A NON-HARMONIC FOURIER SERIES.[1]

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Introduction.

A non-harmonic Fourier series in an expression of the type

$$\sum_n c_n e^{i\lambda_n \eta}, \quad -\pi \leq \eta \leq \pi, \tag{1}$$

in which the numbers λ_n $(n=0, \pm 1, \pm 2, \ldots)$ are not all integers. Paley and Wiener [6] began a systematic study of such series; and Levinson [5] continued their work. The central problem is to discover necessary and sufficient conditions upon the numbers $\{\lambda_n\}$ such that to each real function $f(\eta)$ of a given class there corresponds an expression of the type (1) summable to f for all or almost all η in $-\pi \leq \eta \leq \pi$. So far as I am aware, the best answer to this problem is due to Levinson ([5] Theorems XVIII and XIX), and is to this effect: if the λ_n are real, and if there exists a real constant D such that

$$|\lambda_n - n| \leq D < (p-1)/2p, \quad 1 < p \leq 2, \tag{2}$$

then to every $f(\eta)$ belonging to the Lebesgue class $L^p(-\pi, \pi)$ there corresponds a series (1) which is summable to $f(\eta)$ in the same sense as an ordinary Fourier series $\sum_n c'_n e^{in\eta}$; and that these conclusions are false for the set

$$\lambda_{-n} = -n + (p-1)/2p, \quad \lambda_0 = 0, \quad \lambda_n = n - (p-1)/2p, \quad n = 1, 2, \ldots \tag{3}$$

On account of this last clause, Levinson refers to (2) as a "best possible" result. This phrase is perhaps unfortunate; since, as we shall show, it is not true that every set $\{\lambda_n\}$ which violates (2) does not admit representations of type (1) for every function of $L^p(-\pi, \pi)$. Secondly Levinson's theorem does not cater for the class $L(-\pi, \pi)$, which is the appropriate class for ordinary Fourier analysis and which is wider than and includes the classes $L^p(-\pi, \pi)$ for $p > 1$. Thirdly Levinson's theorem does not admit complex numbers λ_n.

[1] This work was performed under contract of the National Bureau of Standards with American University.

This paper does not attempt a general discussion of the problem; it merely indicates that further work is needed on this question, by showing that a particular set of numbers λ_n, which are in general[1] complex, and which always violate (2), nevertheless allow representation of all functions in $L(-\pi, \pi)$. The resulting series (1) is useful in studying the uptake of β-indolyl acetic acid by plant tissue. The final section of the paper treats of this application.

Statement of theoretical results.

Let Γ consist of the complex numbers $\gamma = 0$ and $\gamma = -\frac{1}{2}(1 + \cos z)$ where $z = \sin z$. This set Γ is discrete and enumerable; and the only real numbers belonging to it are 0 and -1. Let c be any complex number not belonging to Γ. Hereafter we regard c as fixed. The exclusion of the set Γ is, I believe, only a matter of convenience. I suspect, without having attempted to verify these suspicions, that the results of Theorems 1 and 2 would remain true even if c belonged to Γ.

The numbers λ_n which will concern us throughout this paper are the roots of the equation

$$\pi\lambda \cos \pi\lambda + c \sin \pi\lambda = 0. \tag{4}$$

The exclusion of Γ means that this equation has no multiple root. We write $\lambda_0 = 0$ for the zero root of (4). All other roots may be arranged in pairs $(\lambda_n, \lambda_{-n})$ such that $\lambda_n + \lambda_{-n} = 0$, and such that $-\frac{1}{2}\pi < \arg \lambda_n \leq \frac{1}{2}\pi$, $n = 1, 2, \ldots$. Further

Theorem 1. *The suffices of λ_n may be so chosen that the real part of λ_n is not zero for $n = 2, 3, \ldots$ and*

$$\lambda_n = n - \tfrac{1}{2} + (c/\pi^2 n) + O(1/n^2) \quad \text{as} \quad n \to \infty. \tag{5}$$

Hereafter we suppose this system of suffices adopted. Theorem 1 shows that in general the λ_n are complex and that (2) is always violated, since the maximum value of $(p-1)/2p$ in (2) is 1/4.

Theorem 2. *Let $f(\eta)$ be any function of $L(-\pi, \pi)$, and define*

$$\left.\begin{aligned} a_n &= \frac{c^2 + \pi^2 \lambda_n^2}{\pi\{c(c+1) + \pi^2 \lambda_n^2\}} \int_{-\pi}^{\pi} \{\cos \lambda_n \eta - \cos \pi\lambda_n\} f(\eta)\, d\eta \\ b_n &= \frac{c^2 + \pi^2 \lambda_n^2}{\pi\{c(c+1) + \pi^2 \lambda_n^2\}} \int_{-\pi}^{\pi} \sin \lambda_n \eta\, f(\eta)\, d\eta. \end{aligned}\right\} \tag{6}$$

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[1] The possibility that the λ_n may be complex is not, however, responsible; for the λ_n are all real in the particular case [see equation (4)] when c is real and $-1 < c \neq 0$.

Let $s(\eta)$ *denote the Abel sum*[1]

$$s(\eta) = (\mathfrak{A}) \sum_{n=1}^{\infty} (a_n \cos \lambda_n \eta + b_n \sin \lambda_n \eta). \tag{7}$$

Suppose $-\pi < \eta < \pi$. *Then* $s(\eta) = f(\eta)$ *almost everywhere. If for some particular value of* η *there exists a number* f_0 *such that*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^{\varepsilon} \{ f(\eta + \varepsilon') + f(\eta - \varepsilon') - 2 f_0 \} \, d\varepsilon' = 0, \tag{8}$$

then $s(\eta) = f_0$. *In particular, if* $f(\eta)$ *has an ordinary discontinuity at* η, *then* $s(\eta) = \frac{1}{2} \{ f(\eta + 0) + f(\eta - 0) \}$; *and, if* $f(\eta)$ *is continuous at* η, *then* $s(\eta) = f(\eta)$. *Finally, in any closed subinterval of* $(-\pi, \pi)$ *in which* $f(\eta)$ *is continuous, the series* (7) *is uniformly summable to* $f(\eta)$.

These results are so close to those of ordinary Fourier series that it is surprising to find two points of difference. In ordinary Fourier expansions the terms $\cos n \eta$ occur with $n = 0, 1, 2, \ldots$. Here we have $\cos \lambda_n \eta$ for $n = 1, 2, \ldots$: that is to say, no constant function occurs in the expansion. Secondly, in any ordinary Fourier expansion, if $f(\eta)$ is continuous and $f(-\pi) = f(\pi)$, then $s(\pm \pi) = f(\pm \pi)$. That this is no longer true in our case follows from

Theorem 3. *If* t *is any complex number*

$$\sum_{n=1}^{\infty} -2c \left(\cos \pi t + c \frac{\sin \pi t}{\pi t} \right) \frac{\lambda_n (\lambda_n \cos \lambda_n \eta + i t \sin \lambda_n \eta)}{(\lambda_n^2 - t^2) \{ c(c+1) + \pi^2 \lambda_n^2 \} \cos \pi \lambda_n} = \begin{cases} e^{i t \eta}, & -\pi < \eta < \pi \\ e^{i t \eta} - \left(\cos \pi t + c \dfrac{\sin \pi t}{\pi t} \right), & \eta = \pm \pi \end{cases} \tag{9}$$

and in particular, for $t = 0$,

$$-2c(c+1) \sum_{n=1}^{\infty} \frac{\cos \lambda_n \eta}{\{ c(c+1) + \pi^2 \lambda_n^2 \} \cos \pi \lambda_n} = \begin{cases} 1, & -\pi < \eta < \pi \\ -c, & \eta = \pm \pi. \end{cases} \tag{10}$$

Equation (4) has attracted some attention in applied mathematics. In certain solutions of the heat equation (see Sommerfeld [7] p. 28) it is conventional to represent $f(\eta)$ by a sine series $\sum_{n=1}^{\infty} b_n \sin \lambda_n \eta$ over the range $(0, \pi)$. Again the analogous equation $\pi \lambda' \sin \pi \lambda' + c \cos \pi \lambda' = 0$ arises with series $\sum_{n=1}^{\infty} a_n \cos \lambda_n' \eta$. In each

[1] See Hardy [1], p. 71.

of these examples the functions of the series are orthogonal. Over the complete range $(-\pi, \pi)$, when both sines and cosines have to be used, orthogonality of one or other set fails; and this seems to have discouraged analysis. I have little doubt that an analysis, similar to that of this paper, could be given for the equation $\pi\lambda' \sin \pi\lambda' + c \cos \pi\lambda' = 0$.

Proof of theoretical results.

If $z = x + iy$,

$$\left|\frac{c \tan z}{z}\right| = \left|\frac{c}{z}\right| \left(\frac{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y}{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y}\right)^{\frac{1}{2}} \leq \begin{cases} \left|\dfrac{c}{z}\right| |\coth y| \\ \left|\dfrac{c}{z}\right| |\sec x| \end{cases}, \tag{11}$$

since $0 \leq \sinh^2 y \leq \cosh^2 y$. Let $S(\nu)$ denote the square with vertices $\pi\nu(\pm 1 \pm i)$, where ν is a positive integer. On the vertical sides of the square $|\sec x| = 1$, and on the horizontal sides $|\coth y| \leq \coth \pi$. Hence on $S(\nu)$

$$\left|\frac{c \tan z}{z}\right| = O(\nu^{-1}) \text{ as } \nu \to \infty. \tag{12}$$

Thus $|z \cos z| > |c \sin z|$ on $S(\nu)$ for all sufficiently large ν. Now $S(\nu)$ contains $2\nu + 1$ zeros of $z \cos z$, namely $0, \pm\frac{1}{2}\pi, \ldots, \pm(\nu - \frac{1}{2})\pi$. Hence by Rouché's theorem $S(\nu$ contains precisely $2\nu + 1$ zeros of $z \cos z + c \sin z$, provided ν is large enough; and thus there are precisely two such zeros in the annulus between $S(\nu)$ and $S(\nu + 1)$. Suppose $\varepsilon > 0$ prescribed; and let $S'(\nu)$ be the square with vertices $\pi(\nu + \frac{1}{2}) + \varepsilon(\pm 1 \pm i)$. We can choose $\nu_1 = \nu_1(\varepsilon)$ such that

$$|c| |\coth \varepsilon| / \{\pi(\nu + \tfrac{1}{2}) - \varepsilon\} < 1$$

and

$$|c| |\operatorname{cosec} \varepsilon| / \{\pi(\nu + \tfrac{1}{2}) - \varepsilon\} < 1$$

for $\nu \geq \nu_1$. Thereupon, using the upper and lower forms of (11) upon the horizontal and the vertical sides of $S'(\nu)$ respectively, we have $|z \cos z| > |c \sin z|$ upon $S'(\nu)$. It follows that one of the two zeros of $z \cos z + c \sin z$ in the annulus between $S(\nu)$ and $S(\nu + 1)$ lies within $S'(\nu)$. By symmetry the other lies near $-\pi(\nu + \frac{1}{2})$. Hence, we can choose the suffices of λ_n such that $\pi\lambda_n = (n - \frac{1}{2})\pi + o(1)$ as $n \to \infty$. Write $\pi\lambda_n = (n - \frac{1}{2})\pi + \delta_n$. Then, upon substitution in (4), $\{(n - \frac{1}{2})\pi + \delta_n\} \sin \delta_n = c \cos \delta_n$. Expand both sides of this equation in powers of δ_n, and note that $\delta_n \to 0$, so that we can solve and get $\delta_n = c/n\pi + O(n^{-2})$. This gives (5). For the remainder of Theorem 1, suppose $z = iy$ satisfies $z \cos z + c \sin z = 0$. Then $y \cosh y + c \sinh y = 0$. This equation has no real root unless c is real and less than or equal to -1. In the exceptional case $c < -1$ ($c = -1$ being an excluded point of Γ), there is just one positive root, which we may denote by $\pi\lambda_1$, because we have so far only imposed an ordering upon $\pi\lambda_n$ for sufficiently large $|n|$ in satisfying (5). This completes Theorem 1. It follows from Theorem 1 that no zero of $z\cos z + c\sin z$ lies upon $S(\nu)$ provided ν is large enough. Hereafter we restrict ourselves to such sufficiently large ν.

Let $T(\nu)$ denote a given part (possibly the whole) of $S(\nu)$. Let $g(z)$ be an analytic function of z, and let $G(\nu)$ be the least upper bound of $|g(z)/\cos z|$ for z on $T(\nu)$. Then we assert

$$\lim_{\nu\to\infty} \int_{T(\nu)} \frac{g(z)\,dz}{z\cos z + c\sin z} = \lim_{\nu\to\infty} \int_{T(\nu)} \frac{g(z)\,dz}{z\cos z}, \tag{13}$$

provided $G(\nu) = o(\nu)$ as $\nu \to \infty$. For large ν, (12) shows

$$\left|1 - 1\Big/\left(1 + \frac{c\tan z}{z}\right)\right| \leq \frac{\left|c\dfrac{\tan z}{z}\right|}{1 - \left|c\dfrac{\tan z}{z}\right|} = \frac{O(\nu^{-1})}{1 - O(\nu^{-1})} = O(\nu^{-1}). \tag{14}$$

Hence

$$\int_{T(\nu)} \left\{\frac{g(z)}{z\cos z + c\sin z} - \frac{g(z)}{z\cos z}\right\} dz = O(\nu^{-1})\, O\{G(\nu)\}\, O\left\{\int_{T(\nu)} \frac{|dz|}{|z|}\right\} = o(1)$$

and (13) is established.

We shall now suppose (temporarily and until further notice) that c is not a real number less than -1, and that t is any complex number not lying upon the imaginary axis and not equalling λ_n for any integer n. Thus $z\cos z + c\sin z \neq 0$ on the imaginary axis. Let ξ and η be real numbers satisfying $\xi \leq 0$, $-\pi < \eta < \pi$. Wiite $\zeta = \xi + i\eta$, $\zeta' = -\xi + i\eta$. Let $S(\nu+)$ denote the rectangle with vertices $\pm i\pi\nu$ and $\pi\nu(1 \pm i)$, ν being a positive integer, and having a small semi-circular indentation $|z| = \delta$ at the origin, where $\delta < |\pi t|$ and $\delta < |\pi\lambda_n|$ for $n = \pm 1, \pm 2, \ldots$. Let $S(\nu-)$ denote the reflection of $S(\nu+)$ in the imaginary axis. All contour integrals which follow are taken once anti-clockwise. Consider

$$J_0 = \lim_{\delta\to 0+} \lim_{\nu\to\infty} \frac{1}{2\pi i} \left\{ \int_{S(\nu+)} \frac{e^{\zeta z/\pi}\,dz}{z\cos z + c\sin z} + \int_{S(\nu-)} \frac{e^{\zeta' z/\pi}\,dz}{z\cos z + c\sin z} \right\}$$

$$J_t = \lim_{\delta\to 0+} \lim_{\nu\to\infty} \frac{1}{2\pi i} \left\{ \int_{S(\nu+)} \frac{e^{\zeta z/\pi}\,dz}{(z - \pi t)(z\cos z + c\sin z)} + \int_{S(\nu-)} \frac{e^{\zeta' z/\pi}\,dz}{(z - \pi t)(z\cos z + c\sin z)} \right\}$$

We shall show first that the contributions to J_0 and J_t from integrations along parts of $S(\nu)$ are zero. On the common part of $S(\nu)$ and $S(\nu+)$

$$\left|\frac{e^{\zeta z/\pi}}{\cos z}\right| = \frac{e^{(x\xi - y\eta)/\pi}}{\sqrt{(\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y)}} \leq \frac{e^{|y|}}{|\sinh y|} = O(1); \tag{15}$$

and a similar result holds for $|e^{\zeta' z/\pi}/\cos z|$ on the common part of $S(\nu)$ and $S(\nu-)$. The same conclusions hold a fortiori if $(z-\pi t)\cos z$ replaces $\cos z$. Hence by (13) we may disregard the quantity $c \sin z$ in the denominators of the integrands. On the common part of $S(\nu)$ and $S(\nu+)$, we have for $z = R e^{i\theta}$

$$e^{\zeta z/\pi}/\cos z = 2\, e^{x\xi/\pi}/\{\vartheta_1 e^{-R\sin\theta(1-\eta/\pi)} + \vartheta_2 e^{R\sin\theta(1+\eta/\pi)}\}, \tag{16}$$

where $|\vartheta_1| = |\vartheta_2| = 1$. Now $x\xi/\pi$ is non-positive, and $1 \pm \eta/\pi$ are both strictly positive. Hence to each prescribed $\varepsilon > 0$, we can find $\nu_2 = \nu_2(\varepsilon, \eta)$ such that the modulus of the right-hand side of (16) is less than ε for $R \geq \pi\nu_2$, provided $|\theta| \geq \varepsilon$. For the exceptional set $|\theta| < \varepsilon$, the left-hand side of (16) is bounded, by (15). Hence the contribution to the first integral of J_0 must be zero when $\nu \to \infty$. A similar conclusion holds for the common part of $S(\nu-)$ and $S(\nu)$, $e^{\zeta' z/\pi}$ replacing $e^{\zeta z/\pi}$; and a fortiori the contributions to J_t are zero.

Next consider the contributions to J_t from the integrals along the imaginary axis. They are

$$\lim_{\delta\to 0+}\lim_{\nu\to\infty}\frac{1}{2\pi i}\left\{\int_{-\pi\nu}^{-\delta} + \int_{\delta}^{\pi\nu}\right\}\frac{e^{-y\eta/\pi}(e^{-i\xi y/\pi} - e^{i\xi y/\pi})}{(iy-\pi t)(iy\cosh y + ic\sinh y)}\, i\, dy.$$

After a little algebra this reduces to

$$\frac{2}{\pi}\int_0^\infty \frac{\{\pi t\cosh(y\eta/\pi) - iy\sinh(y\eta/\pi)\}\sin(y\xi/\pi)}{(y^2+\pi^2 t^2)(y\cosh y + c\sinh y)}\, dy.$$

Similarly the contribution to J_0 from the integrals along the imaginary axis turns out to be

$$-\frac{2}{\pi}\int_0^\infty \frac{\cosh(y\eta/\pi)\sin(y\xi/\pi)}{y\cosh y + c\sinh y}\, dy.$$

The contributions to J_0 and J_t from the small semicircles at the origin are half the sum of the two residues; and turn out to be $-1/(c+1)$ and $1/\pi t(c+1)$ respectively.

We can, on the other hand, determine J_0 and J_t as the limits of the sum of residues within the contours. Since at $z = \pi\lambda_n$

$$\frac{d}{dz}(z\cos z + c\sin z) = [(1+c) - z\tan z]\cos z = \left(1 + c + \frac{\pi^2\lambda_n^2}{c}\right)\cos\pi\lambda_n,$$

by (4), we have

$$J_0 = \sum_{n=1}^{\infty}\frac{c\,e^{\lambda_n\xi}(e^{i\lambda_n\eta}+e^{-i\lambda_n\eta})}{\{c(c+1)+\pi^2\lambda_n^2\}\cos\pi\lambda_n} = 2c\sum_{n=1}^{\infty}\frac{e^{\lambda_n\xi}\cos\lambda_n\eta}{\{c(c+1)+\pi^2\lambda_n^2\}\cos\pi\lambda_n}.$$

For J_t, the pole at πt will fall within $S(\nu+)$ or $S(\nu-)$ according as $\phi > 0$ or $\phi < 0$ where $\phi = \operatorname{sgn}\{\Re(t)\}$. We therefore find

$$J_t = \frac{2c}{\pi}\sum_{n=1}^{\infty}\frac{e^{\lambda_n\xi}(t\cos\lambda_n\eta + i\lambda_n\sin\lambda_n\eta)}{(\lambda_n^2 - t^2)\{c(c+1)+\pi^2\lambda_n^2\}\cos\pi\lambda_n} + \frac{e^{\phi\xi t + i\eta t}}{\pi t\cos\pi t + c\sin\pi t}.$$

Collection of the results for J_0 yields

$$1 = -2c(c+1)\sum_{n=1}^{\infty}\frac{e^{\lambda_n\xi}\cos\lambda_n\eta}{\{c(c+1)+\pi^2\lambda_n^2\}\cos\pi\lambda_n} - \frac{2(c+1)}{\pi}\int_0^{\infty}\frac{\cosh(y\eta/\pi)\sin(y\xi/\pi)\,dy}{\cosh y + c\sinh y}; \quad (17)$$

and similarly the terms for J_t provide

$$\frac{e^{\phi\xi t + i\eta t}}{\pi t\cos\pi t + c\sin\pi t} = \frac{1}{\pi t(c+1)} - \frac{2c}{\pi}\sum_{n=1}^{\infty}\frac{e^{\lambda_n\xi}(t\cos\lambda_n\eta + i\lambda_n\sin\lambda_n\eta)}{(\lambda_n^2 - t^2)\{c(c+1)+\pi^2\lambda_n^2\}\cos\pi\lambda_n}$$
$$+ \frac{2}{\pi}\int_0^{\infty}\frac{\{\pi t\cosh(y\eta/\pi) - iy\sinh(y\eta/\pi)\}\sin(y\xi/\pi)\,dy}{(y^2+\pi^2 t^2)(y\cosh y + c\sinh y)}. \quad (18)$$

Let us now examine the effect of relaxing the condition that c should not be a negative real number less than -1. In case c is real and $c < -1$ the term $z\cos z + c\sin z$ in the denominators of the original expression for J_0 and J_t would yield poles on the imaginary axis at $\pm\pi\lambda_1$. We could exclude these by small semicircular indentations $|z \pm \pi\lambda_1| = \delta$. The integrals around these semicircles would yield the term $2c\,e^{\lambda_1\xi}\cos\lambda_1\eta/\{c(c+1)+\pi^2\lambda_1^2\}\cos\pi\lambda_1$ in J_0 and a corresponding term in J_t. The integrals along the imaginary axis would yield the same result as before, except that the integral in J_0

$$\int_0^{\infty}\frac{\cosh(y\eta/\pi)\sin(y\xi/\pi)}{y\cosh y + c\sinh y}\,dy$$

would be taken as its principal value at the point $y = -i\pi\lambda_1$. The corresponding integral in J_t must be interpreted in the same way. Again if we relax the condition that t shall not lie on the imaginary axis, we make a small indentation $|z - \pi t| = \delta$; and proceed as before. It follows that (17) and (18) are true, provided that the integrals are taken as principal ones, for all c (except of course the set Γ) and all complex t except λ_n, $n = 0, \pm 1, \pm 2, \ldots$.

Now multiply (17) by $1/\pi t (c+1)$ and add the result to (18). This gives, for $\xi \leq 0$, $-\pi < \eta < \pi$,

$$e^{\phi \xi t + i\eta t} = -2c \sum_{n=1}^{\infty} \left(\cos \pi t + c \frac{\sin \pi t}{\pi t}\right) \frac{e^{\lambda_n \xi} \lambda_n (\lambda_n \cos \lambda_n \eta + i t \sin \lambda_n \eta)}{(\lambda_n^2 - t^2)\{c(c+1) + \pi^2 \lambda_n^2\} \cos \pi \lambda_n}$$

$$- \frac{2}{\pi}\left(\cos \pi t + c \frac{\sin \pi t}{\pi t}\right) \int_0^\infty \frac{\{y \cosh (y\eta/\pi) + i\pi t \sinh (y\eta/\pi)\}\, y \sin (y\xi/\pi)\, dy}{(y^2 + \pi^2 t^2)(y \cosh y + c \sinh y)}, \quad (19)$$

the integral being a principal one. We now assert that this relationship is true for all complex t. We have so far proved it for $t \neq \lambda_n$. In case $t = \lambda_0 = 0$, (19) reduces to (17), the term $\sin \pi 0/\pi 0$ being taken equal to 1 in the usual way. Again if we interpret in the same spirit

$$\frac{\pi \lambda_n \cos \pi \lambda_n + c \sin \pi \lambda_n}{\lambda_n - \lambda_n} = \lim_{t \to \lambda_n} \frac{\pi t \cos \pi t + c \sin \pi t}{\lambda_n - t} = -\frac{\pi}{c}\{c(c+1) + \pi^2 \lambda_n^2\} \cos \pi \lambda_n,$$

then the formal substitution of $t = \lambda_n\,(n \neq 0)$ in (19) reduces it to the trivial identity $e^{\lambda_n \zeta} = e^{\lambda_n \zeta}$.

Putting $\xi = 0$ in (19) gives Theorem 3 in case $-\pi < \eta < \pi$. To deduce the rest of Theorem 3 consider

$$K_0(\pm\pi) = \lim_{\nu \to \infty} \frac{1}{2\pi i} \int_{S(\nu)} \frac{e^{\pm i z}\, dz}{z \cos z + c \sin z}, \quad K_t(\pm\pi) = \lim_{\nu \to \infty} \frac{1}{2\pi i} \int_{S(\nu)} \frac{e^{\pm i z}\, dz}{(z - \pi t)(z \cos z + c \sin z)}.$$

A glance at (15) shows it is valid for $\eta = \pm\pi$. Hence we may omit the term $c \sin z$ in the above integrals. Next $e^{\pm i z}/\cos z$ is bounded on $S(\nu)$, as we may see by multiplying (11) by $|z/c|$. It follows that $K_t(\pm\pi) = 0$, since

$$\int_{S(\nu)} \{e^{\pm i z}/(z - \pi t) z \cos z\}\, dz = O\left\{\int_{S(\nu)} O(\nu^{-1}) \frac{|dz|}{|z|}\right\} = O(\nu^{-1}).$$

On the other hand

$$K_0(\pm\pi) = \lim_{\nu \to \infty} \frac{1}{2\pi i}\left\{\int_{S(\nu)} \frac{dz}{z} \pm \int_{S(\nu)} \frac{\tan z}{z}\, dz\right\} = 1,$$

because $\tan z/z$ is an even function of z and $S(\nu)$ is symmetrical about the origin. Evaluating $K_0(\pm\pi)$ and $K_t(\pm\pi)$ as the sum of their residues, we deduce the rest of Theorem 3 without difficulty.

We now and henceforth take t to be a positive integer. Combining (19) with the result obtained by writing $-\eta$ for η, we get for any pair of numbers a' and b'

$$(a'\cos\eta t + b'\sin\eta t)\,e^{\xi t} = (-)^{t+1}\,2c\sum_{n=1}^{\infty}\frac{\lambda_n(a'\lambda_n\cos\lambda_n\eta + b'\,t\sin\lambda_n\eta)\,e^{\lambda_n\xi}}{(\lambda_n^2 - t^2)\{c(c+1)+\pi^2\lambda_n^2\}\cos\pi\lambda_n}$$

$$+(-)^{t+1}\,2\int_0^{\infty}\frac{(a'\,y\cosh y + b'\,t\sinh y)\,y\sin y\xi\,dy}{(y^2+t^2)(\pi y\cosh\pi y + c\sinh\pi y)},\ \xi\le 0,\ -\pi<\eta<\pi. \tag{20}$$

Here the integral has been modified slightly by the substitution πy for y. Should the integrand have poles, the integral is to have principal value. Let $f(\eta)$ be any given function of $L(-\pi,\pi)$ with ordinary Fourier coefficients a_t' and b_t'

$$a_t' = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\eta)\cos\eta t\,d\eta,\quad b_t' = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\eta)\sin\eta t\,d\eta,\quad t=0,\,1,\,2\,\ldots.$$

Let $s'(\eta)$ denote the Abel sum

$$s'(\eta) = \tfrac{1}{2}a_0' + (\mathfrak{A})\sum_{t=1}^{\infty}(a_t'\cos\eta t + b_t'\sin t\eta).$$

Then

$$s'(\eta) - \tfrac{1}{2}a_0' = \lim_{\xi\to 0-}\lim_{r\to 1-}\sum_{t=1}^{\infty}(a_t'\cos t\eta + b_t'\sin t\eta)(re^{\xi})^t$$

$$= -2c\lim_{\xi\to 0-} S(\xi,\eta) - 2\lim_{\xi\to 0-} I(\xi,\eta) \tag{21}$$

where

$$S(\xi,\eta) = \lim_{r\to 1-}\sum_{t=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-)^t r^t\lambda_n(a_t'\lambda_n\cos\lambda_n\eta + b_t'\,t\sin\lambda_n\eta)\,e^{\lambda_n\xi}}{(\lambda_n^2-t^2)\{c(c+1)+\pi^2\lambda_n^2\}\cos\pi\lambda_n} \tag{22}$$

$$I(\xi,\eta) = \lim_{r\to 1-}\sum_{t=1}^{\infty}\int_0^{\infty}\frac{(-)^t r^t(a_t'\,y\cosh y\eta + b_t'\,t\sinh y\eta)\,y\sin y\xi\,dy}{(y^2+t^2)(\pi y\cosh\pi y + c\sinh\pi y)}. \tag{23}$$

We shall first show that $I(\xi,\eta)\to 0$ as $\xi\to 0-$ uniformly in η for every η in a closed interval $-\pi+\delta\le\eta\le\pi-\delta$ for arbitrary $\delta>0$.

For brevity let us write $Q(y,\xi) = y\sin y\xi/(\pi y\cosh\pi y + c\sinh\pi y)$. Consider first

$$I(\xi,\eta) = S_1 = \lim_{r\to 1-}\sum_{t=1}^{\infty} r^t\int_0^{\infty} Q(y,\xi)\,\frac{(-)^t(a_t'\,y\cosh y\eta + b_t'\,t\sinh y\eta)}{y^2+t^2}\,dy.$$

This as a power series in r^t; and hence by Abel's theorem

$$S_1 = \sum_{t=1}^{\infty} \int_0^{\infty} Q(y, \xi) \frac{(-)^t (a_t' y \cosh y\eta + b_t' t \sinh y\eta)}{y^2 + t^2} dy, \tag{24}$$

provided that the right-hand side of (24) exists. Let

$$S_2 = \sum_{t=1}^{\infty} \int_0^{\infty} \frac{(-)^t b_t'}{t} Q(y, \xi) \sinh y\eta \, dy, \tag{25}$$

where (for the moment) we assume the existence of S_2. Then

$$|S_1 - S_2| = \left| \sum_{t=1}^{\infty} \int_0^{\infty} Q(y, \xi) \left\{ \frac{(-)^t a_t' y \cosh y\eta}{y^2 + t^2} - \frac{(-)^t b_t' y^2 \sinh y\eta}{t(y^2 + t^2)} \right\} dy \right|$$

$$\leq \int_0^{\infty} |Q(y, \xi)| \left\{ y \cosh y\eta \sum_{t=1}^{\infty} \frac{|a_t'|}{t^2} + y^2 |\sinh y\eta| \sum_{t=1}^{\infty} \frac{|b_t'|}{t^3} \right\} dy$$

$$\leq K_1 \int_0^{\infty} y^2 \cosh y\eta \, |Q(y, \xi)| \, dy, \tag{26}$$

for some constant K_1, since a_t' and b_t', being Fourier coefficients, tend to zero as $t \to \infty$ (Hardy and Rogosinski [2] Theorem 30). However, since $|\eta| \leq \pi - \delta$, we can find K_2 such that the right-hand side of (26) is less than

$$K_2 \int_0^{\infty} y^2 e^{-\frac{1}{2}\delta y} |\sin y\xi| \, dy,$$

which tends to zero (obviously without depending upon η) as $\xi \to 0-$. It is therefore enough to show that S_2 exists and tends to zero as $\xi \to 0-$ uniformly in η. But $Q(y, \xi) \sinh y\eta$ is independent of t, and so

$$S_2 = \int_0^{\infty} \sum_{t=1}^{\infty} \frac{(-)^t b_t'}{t} Q(y, \xi) \sinh y\eta \, dy;$$

and, by the same argument as before, S_2 will have the required properties if

$$S_3 = \sum_{t=1}^{\infty} \frac{(-)^t b_t'}{t}$$

exists. But (Hardy and Rogosinski [2] Theorem 44) a Fourier series may be integrated term by term; whence it is easy to verify

$$S_3 = \tfrac{1}{2}\pi a_0' - \frac{1}{2\pi}\int_{-\pi}^{\pi}(\pi+\eta)\, f(\eta)\, d\eta,$$

which exists because $f(\eta)$ belongs to $L(-\pi, \pi)$. We remark in conclusion that the above argument is not affected by the possibility that the integrals may be principal; for the pole of the integrand is simple if existent. Thus if c is real and $c < -1$, we can write the integral in the form $\int_0^\infty = \int_0^{-i\lambda_1} + \int_{-i\lambda_1}^{-2i\lambda_1} + \int_{-2i\lambda_1}^{\infty}$; and then by writing $-y$ for y in the first integral and combining it with the second, the factor $(\pi y \cosh \pi y + c \sinh \pi y)^{-1}$ will be converted into a bounded term. This disposes of $I(\xi, \eta)$, and we now turn to the more difficult treatment of $S(\xi, \eta)$.

For brevity write

$$p_{t,n} = \frac{(-)^t \lambda_n (a_t' \lambda_n \cos \lambda_n \eta + b_t' t \sin \lambda_n \eta)\, e^{(\lambda_n - n)\xi}}{(\lambda_n^2 - t^2)\{c(c+1) + \pi^2 \lambda_n^2\} \cos \pi \lambda_n}; \tag{27}$$

so that

$$S(\xi, \eta) = \lim_{r \to 1-} \sum_{t=1}^{\infty} \sum_{n=1}^{\infty} p_{t,n}\, r^t e^{n\xi}. \tag{28}$$

Until further notice we shall suppose that ξ has a fixed value $\xi < 0$. We notice that $\lambda_n \neq t$; for if this were the case

$$0 = \pi \lambda_n \cos \pi \lambda_n + c \sin \pi \lambda_n = \pi t \cos \pi t + c \sin \pi t = (-)^t \pi t,$$

which is impossible since t is a positive integer. Also, for large n, $\lambda_n - t$ differs from an integer by $\frac{1}{2} + O(n^{-1})$ by (5). Hence for all n and all t, $(\lambda_n - t)^{-1} = O(1)$. Next $\lambda_n/(\lambda_n + t)$ and $t/(\lambda_n + t)$ are both $O(1)$. Equation (5) shows that

$$\cos \pi \lambda_n = (-)^{n+1} \sin \left\{\frac{c}{\pi n} + O(n^{-2})\right\}$$

whence $\lambda_n / \{c(c+1) + \pi^2 \lambda_n^2\} \cos \pi \lambda_n = O(1)$. Finally a_t', b_t', $\cos \lambda_n \eta$, $\sin \lambda_n \eta$, and $e^{(\lambda_n - n)\xi}$ are all bounded. Consequently $p_{t,n}$ is bounded, for all t and all n. Then for every fixed $r < 1$ and every fixed $\xi < 0$, $\sum_{t=1}^{\infty} \sum_{n=1}^{\infty} p_{t,n}\, r^t e^{n\xi}$ is absolutely convergent, and is therefore convergent; and its sums by rows and by columns both exist. Hence we may invert the order of summation and deduce

$$S(\xi, \eta) = \lim_{r \to 1-} \sum_{n=1}^{\infty} \sum_{t=1}^{\infty} p_{t,n}\, r^t e^{n\xi}. \tag{29}$$

The next step is to prove

$$q_n(r) = \sum_{t=1}^{\infty} p_{t,n}\, r^t = O(n) \tag{30}$$

uniformly in r for $r_0 \le r \le 1$ for some value of $r_0 < 1$. Since we may omit multipliers which depend only upon n and are bounded for all n, it is enough to prove

$$u_n(r) = \sum_{t=1}^{\infty} \frac{\lambda_n a_t'(-r)^t}{\lambda_n^2 - t^2} = O(n), \quad v_n(r) = \sum_{t=1}^{\infty} \frac{t\, b_t'(-r)^t}{\lambda_n^2 - t^2} = O(n) \tag{31}$$

uniformly in r for $r_0 \le r \le 1$. Now $S_3(r) = \sum_{t=1}^{\infty} b_t'(-r)^t/t$ is obviously independent of n and it exists because $S_3(1) = S_3$ exists, as already proved. Moreover $\lim_{r \to 1-} S_3(r) = S_3(1)$ by Abel's theorem. Hence there exists $r_0 < 1$ such that $S_3(r)$ is bounded for $r_0 \le r \le 1$, and (obviously) therefore uniformly bounded in n. Thus instead of the second relation of (31) it is sufficient to prove

$$w_n(r) = v_n(r) + S_3(r) = \sum_{t=1}^{\infty} \frac{\lambda_n^2\, b_t'(-r)^t}{t(\lambda_n^2 - t^2)} = O(n)$$

uniformly in $r_0 \le r \le 1$. Since $u_n(r)$ and $w_n(r)$ are both power series in r, they will exist and be $O(n)$ uniformly in $r_0 \le r \le 1$ if

$$U_n = \sum_{t=1}^{\infty} \frac{|\lambda_n|}{|\lambda_n^2 - t^2|} = O(n), \quad W_n = \sum_{t=1}^{\infty} \frac{|\lambda_n^2|}{t\,|\lambda_n^2 - t^2|} = O(n), \tag{32}$$

because a_t', b_t' are bounded. However

$$U_n = \sum_{t=1}^{n} \frac{|\lambda_n|}{|\lambda_n^2 - t^2|} + \sum_{t=n+1}^{\infty} \frac{|\lambda_n|}{|\lambda_n^2 - t^2|} = n.\ \ O(1) + O(\lambda_n) \sum_{t=1}^{\infty} t^{-2} = O(n)$$

and W_n behaves similarly. Thus (30) is established. We now see that for each fixed $\xi < 0$, $\sum_{n=1}^{\infty} q_n(r)\, e^{n\xi}$ is uniformly convergent for $r_0 \le r \le 1$, and hence

$$S(\xi, \eta) = \lim_{r \to 1-} \sum_{n=1}^{\infty} q_n(r)\, e^{n\xi} = \sum_{n=1}^{\infty} \lim_{r \to 1-} q_n(r)\, e^{n\xi}. \tag{33}$$

Next we invoke Abel's theorem once more to show that

$$\lim_{r \to 1-} q_n(r) = \lim_{r \to 1-} \sum_{t=1}^{\infty} p_{t,n}\, r^t = \sum_{t=1}^{\infty} p_{t,n}, \tag{34}$$

for we have already demonstrated the existence of $\sum_{t=1}^{\infty} p_{t,n}$. Collection of the results so far obtained yields

$$S(\xi,\eta)=\sum_{n=1}^{\infty}\left[\left\{\sum_{t=1}^{\infty}\frac{(-)^t a_t'\lambda_n}{\lambda_n^2-t^2}\right\}\cos\lambda_n\eta+\left\{\sum_{t=1}^{\infty}\frac{(-)^t b_t' t}{\lambda_n^2-t^2}\right\}\sin\lambda_n\eta\right]\cdot$$
$$\cdot\left[\frac{\lambda_n e^{\lambda_n\xi}}{\{c(c+1)+\pi^2\lambda_n^2\}\cos\pi\lambda_n}\right]. \quad (35)$$

Now (Hobson [4] p. 581) an ordinary Fourier series may be multiplied by a function of bounded variation and integrated term by term, provided the resulting series converges. Thus

$$F_n=\int_{-\pi}^{\pi}\{f(\eta)-\tfrac{1}{2}a_0'\}\cos\lambda_n\eta\,d\eta=\int_{-\pi}^{\pi}\left\{\sum_{t=1}^{\infty}a_t'\cos t\eta+b_t'\sin t\eta\right\}\cos\lambda_n\eta\,d\eta$$
$$=\sum_{t=1}^{\infty}\int_{-\pi}^{\pi}\{a_t'\cos t\eta+b_t'\sin t\eta\}\cos\lambda_n\eta\,d\eta=\sum_{t=1}^{\infty}a_t'\int_{-\pi}^{\pi}\cos t\eta\cos\lambda_n\eta\,d\eta$$
$$=2\sin\pi\lambda_n\sum_{t=1}^{\infty}\frac{(-)^t a_t'\lambda_n}{\lambda_n^2-t^2}; \quad (36)$$

for we have already shown that the right-hand side of (36) exists. Similarly we have

$$G_n=\int_{-\pi}^{\pi}f(\eta)\sin\lambda_n\eta\,d\eta=2\sin\pi\lambda_n\sum_{t=1}^{\infty}\frac{(-)^t b_t' t}{\lambda_n^2-t^2}. \quad (37)$$

However, $f(\eta)$ belongs to $L(-\pi,\pi)$; and it therefore follows from the Riemann-Lebesgue lemma (Whittaker and Watson [8] p. 172) that

$$F_n=o(1),\ G_n=o(1)\ \text{as}\ n\to\infty.$$

Consequently as $n\to\infty$

$$\alpha_n=\alpha_n(\eta)=[F_n\cos\lambda_n\eta+G_n\sin\lambda_n\eta]\left[\frac{\lambda_n}{\{c(c+1)+\pi^2\lambda_n^2\}\,2\sin\pi\lambda_n\cos\pi\lambda_n}\right]=o(1) \quad (38)$$

and (35) reduces to

$$S(\xi,\eta)=\sum_{n=1}^{\infty}\alpha_n e^{\lambda_n\xi}. \quad (39)$$

We now remove the restriction that $\xi<0$ is fixed; and we assert that $\lim_{\xi\to 0-}S(\xi,\eta)$ exists whenever $s'(\eta)$ exists. This follows from (21): for $c\neq 0$ by hypothesis, and $\lim_{\xi\to 0-}I(\xi,\eta)=0$ in $-\pi<\eta<\pi$ because we have already proved this for $-\pi+\delta\leq\eta\leq$ $\leq\pi-\delta$ for arbitrary $\delta>0$. However, $s'(\eta)$ exists for almost all η (Hardy and Rogosinski [2] Theorem 75). We hereafter[1] consider any fixed value of η for which $s'(\eta)$ exists. For this value of η, $S(\xi, \eta)$ exists for $\xi < 0$; and we define $S(0, \eta) = \lim_{\xi \to 0-} S(\xi, \eta)$. For every given closed interval $\xi_1 \leq \xi \leq \xi_2 < 0$, the series $S(\xi, \eta)$ converges uniformly in ξ. Writing $\phi(\xi) = \sum_{n=1}^{\infty} (\alpha_n / \lambda_n) e^{\lambda_n \xi}$, $\xi < 0$, we have

$$\phi(\xi) - \phi(\xi_1) = \sum_{n=1}^{\infty} \int_{\xi_1}^{\xi} \alpha_n e^{\lambda_n \xi} d\xi = \int_{\xi_1}^{\xi} \sum_{n=1}^{\infty} \alpha_n e^{\lambda_n \xi} d\xi = \int_{\xi_1}^{\xi} S(\xi, \eta) d\xi, \quad \xi \leq \xi_2.$$

This relationship is true for arbitrary $\xi_2 < 0$; and hence for all $\xi < 0$. It follows that

$$\lim_{\xi \to 0-} \phi(\xi) = \phi(\xi_1) + \int_{\xi_1}^{0} S(\xi, \eta) d\xi$$

exists, by virtue of the existence of $S(\xi, \eta)$ up to and including $\xi = 0$. Now

$$\left| \sum_{n=N}^{\infty} \left(\frac{\alpha_n}{n} - \frac{\alpha_n}{\lambda_n} \right) e^{\lambda_n \xi} \right| \leq \sum_{n=N}^{\infty} \left| \frac{\alpha_n}{n} - \frac{\alpha_n}{\lambda_n} \right| = \sum_{n=N}^{\infty} o(n^{-2}) = o(N^{-1}).$$

The existence of $\lim_{\xi \to 0-} \phi(\xi)$ therefore demonstrates the existence of

$$\lim_{\xi \to 0-} \sum_{n=1}^{\infty} \frac{\alpha_n}{n} e^{\lambda_n \xi}. \tag{40}$$

As a consequence of (5) there exists a positive constant M such that

$$\left| e^{\lambda_n \xi} \left(1 - \frac{c\,\xi}{\pi^2 n} \right) - e^{(n - \frac{1}{2})\xi} \right| \leq \frac{M |\xi|}{n^2}$$

for $n = 1, 2, \ldots$ and all sufficiently small non-positive ξ. Hence

$$\left| S(\xi, \eta) - \frac{c\,\xi}{\pi^2} \sum_{n=1}^{\infty} \frac{\alpha_n}{n} e^{\lambda_n \xi} - \sum_{n=1}^{\infty} \alpha_n e^{(n - \frac{1}{2})\xi} \right| \leq M |\xi| \sum_{n=1}^{\infty} \frac{|\alpha_n|}{n^2} = O(|\xi|).$$

Let $\xi \to 0-$, notice the existence of (40), and deduce

$$\lim_{\xi \to 0-} S(\xi, \eta) = \lim_{\xi \to 0-} \sum_{n=1}^{\infty} \alpha_n e^{(n - \frac{1}{2})\xi} = \lim_{\xi \to 0-} e^{-\xi/2} \sum_{n=1}^{\infty} \alpha_n e^{n \xi} = \mathfrak{A} \sum_{n=1}^{\infty} \alpha_n.$$

Since Abelian summation is regular, Theorem 3 gives

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[1] I could have condensed the ensuing analysis, had I known of a Tauberian theorem valid for Abelian summation with *complex* exponents. I am only aware of Tauberian theorems of the appropriate type restricted to real exponents (e.g. Hardy [1] Theorem 104).

$$\tfrac{1}{2} a_0' = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\eta)\, d\eta = (\mathfrak{A}) \sum_{n=1}^{\infty} \frac{-c(c+1)\cos\lambda_n \eta}{\pi\{c(c+1)+\pi^2\lambda_n^2\}\cos\pi\lambda_n} \int_{-\pi}^{\pi} f(\eta)\, d\eta.$$

We collect these last two results with (36), (37) and (38) and deduce

$$\tfrac{1}{2} a_0' - 2c \lim_{\xi\to 0-} S(\xi, \eta) = (\mathfrak{A}) \sum_{n=1}^{\infty} (a_n'' \cos\lambda_n \eta + b_n'' \sin\lambda_n \eta), \tag{41}$$

where

$$a_n'' = -\frac{c(c+1)\int_{-\pi}^{\pi} f(\eta)\, d\eta}{\pi\{c(c+1)+\pi^2\lambda_n^2\}\cos\pi\lambda_n}$$

$$-\frac{c\lambda_n}{\{c(c+1)+\pi^2\lambda_n^2\}\sin\pi\lambda_n\cos\pi\lambda_n} \int_{-\pi}^{\pi} \left\{ f(\eta) - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\eta)\, d\eta \right\} \cos\lambda_n\eta\, d\eta,$$

$$b_n'' = -\frac{c\lambda_n}{\{c(c+1)+\pi^2\lambda_n^2\}\sin\pi\lambda_n\cos\pi\lambda_n} \int_{-\pi}^{\pi} f(\eta)\sin\lambda_n\eta\, d\eta.$$

However (4) shows that

$$\tan\pi\lambda_n = -\pi\lambda_n/c, \quad \cos^2\pi\lambda_n = c^2/(c^2+\pi^2\lambda_n^2).$$

After some algebra we deduce $a_n'' = a_n$, $b_n'' = b_n$. Equations (21) and (41) now yield

$$s'(\eta) = s(\eta) - 2 \lim_{\xi\to 0-} I(\xi, \eta). \tag{42}$$

However the conclusions of Theorem 2 are true if we replace $s(\eta)$ by $s'(\eta)$ (Hardy and Rogosinski [2] Theorem 75). Also $I(\xi, \eta) \to 0$ as $\xi \to 0-$ uniformly in η for $-\pi + \delta \leq \eta \leq \pi - \delta$, for arbitrary $\delta > 0$. Accordingly (42) completes the proof of Theorem 2.

Practical application.

In an experiment to determine the rate of uptake of growth stimulant by plant tissue, some thin disks (of uniform thickness 2δ) of carrot root are plunged into an agitated solution of β-indolyl acetic acid; and, as the acid diffuses into the disks, the concentration of acid in the circumambient solution falls. The object of the experiment is to determine D, the diffusion constant of β-indolyl acid in carrot root, by observing $\psi(t)$, the ratio of concentration of acid in the external solution at time t to that at zero time, when the carrot roots were plunged into the solution. The disks are sufficiently thin to presume that all acid enters normally to the flat surfaces of the disks, and that edge effects are negligible. We shall show that, if c is the ratio of the total volume of the carrot roots to the total volume of the circumambient solution, then

$$\psi(t) = \frac{1}{c+1} + \sum_{n=1}^{\infty} \frac{2\,c\,e^{-D\pi^2\lambda^2_n t/\delta^2}}{\{c\,(c+1) + \pi^2\lambda_n^2\}}, \tag{43}$$

where λ_n are the quantities defined by (4). This series converges very rapidly; and for reasonably large t it is only the first term which matters. Under these circumstances it is easy to determine D from a knowledge of c and $\psi(t)$.

Let v and V denote the actual volumes of carrot root and of external solution respectively; so that $c = v/V$. Let x denote a distance coordinate $(-\delta \leq x \leq \delta)$ taken from the centre of a disk in a direction normal to the flat faces. Let $k(t)$ denote the concentration of acid in the external solution at time t, this concentration at any instant being uniform throughout the acid due to the agitation of the liquid. Let $K(x, t)$ denote the concentration of acid within the disks at the point x and at time t. Since acid diffuses into a disk from both its flat sides, we have to seek an even function of x for $K(x, t)$. At the instant of immersion there is no acid inside the carrot root, so

$$K(x, 0) = 0, \quad 0 \leq x < \delta. \tag{44}$$

At all instants the concentration of acid on the surface of a disk[1] is $k(t)$; so

$$K(\delta, t) = k(t), \quad t \geq 0 \tag{45}$$

The total amount of acid remains constant throughout the experiment; so

$$\frac{v}{2\delta} \int_{-\delta}^{\delta} K(x, t)\,dx + V\,k(t) = \text{const.},$$

whence by (44)

$$c \int_0^{\delta} K(x, t)\,dx + \delta\,k(t) = \delta\,k(0), \quad t \geq 0. \tag{46}$$

Inside the carrot root the diffusion equation holds; so

[1] There is not much practical evidence for either believing or disbelieving (45). In general one may expect discontinuities of concentration at the surfaces of the individual cells of the carrot root (i.e. in the microscopic picture); but this does not of course imply a discontinuity at the surface of the disk in the macroscopic picture. In the practical investigation of absorption by the carrot root, it will be of interest to see whether or not (43) represents observed data: and, if it does not, doubt will be thrown upon the assumptions implicit in (45) and elsewhere.

$$\frac{\partial K(x,\ t)}{\partial t} = D\frac{\partial^2 K(x,t)}{\partial x^2}. \tag{47}$$

Consider the trial solution

$$K(x,\ t) = K_0 + \sum_{n=1}^{\infty} a_n \exp\left(-\frac{D\pi^2\lambda_n^2 t}{\delta^2}\right)\cos\left(\frac{\pi\lambda_n x}{\delta}\right), \tag{48}$$

where the λ_n are parameters to be chosen presently. This solution certainly satisfies (47). Substitution of (48) into (46) yields, by (45),

$$K_0(1+c) + \sum_{n=1}^{\infty} a_n \exp\left(-\frac{D\pi^2\lambda_n^2 t}{\delta^2}\right)\left[\frac{\pi\lambda_n\cos\pi\lambda_n + c\sin\pi\lambda_n}{\pi\lambda_n}\right] = k(0).$$

Thus (46) will be satisfied if we choose the λ_n in accordance with (4); and if we take $K_0 = k(0)/(1+c)$. When $t=0$, we have, by (44),

$$K(x,\ 0) = \frac{k(0)}{1+c} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi\lambda_n x}{\delta}\right) = \begin{cases} 0 & \text{for } 0 \le x < \delta \\ k(0) & \text{for } x = \delta. \end{cases}$$

Thus

$$\sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi\lambda_n x}{\delta}\right) = \begin{cases} -k(0)/(c+1) & \text{for } 0 \le x < \delta \\ k(0)\,c/(c+1) & \text{for } x = \delta. \end{cases}$$

Comparison with Theorem 3 shows that this will be true if

$$a_n = 2\,c\,k(0)/\{c(c+1) + \pi^2\lambda_n^2\}\cos\pi\lambda_n;$$

and (43) follows at once.

This particular problem can be solved by the method of the Laplace transform; but the solution is then very much longer than the above method. Moreover the Laplace transform method becomes awkward in the more general case when the carrot roots already have a given distribution of acid concentration at the initial instant of immersion. Our method deals with this generalisation immediately; for if

$$f(x) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi\lambda_n x}{\delta}\right),\quad 0 \le x < \delta$$

is this initial distribution, then the subsequent distribution at time t is

$$\sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi\lambda_n x}{\delta}\right)\exp\left(-\frac{D\pi^2\lambda_n^2 t}{\delta^2}\right).$$

In fact the method exposes at once the Huygens semi-group property to be expected from partial differential equations of the type (47), (Hille [3], p. 400).

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