left(t(x) + \frac{z(x)}{x^k}\right) \qquad (k = m'/p) \tag{20}$$

in (19). The new variable $z(x)$ will satisfy the equation

$$\begin{gathered} q(x+1)z(x+1) - q(x)z(x) = F(x) \qquad (q(x) = e^{Q(x)}x^{-k}), \\ F(x) = e^{L(x)}h(x) - \Delta e^{Q(x)}t(x) = F_1(x) + F_2(x), \\ F_1(x) = e^{L(x)}h(x) - e^{Q(x)}H'_0(x), \quad F_2(x) = e^{Q(x)}H'_0(x) - \Delta e^{Q(x)}t(x) \end{gathered} \tag{20 a}$$

where $H'_0(x)$ is $H_0(x)$ with the power series factors terminated after a suitable number of terms. The function $H'_0(x)$ is analytic in $(s) + \cdots + (m)$ and

$$\begin{gathered} H'_0(x) \sim H_0(x) \ \ (= \bar{s}'_{\lambda\Gamma}(x)\sigma_{\Gamma+\mu}(x)) \\ (x \text{ in } (s) + \cdots + (m)); \end{gathered} \tag{20 b}$$

hence

$$F_2(x) = \frac{e^{Q(x)}\beta_2(x)}{x^{k''}} \tag{20 c}$$

where $\beta_2(x)$ is analytic and bounded in $(s) + \cdots + (m)$, and k'' can be made arbitrarily great. On the other hand, for x in (s),

$$F_1(x) = \frac{e^{Q(x)}\beta_1(x)}{x^{k'}} \ \ (|\beta_1(x)| \leqq \beta_1)^{1} \tag{20 d}$$

where k' is arbitrarily great.

Writing

$$\underset{t=x}{\mathfrak{S}} F(t) = \underset{t=x}{\mathfrak{S}} F_1(t) + \underset{t=x}{\mathfrak{S}} F_2(t) \tag{21}$$

we evaluate $\underset{t=x}{\mathfrak{S}} F_2(t)$, precisely as in § 4 by means of a contour integral with

[1] The same remark can be made to hold for $\beta_1(x)$, $\beta_2(x)$ as previously made regarding $\beta(x)$ of (3 a; § 4).

path of integration in (m) near the right boundary of (m). Thus $\underset{t=x}{\mathfrak{S}} F_2(t)$ may be considered to be a known function, analytic in $(s) + \cdots + (m)$ such that

$$\underset{t=x}{\mathfrak{S}} F_2(t) = \frac{e^{Q(x)}\beta'_2(x)}{x^{k''_2}} \tag{21 a}$$

$$(|\beta'_2(x)| \leqq \beta'_2 \text{ in } (s) + \cdots + (m);\ k''_2 \to \infty \text{ as } k \to \infty).$$ [1]

Let x be in (s). Write, in accordance with § 4 (Cf. (5)),

$$\underset{t=x}{\mathfrak{S}} F_1(t) = \int_{L_x} \frac{e^{2\pi\sqrt{-1}\lambda(x-t)} F_1(t)dt}{1 - e^{2\pi\sqrt{-1}(x-t)}} \tag{22}$$

where L_x is a contour, formed as in § 4, with the constituent part L situated in (m) near the right boundary of (m). The integer λ is determined depending on the order (with respect to $|e^{2\pi\sqrt{-1}t}|$) along L, of the exponential factor of $F_1(t)$. Existence of such an order may be ascertained as follows. We have $F_1(t) = e^{L(t)}h(t) - e^{Q(t)}H'_0(t)$. A strip V of limited width can certainly be found, in (m), near the right boundary of (m) so that throughout V either $\Re L(t) \geqq \Re Q(t)$ or $\Re Q(t) \geqq \Re L(t)$. Confine L to V. Taking account of the statement in italics, following (10), and noting that by hypothesis $Q(t)$ $(= Q_{r+\mu,\lambda}(t))$ is proper in V (since the proper curve F may be supposed to be to the left of V) it is concluded that in V the exponential factor of $F_1(t)$ has a definite order with respect to $|e^{2\pi\sqrt{-1}t}|$.

By (17)

$$\Re L(x') - \Re Q(x) < -\xi = \xi_1$$
$$(\Im x = \Im x';\ \Re x < \Re x';\ x \text{ in } (s);\ x' \text{ in } (s) + \cdots + (m))$$

where ξ_1 is small (see 17 c). Also, by (15 a),

$$\Re Q(x') - \Re Q(x) < 0$$
$$(\Im x = \Im x';\ \Re x < \Re x';\ x \text{ in } (s);\ x' \text{ in } (s) + \cdots + (m)).$$

As $\Re(x' - x)$ increases, these differences diminish sufficiently rapidly to secure, in virtue of (20 d), the following relation

[1] Here $\beta'_2(x)$ is of the same nature as $\beta_1(x)$ and $\beta_2(x)$.

$$\underset{t=x}{\mathfrak{S}} F_1(t) = \frac{e^{Q(x)} \beta'_1(x)}{x^{k'_1}} \tag{22 a}$$

$$(|\beta'_1(x)| \leqq \beta'_1 \text{ in } (s);\ k'_1 \to \infty \text{ as } k \to \infty).$$

Thus $\underset{t=x}{\mathfrak{S}} F(t)$ may be considered as known and analytic in (s) and such that

$$\underset{t=x}{\mathfrak{S}} F(t) = \frac{e^{Q(x)} \beta(x)}{x^{\bar{k}}} \tag{23}$$

$$(|\beta(x)| \leqq \beta \text{ in } (s);\ \bar{k} \to \infty \text{ as } k \to \infty).$$

A solution $z(x)$ of (20 a) may be given as follows

$$z(x) = x^k e^{-Q(x)} \underset{t=x}{\mathfrak{S}} F(t) \tag{24}$$

where $\underset{t=x}{\mathfrak{S}} F(t)$ is given by (23). Thus, this solution may be considered to be a known function,

$$z(x) = x^{k-\bar{k}} \beta(x) \qquad (|\beta(x)| \leqq \beta \text{ in } (s)), \tag{24 a}$$

analytic in (s). Using this determination of $z(x)$ and (20) we obtain an evaluation of $\underset{t=x}{\mathfrak{S}} e^{L(t)} h(t)$ as a function analytic in (s) such that, to $m(k)$ terms

$$\underset{t=x}{\mathfrak{S}} e^{L(t)} h(t) \sim e^{Q(x)} s(x) \qquad (x \text{ in } (s)) \tag{25}$$

where $s(x)$ is the proper series of (19 b) and $m(k) \to \infty$ as $k \to \infty$.

The integer λ in (22) we define as follows. Let $e^{V(t)}$ be the exponential factor of $F_1(t)$ (for t in V). If $s < m$ let λ be the greatest integer such that, as $|t| \to \infty$ in V,

$$2\pi(\lambda - 1)v + \Re V(t) \to -\infty \qquad (v = \Im t). \tag{25 a}$$

If $s = m$ let λ be the least integer such that

$$2\pi\lambda v + \Re V(t) \to +\infty. \tag{25 b}$$

By (25 a), (25 b) it follows that, for k fixed, (25) holds to infinity of terms (Cf. § 4).

As a consequence of this method of summation and of the asymptotic forms of the $y_{1\lambda}^s(x)$ $(\lambda = 1, \ldots \Gamma)$, known in (s), by (7) it follows that, for x in (s),

$$y_{\Gamma+\mu}^s(x) \sim e^{Q_{\Gamma+\mu}(x)}. \quad s\text{-series.} \tag{26}$$

The second member of (26) is apparently a formal series solution; it has the same exponential factor and, on that account, necessarily essentially the same s-series factor as in the, originally known, formal series solution $e^{Q_{\Gamma+\mu}(x)} s_{\Gamma+\mu}(x)$. This construction can be effected for $\mu = 1, \ldots n - \Gamma$ and for $s = 1, \ldots m$. *Thus, the operator $L_n(y)$ is proper.*

§ 8. Completion of the Proof of the Theorem of § 7.

It remains to prove that $L_n(y)$ is completely proper; i. e., that the periodic functions, connecting the m proper matrix solutions of the system ((6); § 1), related to (1 a; § 7),

$$Y^s(x) = (y_{ij}^s(x)) = (y_j^s(x + i - 1)) \tag{1}$$
$$(i, j = 1, \ldots n; \; s = 1, \ldots m),$$

are proper. We have

$$Y^s(x) \sim S(x) = (e^{Q_j(x)} s_{ij}(x)) = (e^{Q_j(x+i-1)} s_j(x + i - 1)) \tag{1 a}$$
$$(x \text{ in } (s); \; s = 1, \ldots m; \; i, j = 1, \ldots n);$$

$$Y^s(x) = Y^{s+1}(x) G^s(x), \quad G^s(x) = (y_{ij}^s(x)) \tag{1 b}$$
$$(s = 1, \ldots m - 1; \; G^s(x + 1) = G^s(x)).$$

Accordingly, for $s < m$,

$$Y^s(x) = Y^m(x) G(x) \quad (G(x) = (g_{ij}(x))),$$
$$G(x) = G^{m-1}(x) \ldots G^s(x). \tag{2}$$

It will be proved first that the $g_{ij}(x)$ are proper periodic functions.

Consider $\underset{t=x}{\mathfrak{S}} F_1(t)$, as given by (22; § 7). This is a function analytic in $(s) + \cdots + (m)$; in (s) it is given by (22 a; § 7). Let x be in the strip V (§ 7) to the left of L and, of course, not nearly congruent to L (when a position of congruency is approached L is suitably shifted). Taking account of the fact that $Q(x)$ and $L(x)$ [see statement in italics following (10; § 7)] are proper (Def. 3; § 1), in V, and that $F_1(t)$ is given by (20a; § 7), it is concluded that, for x in V, the exponential factor of

$$\int_{L_x} \frac{e^{2\pi\sqrt{-1}\lambda(x-t)} F_1(t)\,dt}{1 - e^{2\pi\sqrt{-1}(x-t)}}$$

is comparable with $|e^{2\pi\sqrt{-1}x}|$. Taking account of this fact, of (21a; § 7) and of (21; § 7) it is observed that the exponential factor of $z(x)$ (see (24; § 7)) has the same property for x in V.

Hence the function $y(x)$, as given by (20; § 7), that is,

$$y(x) = \underset{t=x}{\mathfrak{S}} \bar{y}^s_{\lambda\Gamma}(t+1) z^s_{\Gamma+\mu}(t) \qquad (1 \leqq \lambda \leqq \Gamma;\ 1 \leqq \mu \leqq n - \Gamma)$$

has its exponential factor comparable with $|e^{2\pi\sqrt{-1}x}|$ for x in V. This function is analytic in $(s) + \cdots + (m)$ (to the left of L).

Consideration of (7a; § 7; $e = m$), (7c; § 7; $e = m$) and the fact that the $Q_{ij}(x)$ $(i, j = 1, \ldots n)$ are proper in V make it clear that each of the functions

$$y^s_\lambda(x)\ (= y^s_{1\lambda}(x)) \qquad (\lambda = 1, \ldots \Gamma;\ \text{see } (7;\ \S\ 7))$$

has its exponential factor comparable with $|e^{2\pi\sqrt{-1}x}|$ for x in V. Consequently, an element

$$y^s_{\Gamma+\mu}(x) \qquad (1 \leqq \mu \leqq n - \Gamma;\ (7;\ \S\ 7))$$

will have the same property.

Therefore

$$Y^s(x) \sim S^*(x) \qquad (x \text{ in } V) \tag{3}$$

where the elements of the matrix $S^*(x)$ are formal series with exponential factors comparable with $|e^{2\pi\sqrt{-1}x}|$ for x in V. On the other hand,

$$Y^m(x) \sim S(x)$$

for x in (m) and, in particular, in V. Hence, by (2),

$$G(x) \sim S^{-1}(x)\, S^*(x) = \Gamma(x) \qquad (x \text{ in } V).$$

The exponential factors of the elements of the matrix $S^{-1}(x)$ are the $e^{Q_{ji}(x)}$ and hence are comparable, in V, with $|e^{2\pi\sqrt{-1}x}|$. Consequently the same will be true of the exponential factors in the formal matrix $\Gamma(x)$.

The elements of $G(x)$ are analytic in an upper half plane and, by what precedes, they are of the asymptotic form

$$G(x) = (g_{ij}(x)) \sim (e^{2\pi\sqrt{-1}\, g_{ij}x}\, g_{ij}^{*}) \tag{4}$$

$$(\text{the } g_{ij}, \text{ integers; } g_{ij}^{*}, \text{ constants; } \Im x \geqq \varrho > 0)$$

as in (Def. 5; § 1). Thus the $g_{ij}(x)$ *are proper periodic functions.*

Write for this matrix $G(x)$

$$G(x) = G^{m\,s}(x) = (g_{ij}^{m,s}(x)).$$

The matrices $G^{m\,s}(x)$ $(s = 1, \ldots m-1)$ all consist of proper periodic functions. A matrix $G^{s}(x)$ $(1 \leqq s \leqq m-1)$, occurring in (1 a) is representable as

$$G^{s}(x) = (g_{ij}^{m,\,s+1}(x))^{-1}\,(g_{ij}^{m\,s}(x)) \qquad (s = 1, \ldots m-2). \tag{4 a}$$

The determinant $|(g_{ij}^{m,\,s+1}(x)|$ could not vanish since otherwise at least one of the set of determinants

$$|G^{m-1}(x)|,\ |G^{m-2}(x)|, \ldots\ |G^{s+1}(x)|$$

would vanish. This, in virtue of (1 b), would imply that not all the matrices $Y^{\sigma}(x)$ $(\sigma = 1, \ldots m)$ are fundamental matrix solutions. Thus, the elements of the matrix $(g_{ij}^{m,\,s+1}(x))^{-1}$ are proper periodic functions. The same will be true for the elements of the product (4 a). The matrix $G^{m-1}(x)$ is really $G^{m,\,m-1}(x)$, and its elements are also proper periodic functions. It is seen, then, that the elements of each of the matrices $G^{s}(x)$ $(s = 1, \ldots m-1)$ are of this type. Hence $L_n(y)$ is completely proper, and the Theorem is proved.

An application of Theorem I (§ 5) and of the methods of § 5 will yield the following Lemma.

Lemma 10. *Suppose that the conditions of Theorem II (§ 7) hold. Assume, moreover, that the coefficients in $L_n(y)$ are known and of the right kind (see § 1) not only in $(1) + \cdots + (m)$ but also in a more extensive subregion of Γ,*

$$(1) + \cdots + (m) + \cdots + (\eta) \qquad (\eta > m).$$

It will necessarily follow that $L_n(y)$ is completely proper in $(1) + \cdots + (\eta)$.

§ 9. The Fundamental Existence Theorem.

The results of the preceding sections have prepared the way for the proof of the main result of this paper. This result is embodied in the Theorem.

Fundamental Theorem. *Every equation $L_n(y) = 0$ (or system), with coefficients of the kind specified in § 1 and known in the complete neighborhood of infinity, is completely proper in each of the several quadrants associated with the equation (or system).*[1]

Proof. It is sufficient to give the proof for some quadrant. Let Γ be this quadrant. We may assume the implication of the statement in italics preceding Lemma 1 (§ 2). The methods so far developed indicate how to meet the situation when the condition of that statement does not hold. In fact, it is sufficient, whenever necessary, to replace a 'multiple' B' curve by several corresponding 'simple' B' curves, 'parallel' to each other. This is always possible since, as stated before, B' curves are to be considered to be determined except for a translation.

Certain terms, embodied in the following definition, will be found convenient.

Definition 10. *A set*

$$Q_1(x), \quad \ldots \; Q_n(x),$$

which has a point of division in a region G between the Γ-th and $\Gamma + 1$-st elements of the set, will be said to have a point of separation, in G, if

$$\Re\, Q'_\lambda(x) > \Re\, Q'_{\Gamma+\mu}(x)$$
$$(\lambda = 1, \; \ldots \Gamma; \; \mu = 1, \; \ldots n - \Gamma; \; x \text{ in } G).$$

The Q-factorization corresponding to a point of separation will be called Q^*-factorization.

We note also the following simple fact. If $\Re\, Q'_i(x) \not\equiv \Re\, Q'_j(x)$, while interior to G (along some curve) $\Re\, Q'_i(x) = \Re\, Q'_j(x)$, then in some portion of G

$$\Re\, Q'_i(x) > \Re\, Q'_j(x),$$

[1] A similar result will hold, for more restricted regions, when the coefficients of $L_n(y)$ (or those involved in a system) are of the right kind (§ 1) in certain portions of the plane only.

while in another part of G

$$\Re\, Q_i'(x) < \Re\, Q_j'(x).$$

If the theorem is not true, there are equations $L_n(y) = 0$ $(n \geqq 2)$ of least order n for which the theorem fails in Γ. Let

$$L_n(y) = 0 \tag{1}$$

be an equation of this kind. If the set of $Q(x)$'s

$$Q_1(x), \ \ldots \ Q_n(x), \tag{1 a}$$

belonging to this equation, is such that $\Re\, Q_1'(x) \equiv \Re\, Q_2'(x) \equiv \cdots \equiv \Re\, Q_n'(x)$ the equation will be seen to be completely proper in Γ (see Theorem I; § 5). Hence not all the $\Re\, Q_j'(x)$ $(j = 1, \ldots n)$ are identical.

Suppose now that in Γ the set (1 a) has a point of division. Necessarily there will be a point of separation in Γ. By Lemma 9 (§ 6) there will be a factorization

$$L_n(y) \equiv L_{n-\Gamma}\, L_\Gamma(y) = 0 \qquad (1 \leqq \Gamma < n;\ x \text{ in } \Gamma) \tag{1 b}$$

corresponding to the point of separation. In Γ, near enough to the positive axis of imaginaries, there is a curve F which is proper (Def. 9; § 1) for the set (1 a).[1] Now, the operators $L_{n-\Gamma}(z)$, $L_\Gamma(y)$ are completely proper in Γ, since $n - \Gamma < n$ and $\Gamma < n$. Theorem II (§ 7) is applicable and, consequently, $L_n(y)$ is completely proper in Γ. Hence there can be no point of division in Γ.

Accordingly, assume that the set (1 a) has a point of division in $(1) + \cdots + (m)$ (a subregion of Γ) and has no point of division in $(1) + \cdots + (m + 1)$.[2] Necessarily there can be only one point of division in $(1) + \cdots + (m)$. Since the $\Re\, Q_j'(x)$ $(j = 1, \ldots n)$ are not all identical this point of division is necessarily a point of separation in $(1) + \cdots + (m)$. $L_n(y)$ is correspondingly Q^*-factorable. Let

$$L_n(y) \equiv L_{n-\Gamma}\, L_\Gamma(y) = 0 \qquad (1 \leqq \Gamma < n;\ x \text{ in } (1) + \cdots + (m)) \tag{2}$$

be the corresponding factorization according to Lemma 9 (§ 6). By Lemma 3

[1] This will be true, of course, for any set of $Q(x)$'s.

[2] Incidentally, this would mean that $\Re\gamma_1 = \cdots = \Re\gamma_n$ and $\mu_1 = \mu_2 = \cdots = \mu_n$.

(§ 2) there exists a curve F, in (m), which is proper for the set (1 a). Suppose that $L_{n-r}(z)$, $L_r(y)$ are both completely proper in $(1) + \cdots + (m)$. By Theorem II (§ 7) it would then follow that $L_n(y)$ is completely proper in $(1) + \cdots + (m)$. Furthermore, by Lemma 10 (§ 9), $L_n(y)$ would be completely proper in Γ. Hence at least one of the above two factors is not completely proper in $(1) + \cdots + (m)$. Let it be denoted by L^1.

Consider the equation

$$L^1(y) = 0.$$

Its coefficients are known in $(1) + \cdots + (m)$. Now, $L^1(y)$ is not Q-factorable in $(1) + \cdots + (m)$. There exists an integer m_1, $2 \leqq m_1 < m$, such that $L^1(y)$ is Q-factorable (and, of course, correspondingly factorable) in $(1) + \cdots + (m_1)$ and is not Q-factorable in $(1) + \cdots + (m_1 + 1)$.[1] This Q-factorization of $L^1(y)$ is necessarily unique in $(1) + \cdots + (m_1)$. The $\Re Q'(x)$ belonging to $L^1(y)$ cannot, of course, be all identical. Hence this Q-factorization is a Q^*-factorization. By Lemma 3 (§ 2) the set of $Q(x)$'s belonging to $L^1(y)$ will be proper (Def. 3; § 1) to the right of a curve F_1, lying in (m_1). If both factors of $L^1(y) = 0$, known in $(1) + \cdots + (m_1)$ and corresponding to the Q^*-factorization, were completely proper in $(1) + \cdots + (m_1)$ it would follow by Theorem II (§ 7) that $L^1(y)$ is completely proper in $(1) + \cdots + (m_1)$. By Lemma 10 (§ 8) $L^1(y)$ will be completely proper in $(1) + \cdots + (m)$. Hence at least one of the two factors of $L^1(y)$ is not completely proper in $(1) + \cdots + (m_1)$. Denote it by L^2.

The equation

$$L^2(y) = 0,$$

with coefficients known in $(1) + \cdots + (m_1)$ is not Q-factorable in $(1) + \cdots + (m_1)$. There exists an integer m_2, $2 \leqq m_2 < m_1$, such that $L^2(y) = 0$ is Q-factorable in $(1) + \cdots + (m_2)$ and is not Q-factorable in $(1) + \cdots + (m_2 + 1)$. By the reasoning applied to $L^1(y) = 0$, previously, it is shown that of the two factors of $L^2(y)$, which correspond to the Q-factorization (necessarily Q^*-factorization), at least one is not completely proper in $(1) + \cdots + (m_2)$. Denote this factor by L^3.

Continuing the indicated process, we obtain a sequence of integers m_i

$$2 \leqq \cdots < m_i < \cdots < m_1 < m$$

and a sequence of equations $L^i(y) = 0$ such that the following conditions hold.

[1] An equation which is not Q-factorable might be factorable (in the usual sense).

1°. $L^i(y)$ is not completely proper in $(1) + \cdots + (m_{i-1})$ $(m_0 = m)$.

2°. $L^i(y)$ is Q^*-factorable in $(1) + \cdots + (m_i)$ and is not Q-factorable in $(1) + \cdots + (m_i + 1)$.

For a certain i $(= i')$ we have $m_{i'} = 2$. Correspondingly there will exist an equation $L^{i'}(y) = 0$ with coefficients of the right kind (Cf. § 1) in $(1) + \cdots + (m_{i'-1})$ $(2 < m_{i'-1})$, Q^*-factorable in $(1) + (2)$ and not Q-factorable in $(1) + (2) + (3)$. By Lemma 9 (§ 6) there will be a corresponding factorization

$$L^{i'}(y) \equiv L_{k-\Gamma} L_\Gamma(y) = 0$$
$$(1 \leqq \Gamma < k;\ x \text{ in } (1) + (2)).$$

Necessarily the factors will be of order one and two. The B' curve, separating (1) from (2), will correspond to the two $Q(x)$'s belonging to the factor of order two; that is, the subscripts associated with this curve will be those of the two mentioned $Q(x)$'s. Hence, in view of Lemma 1 (§ 2), Theorem I (§ 5) will certainly be applicable to the factor of order two. Thus the two factors are completely proper in $(1) + (2)$. Now $L^{i'}(y)$ is Q^*-factorable in $(1) + (2)$ and is not Q-factorable in $(1) + (2) + (3)$. Hence, by Lemma 3 (§ 2), the set of $Q(x)$'s belonging to $L^{i'}(y)$ is proper to the right of a curve F, lying in (2). By Theorem II (§ 7) the operator $L^{i'}(y)$ will be completely proper in $(1) + (2)$. Moreover, in virtue of Lemma 10 (§ 8), $L^{i'}(y)$ will be seen to be completely proper in $(1) + \cdots + (m_{i'-1})$. We have thus arrived at a contradiction. Thus, the Theorem has been proved for Γ.

For the several other quadrants (below the axis of reals, to the right of the axis of imaginaries, and for various ranges of $\arg x$) the demonstration would be entirely analogous and structurally identical with the one just given. Thus the Theorem is seen to be true.

It is essential to note that given any particular equation the preceding sections give actual methods for construction of those solutions and of those periodic functions whose existence has been established in the Fundamental Theorem.

If we consider two adjacent quadrants with a common strip V along, say, the positive axis of imaginaries it is noted that the two proper sets of solutions corresponding to the two overlapping sub-regions of the quadrants are connected by proper periodic functions. This is seen to hold because every set of $Q(x)$'s is proper in such a strip. In this sense, *every equation* $L_n(y) = 0$ (*or system*), *with coefficients of the right kind in the complete neighborhood of infinity, is completely proper in each of the several upper and lower half-planes associated with the equation.*

§ 10. Connection between 'upper' and 'lower' Solutions.

Consider two adjacent quadrants Γ and Ω above and below the negative axis of reals, respectively. We shall write

$$\Gamma = (1) + (2) + \cdots, \quad \Omega = [1] + [2] + \cdots$$

(see § 3). Let K denote the combined region $(1) + [1]$; this region extends indefinitely upwards and downwards from the negative axis of reals. It may happen that the negative axis of reals is a B' curve. In any case, (1) and [1] may be considered as overlapping along a strip H:

$$-\varrho \leqq \Im x \leqq \varrho \ (\varrho > 0), \ |x| \geqq g > 0.$$

An ordering

$$\Re Q_1'(x) \geqq \cdots \geqq \Re Q_n'(x) \tag{1}$$

will be maintained in K, if the negative axis of reals is not a B' curve. In the contrary case assume this ordering in the region (1) down including the negative axis. The lower boundary of (1), h, will consist of a portion ot the line $\Im x = -\varrho$. The upper boundary of [1], h^*, will consist of a portion of the line $\Im x = \varrho$.

Let $Y^u(x)$ be a matrix solution, consisting of elements analytic in (1), such that

$$Y^u(x) = (y_{ij}^u(x)) \sim S(x) = (e^{Q_j(x)} s_{ij}(x)) \tag{1 a}$$
$$(i, j = 1, \ldots n; \ x \text{ in } (1)).$$

Let $Y^l(x)$ be a matrix solution, with elements analytic in [1], such that

$$Y^l(x) = (y_{ij}^l(x)) \sim S(x). \qquad (x \text{ in } [1]) \tag{1 b}$$

The matrix $P(x) (= (p_{ij}(x)))$ of periodic functions, defined by the relation

$$Y^u(x) = Y^l(x) P(x), \tag{2}$$

consists of elements analytic in H. By (1 a) and (1 b) it follows that

(2 a) $$(p_{ij}(x)) = \left(e^{Q_{ji}(x)}\left(\delta_{ij} + \frac{\varrho_{ij}(x)}{x^k}\right)\right),$$

where k can be made arbitrarily great while the $\varrho_{ij}(x)$ are bounded in H.

Along the negative axis of reals the ordering (1) holds, that is

$$\Re\, Q'_{ji}(x) \geqq 0. \qquad (j < i)$$

Hence $\Re\, Q_{ji}(x)$ $(j < i)$ is non increasing as $|x| \to \infty$ along the negative axis. In virtue of (2 a) and the periodicity of the $p_{ij}(x)$ we shall have

$$p_{ij}(x) = \lim_{|x'| \to \infty} p_{ij}(x') = 0$$

$$(\Im\, x = \Im\, x' = 0;\ \Re\, x > \Re\, x';\ x - x',\ \text{integer})$$

for $i > j$; on the other hand,

$$p_{ii}(x) = \lim_{|x'|} p_{ii}(x') = 1 \qquad (i = 1, \ldots n).$$

Since the $p_{ij}(x)$ are analytic it would follow that

(2 b) $$p_{ij}(x) \equiv 0 \qquad (i > j),$$

$$p_{ii}(x) \equiv 1 \qquad (i = 1, \ldots n).$$

Definition 11. *A matrix $(h_{ij}(x))$ will be termed a half matrix if $h_{ij}(x) \equiv 0$ $(i > j)$, while $h_{ii}(x) \equiv 1$ $(i = 1, \ldots n)$.*

Use the transformation

$$z = e^{2\pi\sqrt{-1}\,x}.$$

Write

(2 c) $$P(x) \equiv G(z) \equiv (g_{ij}(z)).$$

The $g_{ij}(z)$ are analytic for $e^{-2\pi\varrho} \leqq |z| \leqq e^{2\pi\varrho}$. Letting c_1 denote the circle $|z| = e^{-2\pi\varrho}$ and c_2 denote the circle $|z| = e^{2\pi\varrho}$ the following will hold for any function $g(z)$ analytic for $e^{-2\pi\varrho} \leqq |z| \leqq e^{2\pi\varrho}$.

(3) $$g(z) = a(z) + b(z),$$

$$a(z) = \frac{1}{2\pi\sqrt{-1}} \int_{c_2} \frac{g(\zeta)\,d\zeta}{\zeta - z}, \quad b(z) = \frac{1}{2\pi\sqrt{-1}} \int_{c_1} \frac{g(\zeta)\,d\zeta}{\zeta - z}$$

where c_2 is described in the counter clockwise direction and c_1 is described in the clockwise direction. The function $a(z)$ will be analytic interior to c_2, while $b(z)$ will be analytic exterior to c_1.

We seek to determine half matrices $B(z)\,(=(b_{ij}(z)))$ and $A(z)\,(=((a_{ij}(z)))$, with the $b_{ij}(z)$ analytic exterior to c_1^* and the $a_{ij}(z)$ analytic interior to c_2^*, so that

$$P(x) \equiv G(z) = B(z)\,A(z); \tag{4}$$

here the radius of the circle c_1^* is to be slightly greater than $e^{-2\pi\varrho}$ while the radius of the circle c_2^* is to be slightly less than $e^{2\pi\varrho}$.

From (4) we have

$$g_{ij}(z) = \sum_{\lambda=1}^{n} b_{i\lambda}(z)\,a_{\lambda j}(z). \tag{4 a}$$

Let $i > j$. In the second member $b_{i\lambda}(z) = 0$ for $i > \lambda$ so that $\sum_{\lambda=1}^{n} = \sum_{\lambda=i}^{n}$. In the latter sum the subscripts of the $a_{\lambda j}(z)$ satisfy the inequalities

$$\lambda \geqq i > j;$$

thus these $a_{\lambda j}(z)$ are zero. Hence (4 a) is satisfied for $i > j$. For $i = j$ we should have

$$1 = \sum_{\lambda=1}^{n} b_{j\lambda}(z)\,a_{\lambda j}(z) \qquad (i = 1, \ldots n). \tag{4 b}$$

The equations (4 b) are obviously satisfied since $b_{i\lambda}(z) = 0$ $(i > \lambda)$, $a_{\lambda i}(z) = 0$ $(\lambda > i)$ while $b_{ii}(z) = a_{ii}(z) = 1$. It remains to consider (4 a) for $i < j$. These equations take the form

$$g_{ij}(z) = \sum_{\lambda=i}^{j} b_{i\lambda}(z)\,a_{\lambda j}(z) \quad (i < j;\ i, j = 1, \ldots n). \tag{4 c}$$

They will be grouped, for $\sigma = 1, 2, \ldots$, as follows

$$g_{i,i+\sigma}(z) = \sum_{\lambda=i}^{i+\sigma} b_{i\lambda}(z)\,a_{\lambda, i+\sigma}(z) \qquad (i = 1, \ldots n - \sigma). \tag{5}$$

An equation

$$g_{i,i+1}(z) = a_{i,i+1}(z) + b_{i,i+1}(z)$$

of the set (5; $\sigma = 1$) will be solved by letting

(5 a)
$$a_{i,i+1}(z) = \frac{1}{2\pi\sqrt{-1}} \int_{c_2} \frac{g_{i,i+1}(\zeta)\,d\zeta}{\zeta - z},$$

$$b_{i,i+1}(z) = \frac{1}{2\pi\sqrt{-1}} \int_{c_1} \frac{g_{i,i+1}(\zeta)\,d\zeta}{\zeta - z}.$$

These are functions of the desired type. The function

(5 b)
$$g^2_{i,i+2}(z) \equiv g_{i,i+2}(z) - b_{i,i+1}(z)\,a_{i+1,i+2}(z)$$

is consequently known and analytic in a closed Laurent ring (c_1^2, c_2^2) slightly interior to the ring (c_1, c_2). The set (5; $\sigma = 2$) can be written in the form

(5 c)
$$g^2_{i,i+2}(z) = a_{i,i+2}(z) + b_{i,i+2}(z)$$
$$(i = 1, \ldots n-2).$$

Solutions of this set, of the desired kind, are obtained by writing

(5 d)
$$a_{i,i+2}(z) = \frac{1}{2\pi\sqrt{-1}} \int_{c_2^2} \frac{g^2_{i,i+2}(\zeta)\,d\zeta}{\zeta - z},$$

$$b_{i,i+2}(z) = \frac{1}{2\pi\sqrt{-1}} \int_{c_1^2} \frac{g^2_{i,i+2}(\zeta)\,d\zeta}{\zeta - z}.$$

Suppose that desired solutions of (5) have been obtained for $\sigma = 1, 2, \ldots m-1$, with reference to a Laurent ring (c_1^{m-1}, c_2^{m-1}), slightly interior to the ring (c_1, c_2). As a consequence, the function

(6)
$$g^m_{i,i+m}(z) \equiv g_{i,i+m}(z) + [b_{i,i+1}(z)\,a_{i+1,i+m}(z) + b_{i,i+2}(z)\,a_{i+2,i+m}(z) + \cdots + b_{i,i+m-1}(z)\,a_{i+m-1,i+m}(z)]$$

will be known and analytic in a closed ring (c_1^m, c_2^m), interior to (c_1^{m-1}, c_2^{m-1}). The set (5; $\sigma = m$) can then be written in the form

(6 a)
$$g^m_{i,i+m}(z) = a_{i,i+m}(z) + b_{i,i+m}(z)$$
$$(i = 1, \ldots n-m)$$

and can be solved by writing

(6 b)
$$a_{i,i+m}(z) = \frac{1}{2\pi\sqrt{-1}} \int\limits_{c_2^m} \frac{g_{i,i+m}^m(\zeta)\, d\zeta}{\zeta - z},$$

$$b_{i,i+m}(z) = \frac{1}{2\pi\sqrt{-1}} \int\limits_{c_1^m} \frac{g_{i,i+m}^m(\zeta)\, d\zeta}{\zeta - z}.$$

Thus the matrix equation (4) may be considered solved as required. The elements of $A(z)$ will be analytic for $|z| \leqq e^{2\pi c}$ and those of $B(z)$ will be analytic for $|z| \geqq e^{-2\pi c}$ $(\varrho > c > 0)$. In virtue of (6 b) the $\alpha_{ij}(x)$ $(\equiv a_{ij}(z))$ are of period unity, analytic for $\Im x \geqq -c$ and of the form

(7)
$$\alpha_{ij}(x) = e^{2\pi\sqrt{-1}\,\alpha_{ij}\,x}\, \alpha_{ij}^* + \cdots$$
$$(\alpha_{ij}, \text{ integer}, \geqq 0;\ \Im x \geqq -c)$$

where the second member is a convergent series in positive powers of $e^{2\pi\sqrt{-1}\,x}$. Similarly,

(7 a)
$$\beta_{ij}(x) = e^{-2\pi\sqrt{-1}\,\beta_{ij}\,x}\, \beta_{ij}^* + \cdots$$
$$(\beta_{ij}, \text{ integer}, \geqq 1;\ \Im x \leqq c)$$

where the second member is a series in positive powers of $e^{-2\pi\sqrt{-1}\,x}$. Consequently, the $\alpha_{ij}(x)$ are proper in an upper half plane, and the $\beta_{ij}(x)$ are 'proper in a lower half plane'.[1] That is, in suitable regions

(8)
$$\alpha_{ij}(x) \sim e^{2\pi\sqrt{-1}\,\alpha_{ij}\,x}\, \alpha_{ij}^*$$
$$(\alpha_{ij}, \text{ integer}, \geqq 0;\ \Im x \geqq -c;\ \alpha_{ii} = 0;\ \alpha_{ii}^* = 1;\ i, j = 1, \ldots n),$$

(8 a)
$$\beta_{ij}(x) \sim e^{-2\pi\sqrt{-1}\,\beta_{ij}\,x}\, \beta_{ij}^*$$
$$(\beta_{ij}, \text{ integer}, \geqq 1;\ \Im x \leqq c;\ \beta_{ii} = 0;\ \beta_{ii}^* = 1;\ i, j = 1, \ldots n).$$

Besides, $(\alpha_{ij}(x))$, $(\beta_{ij}(x))$ are half matrices. It can be easily shown that $(\alpha_{ij}(x))^{-1}$ $(\equiv ((\bar{\alpha}_{ij}(x)))$ also is a half matrix, with elements analytic for $\Im x \geqq -c$ and proper in an upper half plane;

---

[1] The meaning of the latter term is made obvious by analogy to (Def. 5: § 1), which was given for an upper half plane.

(8 b)
$$\alpha_{ij}(x) \sim e^{2\pi\sqrt{-1}\bar{\alpha}_{ij}x}\bar{\alpha}^*_{ij}$$
$$(\bar{\alpha}_{ij}, \text{ integer}; \ \Im x \geqq -c; \ \bar{\alpha}_{ii} = 0; \ \bar{\alpha}^*_{ii} = 1).$$

Examine the situation when

(9)
$$\Re\, Q_\sigma(x) = \cdots = \Re\, Q_j(x)$$
$$(\sigma < j; \ x \text{ on negative axis of reals}).$$

In this case, by (2 a), it would follow that along the negative axis, and hence identically,

(9 a)
$$p_{im}(x) = 0 \qquad (i < m; \ \sigma \leqq i, m \leqq j).$$

It will follow that $g_{j-1,j}(z) \ (= g'_{j-1,j}(z)) = 0$ so that, by (5 a),

$$a_{j-1,j}(z) = b_{j-1,j}(z) = 0.$$

If $\sigma \leqq j-2$, by (9 a), $g_{j-2,j}(z) = 0$. The relations

$$g_{j-2,j}(z) = a_{j-1,j}(z) = 0$$

would imply that $g^2_{j-2,j}(z)$ (see 5 b) is zero. This, in turn, would mean that

$$a_{j-2,j}(z) = b_{j-2,j}(z) = 0 \qquad (\text{see (5 d)}).$$

By induction, it could be shown without difficulty that (9) implies the relations

(10)
$$\alpha_{\sigma,j}(x) = \alpha_{\sigma+1,j}(x) = \cdots = \alpha_{j-1,j}(x) \equiv 0,$$
$$\beta_{\sigma,j}(x) = \beta_{\sigma+1,j}(x) = \cdots = \beta_{j-1,j}(x) \equiv 0.$$

These relations would necessarily follow if in place of (9) it were merely assumed that

(10 a)
$$\Re\, Q_{j,j-1}(x), \ \Re\, Q_{j,j-2}(x), \ \ldots \ \Re\, Q_{j,\sigma}(x) \leqq k$$

along the negative axis of reals.

Now, for $s = 1, 2, \ldots$,

(11)
$$\sum_{\lambda=i+1}^{j+s-1} \bar{\alpha}_{i,\lambda}(x)\, \alpha_{\lambda,i+s}(x) + \alpha_{i,i+s}(x) + \bar{\alpha}_{i,i+s}(x) = 0$$
$$(i = 1, \ldots n - s).$$

Suppose (10) holds. By (11; $s = 1$)

(11 a) $$\bar{\alpha}_{i,i+1}(x) = -\alpha_{i,i+1}(x) \qquad (i = 1, \ldots n-1)$$

so that, since $\alpha_{j-1,j}(x) = 0$, we have $\bar{\alpha}_{j-1,j}(x) = 0$. By (11; $s = 2$), for $i = 1, \ldots n-2$,

$$\alpha_{i,i+2}(x) + \bar{\alpha}_{i,i+2}(x) + \bar{\alpha}_{i,i+1}(x)\,\alpha_{i+1,i+2}(x) = 0.$$

Thus

$$\alpha_{j-2,j}(x) + \bar{\alpha}_{j-2,j}(x) + \bar{\alpha}_{j,j-1}(x)\,\alpha_{j-1,j}(x) = 0$$

so that, if in (10) $\sigma \leqq j-2$, necessarily $\bar{\alpha}_{j-2,j}(x) = 0$. By induction it follows easily that the first line in (10) implies that

(12) $$\bar{\alpha}_{0,j}(x) = \bar{\alpha}_{\sigma+1,j}(x) = \cdots = \bar{\alpha}_{j-1,j}(x) \equiv 0.$$

Thus, (10 a) *implies* (10) *and* (12).

Consider the matrix $Z(x)$ $(= z_{ij}(x))$ defined by the relations

(13) $$Y^u(x)\,(\bar{\alpha}_{ij}(x)) = Y^l(x)\,(\beta_{ij}(x)) = Z(x).$$

This is apparently a matrix solution. Its elements are analytic in $K (= (1) + [1])$. This follows by (2) and in virtue of the relation (4),

$$P(x) = (\beta_{ij}(x))\,((\alpha_{ij}(x)).$$

For x in K and $\Im x \geqq -c$, on account of (1 a) and (8 b),

(13 a) $$z_{ij}(x) = e^{Q_j(x)}\left[t_{ij}(x) + \frac{\Gamma^u_{ij}(x)}{x^{k_1}} + r_{ij}(x)\right],$$

(13 b) $$r_{ij}(x) = \sum_{\lambda=1}^{j-1} e^{Q_{\lambda j}(x) + 2\pi\sqrt{-1}\,\bar{\alpha}_{\lambda j}x}\left(t_{i\lambda}(x) + \frac{\Gamma^u_{i\lambda}(x)}{x^k}\right)(\bar{\alpha}^*_{\lambda i} + \cdots).$$

Here $t_{ij}(x)$ denotes $s_{ij}(x)$ with the power series factors terminated after k terms (k sufficiently large); for a fixed k the $\Gamma^u_{ij}(x)$ are bounded in K for $\Im x \geqq -c$, while k_1 can be made arbitrarily great by taking k sufficiently great. The $\Re Q_{\lambda j}(x)$ $(\lambda = 1, \ldots j-1)$ are non-increasing, in virtue of (1), as $x \to \infty$ along the negative axis of reals. In fact, at least for $|x|$ sufficiently large, the inequalities (1) would imply that, for x in K in a strip

(14) $$0 \leqq \Im x \leqq d \qquad (0 < d),$$

wa have

(14 a) $$\Re\,Q_1(x) \leqq \Re\,Q_2(x) \leqq \cdots \leqq \Re\,Q_n(x);$$

moreover, (14) will hold, in K, in a wider strip

(14 b) $$-d \leqq \Im x \leqq d$$

if (1) is maintained throughout K.

If $\Re Q_{j-1,j}(x)$ approaches $-\infty$ along the negative axis, the same will be true of the $\Re Q_{\lambda j}(x)$ $(\lambda = 1, \ldots j-2)$, and, more generally, this would be the situation in a strip (14) or (14 b), as the case may be. Thus, if $\Re Q_{j,j-1}(x)$ is not bounded along the negative axis of reals, all the exponential factors in (13 b) will diminish rapidly enough, as $x \to \infty$ in K in the strip (14), to insure the asymptotic form

(15) $$z_{ij}(x) \sim e^{Q_j(x)} s_{ij}(x)$$

in this strip. If $\Re Q_{j,j-1}(x)$ is not bounded along the negative axis of reals and (1) is maintained in K, (15) will hold in the strip (14 b).

Suppose that the functions

$$\Re Q_{j,j-1}(x),\ \Re Q_{j,j-2}(x),\ \ldots \Re Q_{j,\sigma}(x)$$

are bounded along the negative axis of reals. In view of the statement in italics, following (12), it is clear that (12) will hold. Thus, if $\sigma = 1$,

$$r_{ij}(x) \equiv 0.$$

On the other hand, if $\sigma > 1$

(16) $$r_{ij}(x) = \sum_{\lambda=1}^{\sigma-1} \cdots$$

The exponential factors in the latter sum may be supposed to approach zero in the strip (14 a) (or in the strip (14 b)). Consequently the asymptotic relation (15) will certainly hold in the strip (14), in K. It will be maintained in a strip (14 b) if (1) holds throughout K.

Consider all those curves which lie in Γ and satisfy equations

(17) $$\Re [Q_{ij}(x) + 2\pi \sqrt{-1}\, \bar{\alpha}_{ij} x] = 0 \qquad (i < j)$$

formed for all such i and j that $\Re Q_{ij}(x)$ approaches $-\infty$ along the negative axis of reals. Let B_1^u denote the lowest of these curves. In the case of no curves (17) in Γ we let B_1^u denote the right boundary of Γ. If the limiting direction of B_1^u is that of the negative axis of reals, this curve will be necessarily of the form

$$v = h(-u)^{e_1} + \cdots \qquad (x = u + \sqrt{-1}\, v;\ h > 0;\ 1 > e_1 > 0). \tag{17 a}$$

Let B^u denote some curve in Γ, to the left of B_1^u, with a limiting direction slightly different from that of B_1^u if B_1^u is not of the form (17 a). In the contrary case let B^u satisfy an equation

$$v = h(-u)^{e} \cdot \qquad (e_1 > e > 0). \tag{17 b}$$

In any case B^u is to be in Γ.

We shall have

$$Z(x) \sim S(x) \tag{18}$$

in the closed region K^u consisting of the part of K bounded below by the negative axis of reals and bounded above by the lower one of the two curves B' (upper boundary of K), B^u.

If the ordering (1) is maintained in K the relation

$$Z(x) = Y^l(x)\,(\beta_{ij}(x))$$

may be used to show that (18) holds also in a closed region K^l consisting of the part of K bounded above by the negative axis and bounded below by a curve B^l, lying in Ω, or by the lower boundary of K. The curve B^l is to be considered as determined with reference to the set of equations

$$\Re\,[Q_{ij}(x) - 2\pi\sqrt{-1}\,\beta_{ij}\,x] = 0 \qquad (i < j)$$

just as B^u has been specified on the basis of the equations (17).

Consequently the following has been made evident. *Given matrix solutions* $Y^u(x)$, $Y^l(x)$ *proper in* (1) *and* [1], *respectively, there exists a matrix solution* $Z(x)$, *with elements analytic in* K $(= (1) + [1])$, *such that* $Z(x) = Y^u(x)\,P^u(x) = Y^l(x)\,P^l(x)$. Here $P^u(x)$, $P^l(x)$ *are half matrices consisting of periodic functions proper in an upper and lower half plane, respectively. Moreover,* $Z(x)$ *may be so constructed that* $Z(x) \sim S(x)$ *in a sub-region of* K, *extending from the negative axis upwards* (or downwards) *at least to a curve of form* $v = h(-u)^e$. *In the latter equation* h *is positive* (or negative) *while* $1 > e > 0$. *In particular, if the negative axis is not* B' *curve,* $Z(x) \sim S(x)$ *in a sub-region of* K (*or in* K *itself*), *extending from the negative axis upwards and downwards at least to curves of the form*

$$v = h(-u)^e \qquad (h \text{ real}, \neq 0;\ 1 > e > 0).$$

It is clear that analogous facts will hold with reference to quadrants, above and below, the positive axis, to the right of the axis of imaginaries.

§ 11. The Converse Theorem.

In this section the problem, inverse to the one solved in the Fundamental Theorem, will be considered. The result, in this connection, will be embodied in the theorem.

Theorem III. *Let*

$$(1) \qquad e^{Q_1(x)} s_1(x),\ e^{Q_2(x)} s_2(x), \ldots e^{Q_n(x)} s_n(x)$$

be a linearly independent set of formal series where the $Q_j(x)$ *and the formal s-series* $s_j(x)$ $(j = l, \ldots n)$ *are of the same general description as might occur in connection with a difference equation of order* n. *Let* $R_1, R_2, \ldots R_s, R_{s+1}, \ldots$ *be the set of consecutive regions, formed with reference to the set of* $Q_j(x)$ $(j = l, \ldots n)$ *as on the preceding pages. Let two such consecutive regions have at least a strip of, say, unit width in common.*

Let there be associated with each region R_s $(s = 1, 2, \ldots)$ *a set of* n *functions*

$$(2) \qquad y_1^s(x),\ y_2^s(x), \ldots y_n^s(x)$$

analytic in R_s, *and such that*

$$(2\text{ a}) \qquad y_j^s(x) \sim e^{Q_j(x)} s_j(x) \qquad (j = 1, \ldots n;\ x \text{ in } R_s).$$

Assume that, for $s = 1, 2, \ldots$,

$$(2\text{ b}) \qquad y_j^s(x) = \sum_{\lambda=1}^{n} y_\lambda^{s+1}(x)\, p_{\lambda j}^{s,s+1}(x) \qquad (j = 1, \ldots n)$$

where the $p_{ij}^{s,s+1}(x)$ *are of period one.*

It will necessarily follow that there exists a difference equation of order n

$$(3) \qquad L_n(y) \equiv y(x+n) + a_1(x)\, y(x+n-1) + \cdots + a_n(x)\, y(x) = 0,$$

with coefficients of the same kind in the complete neighborhood of infinity as postu-

lated in § 1, possessing the following properties. The series (1) *will constitute a set of formal solutions of* (3). *Each set* (2) *will be a fundamental set of solutions of* (3).

Proof. Form the determinant

$$(4)\qquad D_n^s(y) \equiv \begin{vmatrix} y(x+n), & y_1^s(x+n), & \ldots y_n^s(x+n) \\ y(x+n-1), & y_1^s(x+n-1), & \ldots y_n^s(x+n-1) \\ \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdots\quad\cdot\;\cdot\;\cdot \\ y(x), & y_1^s(x) & \ldots y_n^s(x) \end{vmatrix}$$

$$\equiv d_0^s(x)\,y(x+n) + d_1^s(x)\,y(x+n-1) + \cdots + d_n^s(x)\,y(x).$$

On account of (2 a) and of the linear independence of the series (1),

$$d_0^s(x) \not\equiv 0, \quad d_n^s(x) \not\equiv 0.$$

The coefficients $d_j^s(x)$ $(j = 0, 1, \ldots n)$ are analytic in R_s. They will be asymptotic, for x in R_s, to the formal series obtained by replacing the elements in the determinants, expressing these coefficients, by corresponding formal series. Thus write

$$(4\text{ a})\qquad d_j^s(x) \sim e^{Q_1(x)+\cdots+Q_n(x)}\,\delta_j(x)$$

$$(j = 0, 1, \ldots n;\; x \text{ in } R_s)$$

where $\delta_j(x)$ is an s-series. Now the logarithms in the s-series of factors in (1) and the $Q_j(x)$ enter in such a way that it is possible to combine the columns in the mentioned determinants so that the logarithms will not enter in the $\delta_j(x)$ $(j = 0, 1, \ldots n)$. On the other hand, if we write

$$(5)\qquad a_j^s(x) = \frac{d_j^s(x)}{d_0^s(x)} \qquad (s = 1, \ldots n),$$

we shall have $a_j^s(x)$ analytic in R_s (for $|x|$ sufficiently great) and

$$(5\text{ b})\qquad a_j^s(x) \sim \alpha_j(x)$$

$$(j = 1, \ldots n;\; x \text{ in } R_s;\; a_n^s(x) \not\equiv 0;\; s = 1, \ldots n)$$

where the formal series $\alpha_j(x)$ are in negative powers of $x^{\frac{1}{p}}$ (p, positive integer) with, possibly, a few positive powers present.

Noting that

$$(-1)^j d_j^s(x) = \begin{vmatrix} y_1^s(x+n) & \cdots & y_n^s(x+n) \\ \cdots & \cdots & \cdots \\ y_1^s(x+n-j+1) & \cdots & \cdots \\ y_1^s(x+n-j-1) & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ y_1^s(x) & \cdots & \cdots \end{vmatrix}$$

and combining columns, having (2 b) (with s replaced by $s-1$) in view, we conclude that

(6) $$d_j^s(x) = p^{s-1,s}(x)\, d_j^{s-1}(x)$$

$$(j = 0, 1, \ldots n;\ s = 2, 3, \ldots;\ p^{s-1,s}(x) \text{ periodic}, \not\equiv 0.)$$

If R_s and R_{s-1} have a strip along the negative axis of reals in common, necessarily $p^{s-1,s}(x)$ is analytic near the real axis. Otherwise, R_s and R_{s-1} will have a strip in common extending indefinitely upwards (or downwards) from the axis of reals. In this case

$$p^{s-1,s}(x),$$

in virtue of periodicity, is analytic in an upper (or lower) half plane.

By (5) and (6)

(7) $$a_j^s(x) = a_j^{s-1}(x) \qquad (j = 1, \ldots n;\ s = 2, 3, \ldots).$$

A function $a_j(x)$, defined as the analytic extension of, say, $a_j^1(x)$, will be analytic in virtue of (7) in each of the regions

$$R_1,\ R_2, \ldots.$$

Moreover, the asymptotic relations

(8) $$a_j(x) \sim \alpha_j(x) \qquad (j = 1, \ldots n)$$

will be maintained in these regions.

The equation

(9) $$L_n(y) \equiv y(x+n) + a_1(x)y(x+n-1) + \cdots + a_n(x)y(x) = 0$$

will be actually of order n and with coefficients of the required type. Each of the sets (2) will be a fundamental set of solutions of (9). This follows from the form of the determinant expression for the operator $L_n(y)$. In view of (2 a) and the definition of the $a_k(x)$ ($k = 1, \ldots n$) in terms of the $y_j^s(x)$ the series (1) will formally satisfy (9) in $R_1, R_2, \ldots$ The proof of the Theorem is thus completed.

§ 12. The Related Riemann Problem.

Let $T(x)$ $(= (e^{Q_j(x)} t_{ij}(x)))$ denote a matrix whose elements are those of a formal matrix $S(x)$ $(\equiv (e^{Q_j(x)} s_{ij}(x)))$ with the power series factors terminated after a number of terms. Suppose that $S(x)$ has the general character of a matrix of formal solutions which might occur in connection with a difference system of order n, of the type indicated in § 1. The set of $Q(x)$'s,

$$Q_1(x), \ldots Q_n(x) \tag{1}$$

defines a sequence of consecutive regions

$$R_1, R_2, \ldots R_m \qquad (|x| > \varrho > 0), \tag{2}$$

as indicated on the preceding pages. For definiteness suppose that R_1 is the lower one of the regions constituting the quadrant

$$-\varepsilon \leqq \arg x \leqq \frac{\pi}{2} + \varepsilon.$$

The regions (2) will cover (outside the circle $|x| = \varrho$) the extended complex plane over the range

$$-\varepsilon \leqq \arg x \leqq 2\pi p + \varepsilon, \tag{2 a}$$

where p is a suitable integer depending on the $Q_j(x)$. When a particular region R_s is considered it is essential to keep in view the corresponding range of $\arg x$. As we proceed in the counter clockwise direction, let B^s denote the last one of the boundaries of R_s encountered. In R_1 a fixed determination of $T(x)$ will be supposed as given. As a consequence, $T(x)$ will be known uniquely in each of the regions (2). For x in R_m, in the neighborhood of B^m (a portion of a line $\Im x = c > 0$), the $Q_j(x)$ ($j = 1, \ldots n$) will correspondingly be the same as in R_1. On the other hand, the $t_{ij}(x)$ may be different. The latter situation will take place if a factor $x^{r_{ij}}$ has r_{ij} not equal to an integral multiple of $1/p$ or if there are logarithms present in some of the $t_{ij}(x)$.

Assume that associated with B^s we have a matrix

$$P^{s,s+1}(x) = (p_{ij}^{s,s+1}(x)) \qquad (s = 1, 2, \ldots), \tag{3}$$

consisting of elements of period one. Suppose that $P^{s,s+1}(x)$ has the general character of a matrix of periodic functions which might occur, as indicated on the preceding pages, in connection with a different system of order n, formally satisfied by $S(x)$. More specifically, the following is assumed.

The $p_{ij}^{s,s+1}(x)$ $(i, j = 1, \ldots n)$ are proper, unless B^s is a portion of a line parallel to the axis of reals. In the latter case they are analytic in a strip along the axis of reals. Along B^s (and within a limited distance of B^s), in the vicinity of $x = \infty$,

$$A^{s,s+1}(x) = (a_{ij}^{s,s+1}(x)) = T(x) P^{s,s+1}(x) T^{-1}(x) \sim I, \tag{3 a}$$

while the derivatives of all orders of the matrix $A^{s,s+1}(x)$ are asymptotic, along B^s, to zero.

The following theorem will be proved.

Theorem 4. *Let $T(x)$, the corresponding regions (2) and the matrices (3) of periodic functions,*

$$P^{s,s+1}(x) \qquad (s = 1, 2, \ldots),$$

be given. Concerning these periodic functions assume the statement in italics preceding this theorem.

There exist then matrices

$$Y_1(x),\ Y_2(x), \ldots\ Y_s(x), \ldots \tag{4}$$

such that

$$Y_s(x) = Y_{s+1}(x) P^{s,s+1}(x) \qquad (s = 1, 2, \ldots). \tag{5}$$

Furthermore, for x in R_s, the elements of $Y_s(x)$ will be analytic (save at $x = \infty$) and $|Y_s(x)| \not\equiv 0$, while

$$Y_s(x) \sim \overline{S}(x) \qquad (x \text{ in } R_s). \tag{5 a}$$

The exponential factors $e^{Q(x)}$, occurring in the elements of the formal matrix $\overline{S}(x)$, are correspondingly the same as in the matrix $T(x)$.

Proof. For the purpose at hand the following theorem established by Birkhoff[1] will be used.

'Let $C_1, \ldots C_r$ be r simple closed curves in the extended complex plane. Let $A_1(x), \ldots A_r(x)$ be matrices of functions defined and indefinitely differentiable along $C_1, \ldots C_r$ respectively, analytic save at a finite number of points of these curves and of determinant not zero. If furthermore at any point of intersection of C_α, C_β the matrices $A_\alpha(x)$, $A_\beta(x)$ are such that the formal derivatives of all orders of the matrix

$$A_\alpha(x) A_\beta(x) - A_\beta(x) A_\alpha(x) \tag{6}$$

vanish, there exists a matrix $\Phi(x)$ with the following properties:

(1) each element of $\Phi(x)$ is analytic except along $C_1, \ldots C_r$ and at an arbitrary point $x = a$ where the elements may become infinite to finite order; $|\Phi(x)|$ nowhere vanishes save possibly at $x = a$;

(2) the elements of $\Phi(x)$ are continuous and indefinitely differentiable along each curve C_i from either side, analytic from either side save at points of intersection of the curves, or at those points where an element of $A_i(x)$ fails to be analytic, or at $x = a$; if a lies on a curve C_i, the matrix $(x-a)^l A_i(x)$ [or $x^{-l} A_i(x)$ if $a = \infty$] is indefinitely differentiable along C_i for a suitable l;

(3) if a + and — side of each curve C_i are chosen, then,

$$\lim_{x \to x_i^+} \Phi(x) = [\lim_{x \to x_i^-} \Phi(x)]\, A_i(x_i) \qquad (i = 1, \ldots r) \tag{7}$$

where the approach to the arbitrary point x_i of C_i is along the + and — side respectively.'

The curves C_s may be subjected to weaker restrictions and may extend to infinity. Thus, for instance C_s $(s = 1, 2, \ldots)$, may be defined as consisting of B^s up to a point on the circle $|x| = \varrho$; inside of this circle a portion of C_s will consist of a curve I^s, through $x = 0$, joining the mentioned point on the rim of the circle with another suitable point on the rim of the circle; from the latter point on, C_s will consist of a curve $\bar{B}^s$ extending to infinity (and analytic in every finite part of the plane); moreover, the component parts B^s, I^s, $\bar{B}^s$ of C_s will be supposed so joined that C_s is a simple curve with a continuously turning

[1] Cf. III. Methods developed with a view of application to the classical Riemann problem, can be found in papers by Hilbert (Gött. Nachr., 1905, pp. 307—338) and Plemelj (Monatsch. f. Math. u. Phys., 1908, pp. 205—246).

tangent; furthermore, if the limiting direction of B^s is α_s that of $\bar{B}^s$ is to be considered as $\alpha_s + \pi$; finally, the several curves C_s are to have points in common only at $x = 0$ and at $x = \infty$.

The matrices $A_s(x)$ may be defined as follows. Along B^s

$$A_s(x) = A^{s,s+1}(x); \tag{8}$$

along $\bar{B}^s$

$$A_s(x) = I. \tag{8 a}$$

We take ϱ sufficiently great. On the other hand, along I^s the elements of $A_s(x)$ are defined so that the conditions of the above theorem of Birkhoff hold with respect to $A_s(x)$. Thus, except at infinity, the elements of $A_s(x)$ are analytic along C_s for $|x| > \varrho$; for x on C_s

$$|A_s(x)| \neq 0.$$

The condition stated with respect to (6) will certainly hold at $x = \infty$. In the neighborhood of $x = 0$ the elements of the matrices $A_s(x)$ may be so determined that this condition will hold at $x = 0$ as well.[1] The point $x = a$ will be supposed to lie interior the circle $|x| = \varrho$.

There will exist a matrix $\Phi(x)$, with elements analytic for $|x| > \varrho$ except along $B^1, B^2, \ldots$ and except at $x = \infty$, such that $|\Phi(x)| \neq 0$ ($|x| > \varrho$). Along each curve B^s the elements of $\Phi(x)$ will be analytic from either side (for $|x| > \varrho$, and excepting $x = \infty$). Moreover, for x_s on B^s,

$$\lim_{x \to x_s^+} \Phi(x) = [\lim_{x \to x_s^-} \Phi(x)]\, A^{s,s+1}(x_s). \tag{9}$$

Here $A^{s,s+1}(x)$ is given by (3 a). The $+$ side of B^s will be taken corresponding to an approach from the interior of R_s. The asymptotic form, at $x = \infty$, of $\Phi(x)$ will be the same along both sides of B^s since $A^{s,s+1}(x_s) \sim I$. Thus, there exist matrices

$$U_1(x),\ U_2(x), \ldots \tag{9 a}$$

[1] Determination of $A_s(x)$ as stated, involves an approximation problem a solution of which had been given by A. Besikowitsch (Mathematische Zeitschrift, Band 21, Heft 1/2, 1924). Another solution had been given by Trjitzinsky. (Cf. forthcoming paper 'Approximation by analytic functions with prescribed derivatives' to apper in the Am. Jour. of Math.).

such that the elements of $U_s(x)$ $(s=1, 2, \ldots)$ are analytic (save at $x=\infty$) in R_s and such that $|U_s(x)| \neq 0$ $(|x| > \varrho)$ in R_s. Moreover,

$$U_s(x) = U_{s+1}(x)\, A^{s,\, s+1}(x) \qquad (s=1, 2, \ldots). \tag{9 b}$$

It is evident, in the light of the papers referred to above and treating Riemann problems, that the elements of $U_s(x)$ $(s=1, 2, \ldots)$ behave at infinity essentially as rational functions. For x in R_s

$$U_s(x) \sim U(x) = (u_{ij}(x)) \qquad (s=1, 2, \ldots). \tag{9 c}$$

Whether the elements of the formal matrix $U(x)$ can be made to be formal s-series is left, for the present, undecided.

Write

$$Y_s(x) = U_s(x)\, T(x). \tag{10}$$

Then (5) will hold. Moreover, for x in R_s,

$$\begin{aligned} U_s(x)\, T(x) &\sim (u_{ij}(x))\, (e^{Q_j(x)}\, t_{ij}(x)) \\ &= \bar{S}(x) = (e^{Q_j(x)} \sum_{\lambda} u_{i\lambda}(x)\, t_{\lambda j}(x)). \end{aligned}$$

It follows immediately that the theorem is true.

In a future paper one of the present authors (Trjitzinsky) proposes to develop the analytic theory of linear difference equations with rational coefficients which is a case of particular interest in which more special results can be obtained. Here it would be desirable to find those 'principal solutions' which stand out because of their peculiar analytic simplicity and to formulate the corresponding Riemann problem. The same author proposes also to develop in an analogous manner the analytic theory of q-difference equations and of the ordinary linear differential equations.