THE SPACE OF GROUPOIDS OVER A COMPACTUM*.

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1. Introduction. Numerous studies have been made concerning the topology of spaces of functions of various types (continuous, measurable, etc.) over various types of topological spaces. These functions generally take values in a linear normed space. It seems also desirable to consider the set of algebraic functions (that is, binary, single-valued operations) defined in a given space. This is the object of the present paper.

We restrict attention to the case when our underlying space is a compactum (= compact metric space). Our definition of distance in the space of groupoids applies to any bounded semimetric space and a similar function may be defined even in the case of an unbounded semimetric space by the introduction of a "convergence factor" such as is used by Busemann[1] in the group of motions of a distance space. However, the restriction to a compactum is used in the present paper to obtain our desired results.

Throughout, we use the axiom of choice without further comment. We also employ as a Lemma in Section 3 the well-known Policeman's theorem:

Lemma 0. *Let M be a compactum and $\varepsilon > 0$. There is a finite subset of M which is ε-dense in M.*

We consider first some fundamental metric and topological properties of the space of groupoids and then turn to the closure of certain subsets of this space whose elements are of particular interest in topological algebra. The final section discusses the major unsolved problem and a suggestion for related study.

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* Presented to the American Mathematical Society; April 20, 1951.

[1] HERBERT BUSEMANN, *Local metric geometry*, Trans. Am. Math. Soc., vol. 56 (1944), pp. 200—274.

2. Terminology and notation. M is a compactum with distance function $\delta(x, y)$. Let $\mathfrak{G}$ be the set of all groupoids over the point set of M (that is, $\mathfrak{G}$ is the set of all single-valued functions on MM to M). Throughout, small letters refer to elements and capitals to sets. English letters refer to M, german letters to $\mathfrak{G}$, and greek letters to real numbers. The elements of $\mathfrak{G}$ are defined by their labels; that is, $\mathfrak{a}$ is the element of $\mathfrak{G}$ whose operation is denoted by $x\mathfrak{a}y$. It is clear that $\mathfrak{G}$ is finite when M is finite and that $\mathfrak{G}$ has cardinal $\mathbf{2}^{\boldsymbol{N}}$ when M is infinite of cardinal $\boldsymbol{N}$ ($\boldsymbol{N} \leq \boldsymbol{c}$ since M compact).

We define distance in $\mathfrak{G}$ by

$$\delta(\mathfrak{a}, \mathfrak{b}) = \sup_{x, y \in M} \delta(x\mathfrak{a}y, x\mathfrak{b}y).^{1}$$

It is clear that $\mathfrak{G}$ forms a semimetric space under this distance function and the triangle inequality is easily verified. The resulting metric space, $\mathfrak{G}$, is referred to as *the space of groupoids over M*.

Only the metric topologies are employed in M and $\mathfrak{G}$. Thus, in M, $\lim_{\eta\to\infty} p_\eta = p \cdot \sim \cdot \lim_{\eta\to\infty} \delta(p, p_\eta) = 0$ [2], and similarly in $\mathfrak{G}$.

We denote by $\mathfrak{A}$ the subset of $\mathfrak{G}$ consisting of commutative groupoids, by $\mathfrak{S}$ the subset consisting of semigroups, by $\mathfrak{Q}$ the set of quasigroups, by $\mathfrak{L}$ the set of loops, and by $\mathfrak{G}^*$ the set of groups.

Other terminology and notation will be introduced as needed.

3. Fundamental metric and topological properties of $\mathfrak{G}$.

Theorem 3.1. $\mathfrak{a}, \mathfrak{b} \in \mathfrak{G} \cdot) \cdot \exists\, a, b \in M \ni \delta(\mathfrak{a}, \mathfrak{b}) = \delta(a, b)$. (For an explanation of the notation see footnote 2).

Proof. By definition of $\delta(\mathfrak{a}, \mathfrak{b})$ there is a pair of sequences $\{x_\eta\}$, $\{x_\eta\}$ with $\lim_{\eta\to\infty} \delta(x_\eta, y_\eta) = \delta(\mathfrak{a}, \mathfrak{b})$. Since M is compact, there are convergent subsequences of these sequences $\{x_{\varkappa_\eta}\}$, $\{y_{\varkappa_\eta}\}$ (the usual process yields convergent subsequences having the same original indices on x's and y's). Let $\lim_{\eta\to\infty} x_{\varkappa_\eta} = a$, $\lim_{\eta\to\infty} y_{\varkappa_\eta} = b$. Then, since the metric of M is continuous, $\delta(\mathfrak{a}, \mathfrak{b}) = \delta(a, b)$.

[1] We employ δ for the distance function in both M and $\mathfrak{G}$. This causes no confusion since the argument indicates which function is meant.

[2] We employ the following logical symbols most of which are due to E. H. Moore:

"$\exists$" is read "there exist(s)",

"$\forall$" is read "for all",

"$\cdot)\cdot$", "$:):$", etc. are read "implies",

"$\cdot \sim \cdot$" is read "if and only if",

"$\ni$" is read "such that",

"$\wedge$" is read "and".

Corollary. $\mathfrak{G} \approx M \pmod{\triangle}$.

Explanation. If S is a semimetric space we denote by $\triangle(S)$ the distance set of S. $\mathfrak{G} \approx M \pmod{\triangle}$ is read "$\mathfrak{G}$ is distancial to M" and means $\triangle(\mathfrak{G}) = \triangle(M)$.

Proof of Corollary. The corollary follows immediately from Theorem 3.1 and the obvious fact that $\mathfrak{K}$, the set of constant elements of $\mathfrak{G}$, is congruent to M, $\mathfrak{K} \approx M$. Thus, $\triangle(M) \subset \triangle(\mathfrak{G})$ and $\triangle(\mathfrak{G}) \subset \triangle(M)$.

Theorem 3.2. *$\mathfrak{G}$ is complete (in the sense of Fréchet).*

Proof. Let $\mathfrak{a}_1, \mathfrak{a}_2, \ldots$ be a Cauchy sequence of elements of $\mathfrak{G}$. We have then

$$(1) \qquad \varepsilon > 0:):\exists\, \eta \,_{\xi}\, \alpha, \beta > \eta \cdot) \cdot \delta(x\mathfrak{a}_\alpha y, x\mathfrak{a}_\beta y) < \varepsilon; \quad \forall x, y \in M.$$

Select $x, y \in M$. From (1), the sequence $\{x\mathfrak{a}_\alpha y\}$ is a Cauchy sequence of points of M (which is compact and hence complete). Let $\lim_{\alpha\to\infty} x\mathfrak{a}_\alpha y = x\mathfrak{a}y$ to define $\mathfrak{a} \in \mathfrak{G}$; $\forall x, y \in M$. Now, the metric of M is continuous so that (1) yields

$$(2) \qquad \varepsilon > 0:):\exists\, \eta \,_{\xi}\, \alpha > \eta \cdot) \cdot \delta(x\mathfrak{a}y, x\mathfrak{a}_\alpha y) \leq \varepsilon; \quad \forall x, y \in M.$$

And, hence,

$$(3) \qquad \varepsilon > 0:):\exists\, \eta > 0 \,_{\xi}\, \alpha > \eta \cdot) \cdot \delta(\mathfrak{a}, \mathfrak{a}_\alpha) \leq \varepsilon.$$

But (3) implies $\lim_{\eta\to\infty} \mathfrak{a}_\eta = \mathfrak{a}$, and the theorem is proved.

Remark. It is clear that the preceding proof applies if M is merely complete and bounded (not necessarily compact). It is also clear that $\mathfrak{G}$, although complete, is not, in general, either compact or separable. This may be seen either from consideration of the cardinal of $\mathfrak{G}$ or by construction of actual examples. Theorem 3.3 below, is, however, of interest in this respect.

Corollary. *Every closed subset of $\mathfrak{G}$ is complete.*

Theorem 3.3. *If M is infinite, then $\mathfrak{G}$ contains a subset $\mathfrak{D}$ with the properties:* (1) *$\mathfrak{D}$ consists of finite-valued groupoids and* (2) *$\mathfrak{D}$ is dense in $\mathfrak{G}$.*

Proof. We shall exhibit the required set $\mathfrak{D}$. It is then easy to verify that every neighborhood of an arbitrary element of $\mathfrak{G}$ contains an element of $\mathfrak{D}$ so that $\mathfrak{G} = \bar{\mathfrak{D}}$. Let M be infinite and let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\eta, \ldots$ be a sequence of positive real numbers with $\lim_{\eta\to\infty} \varepsilon_\eta = 0$. By Lemma 0 (Policeman's Theorem) there is a subset (finite) $p^1{}_\eta$, $p^2{}_\eta, \ldots, p_\eta^{\lambda(\eta)}$ of M which is ε_η-dense in M. Select such a subset for each η and denote it by P_η. Let $\mathfrak{F}_\eta$ be the set of all elements of $\mathfrak{G}$ all of whose values ($x\mathfrak{a}y$; $\forall x, y \in M$) lie in P_η. Let $\mathfrak{D} = \bigcup_\eta \mathfrak{F}_\eta$.

Theorem 3.4. *If M is convex*[1], *then $\mathfrak{G}$ is convex.*

Proof. Let $\mathfrak{a}, \mathfrak{b} \in \mathfrak{G}$. For $x, y \in M$ consider $x \mathfrak{a} y$ and $x \mathfrak{b} y$. If these coincide, let $x \mathfrak{c} y = x \mathfrak{a} y$. If $x \mathfrak{a} y \neq x \mathfrak{b} y$, let $x \mathfrak{c} y$ be the midpoint of $x \mathfrak{a} y$ and $x \mathfrak{b} y$ (such midpoints exist since M is metrically convex[1]). Then, for any $x, y \in M$:

$$\delta(x \mathfrak{a} y, x \mathfrak{c} y) = \delta(x \mathfrak{b} y, x \mathfrak{c} y) = \tfrac{1}{2} \delta(x \mathfrak{a} y, x \mathfrak{b} y), \text{ and,}$$

clearly, $\mathfrak{c}$ is a midpoint in $\mathfrak{G}$ for $\mathfrak{a}$ and $\mathfrak{b}$ unless $\mathfrak{a}$ and $\mathfrak{b}$ coincide.

Corollary. *If M is convex, $\mathfrak{G}$ is segmentally connected.*

Proof. This is immediate from Theorems 3.2 and 3.4 and the fundamental theorem of the theory of metric convexity.[1]

4. Continuity of elements of $\mathfrak{G}$.

By *continuity* of $\mathfrak{a} \in \mathfrak{G}$ *at* $x, y \in M$ we mean

$$\lim_{\eta \to \infty} x_\eta = x \wedge \lim_{\eta \to \infty} y_\eta = y \cdot) \cdot \lim_{\eta \to \infty} x_\eta \mathfrak{a} y_\eta = x \mathfrak{a} y.$$

That is, by continuity, we mean simultaneous continuity in both arguments. $\mathfrak{a} \in \mathfrak{G}$ is continuous if it is continuous at each pair of points of M.

We denote by $\mathfrak{C}(x, y)$ the set of all elements of $\mathfrak{G}$ which are continuous at x, y, and by $\mathfrak{C}$ the set of all continuous elements of $\mathfrak{G}$.

Theorem 4.1. *If $\mathfrak{a} \in \mathfrak{G}$ is discontinuous at $x, y \in M$ then there is an $\varepsilon > 0$ so that $\mathfrak{b} \in \mathfrak{G} \wedge \delta(\mathfrak{a}, \mathfrak{b}) < \varepsilon$ imply the discontinuity of $\mathfrak{b}$ at x, y.*

Proof. Let $\mathfrak{a}$ be discontinuous at x, y. Then there are sequences $\{x_\eta\}$ and $\{y_\eta\}$ with $\lim_{\eta \to \infty} x_\eta = x \wedge \lim_{\eta \to \infty} y_\eta = y$ so that either $\lim_{\eta \to \infty} x_\eta \mathfrak{a} y_\eta$ fails to exist or does exist and is distinct from $x \mathfrak{a} y$. Suppose that for each $\varepsilon > 0$ there is $\mathfrak{b} \varepsilon \mathfrak{G}$ with $\delta(\mathfrak{a}, \mathfrak{b}) < \varepsilon$ and with $\mathfrak{b}$ continuous at x, y.

Now,

$$(1) \qquad \delta(x_\alpha \mathfrak{a} y_\alpha, x \mathfrak{a} y) \leq \delta(x_\alpha \mathfrak{a} y_\alpha, x_\alpha \mathfrak{b} y_\alpha) + \delta(x_\alpha \mathfrak{b} y_\alpha, x \mathfrak{a} y)$$
$$\leq \delta(x_\alpha \mathfrak{a} y_\alpha, x_\alpha \mathfrak{b} y_\alpha) + \delta(x_\alpha \mathfrak{b} y_\alpha, x \mathfrak{b} y) + \delta(x \mathfrak{a} y, x \mathfrak{b} y).$$

Also,

$$(2) \qquad \eta > 0 :) : \exists \beta > 0_\xi \alpha > \beta \cdot) \cdot \delta(x_\alpha \mathfrak{b} y_\alpha, x \mathfrak{b} y) < \eta.$$

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[1] See Karl Menger, *Untersuchungen über allgemeine Metrik*, Math. Annalen, vol. 100 (1928), pp. 75—163; or L. M. Blumenthal, *Distance geometries*, University of Missouri Studies, vol. XIII (1938), pp. 1—142.

Hence, for $\alpha > \beta$,

$$\delta(x_a \mathfrak{a} y_a, x \mathfrak{a} y) < 2\varepsilon + \eta. \tag{3}$$

However, ε and η are both arbitrarily small so that (3) implies $\lim_{a\to\infty} x_a \mathfrak{a} y_a = x \mathfrak{a} y$, contrary to assumption. We infer the validity of the theorem.

Corollary. $\mathfrak{C}(x,y)$ *is closed*; $\forall x, y \in M$.

Corollary. $\mathfrak{C}$ *is closed.*

Proof. The first corollary follows from the Theorem immediately since it has been shown that each point of the complement of $\mathfrak{C}(x,y)$ is an interior point. The second corollary follows from the first since a (set-theoretic) product of closed subsets of a metric space is closed.

For $\mathfrak{E} \subset \mathfrak{G}$ we denote $\mathfrak{E} \cap \mathfrak{C}$ by $\mathfrak{E}_\gamma$.

Let $\mathfrak{a} \in \mathfrak{Q}$. We define $\mathfrak{a} \in \mathfrak{Q}_\tau$ provided $\mathfrak{a} \in \mathfrak{Q}_\gamma$ and

$$\lim_{a\to\infty} a_a = a \wedge \lim_{a\to\infty} b_a = b \wedge (a_a \mathfrak{a} x_a = b_a \wedge y_a \mathfrak{a} a_a = b_a (\forall \alpha))$$
$$\cdot) \cdot \exists x, y \in M_\xi \lim_{a\to\infty} x_a = x \wedge \lim_{a\to\infty} y_a = y \wedge a \mathfrak{a} x = y \mathfrak{a} a = b.$$

We also define $\mathfrak{L}_\tau = \mathfrak{L} \cap \mathfrak{Q}_\tau$ and $\mathfrak{G}^*_\tau = \mathfrak{G}^* \cap \mathfrak{Q}_\tau (= \mathfrak{S} \cap \mathfrak{Q}_\tau)$. Clearly, the elements of $\mathfrak{L}_\tau$ and $\mathfrak{G}^*_\tau$ are topological loops and groups, respectively, in the usual sense of these terms.

5. Closure of certain important subsets of $\mathfrak{G}$.

In the preceding section, we showed that $\mathfrak{C}$ is closed. In the present section we show that certain subsets of $\mathfrak{C}$ (namely $\mathfrak{A}$, $\mathfrak{A}_\gamma$, $\mathfrak{S}_\gamma$, $\mathfrak{Q}_\tau$, $\mathfrak{L}_\tau$, and $\mathfrak{G}^*_\tau$) which are of interest from the viewpoint of topological algebra are also closed. One may construct examples to show that some of the larger algebraically defined sets such as $\mathfrak{S}$, $\mathfrak{Q}$, $\mathfrak{Q}_\gamma$, $\mathfrak{L}_\gamma$, and $\mathfrak{G}^*_\gamma$ are not, in general, closed. Of course, one would not expect the limit of quasigroups to be a quasigroup without assumption of continuity of inverses so that this is not an unsatisfactory state of affairs.

We shall prove the theorems concerning the closure of $\mathfrak{A}$ and of $\mathfrak{Q}_\tau$. (The proof of the closure of $\mathfrak{Q}_\tau$ is really only outlined since a full proof is very long by our methods). The case for $\mathfrak{S}_\gamma$ is quite straightforward and easy. The closure of $\mathfrak{L}_\tau$ may be inferred from the proof of that of $\mathfrak{Q}_\tau$ since this proof shows that unit elements will be preserved in the limit. The closure of $\mathfrak{G}^*_\tau$ follows from the closures of $\mathfrak{S}_\gamma$ and $\mathfrak{Q}_\tau$.

Theorem 5.1. $\mathfrak{A}$ *is closed.*

Proof. Let $\lim_{\eta\to\infty} \mathfrak{a}_\eta = \mathfrak{a}$, $\mathfrak{a}_\eta \in \mathfrak{A}$. This implies $\lim_{\eta\to\infty} x\,\mathfrak{a}_\eta\, y = x\,\mathfrak{a}\, y$; $\forall\, x, y \in M$. But $\lim_{\eta\to\infty} x\,\mathfrak{a}_\eta\, y = \lim_{\eta\to\infty} y\,\mathfrak{a}_\eta\, x = y\,\mathfrak{a}\, x$, since $\mathfrak{a}_\eta \in \mathfrak{A}$. Hence, $\mathfrak{a} \in \mathfrak{A}$.

Corollary. $\mathfrak{A}_\gamma$ *is closed.*

Theorem 5.2. $\mathfrak{S}_\gamma$ *is closed.*

Theorem 5.3. $\mathfrak{Q}_\tau$ *is closed.*

Proof. Let $\lim_{\eta\to\infty} \mathfrak{a}_\eta = \mathfrak{a}$, $\mathfrak{a}_\eta \in \mathfrak{Q}_\tau \subset \mathfrak{C}$. Then $\mathfrak{a} \in \mathfrak{C}$. Now, $\lim_{\eta\to\infty} \mathfrak{a}_\eta = \mathfrak{a}$ implies

$$(1) \qquad \lim_{\eta\to\infty} x\,\mathfrak{a}_\eta\, y = x\,\mathfrak{a}\, y; \quad \forall\, x, y \in M.$$

Select $p, q \in M$. There is (by definition of quasigroup) a unique x_η for each η so that $p\,\mathfrak{a}_\eta\, x_\eta = q$. Since M is compact, there is a convergent subsequence $\{x_{\varkappa_\eta}\}$ of $\{x_\eta\}$ · Let $\lim_{\eta\to\infty} x_{\varkappa_\eta} = x$.

Now,

$$(2) \qquad \lim_{\eta\to\infty} p\,\mathfrak{a}_{\varkappa_\eta}\, x = p\,\mathfrak{a}\, x, \text{ by (1).}$$

However, $\mathfrak{a}_\eta \in \mathfrak{C}$ so that for η sufficiently large and $\varepsilon > 0$,

$$(3) \qquad \delta(p\,\mathfrak{a}_{\varkappa_\eta}\, x,\; p\,\mathfrak{a}_{\varkappa_\eta}\, x_{\varkappa_\eta}) < \varepsilon.$$

But $p\,\mathfrak{a}_{\varkappa_\eta}\, x_{\varkappa_\eta} = q$; $\forall_\eta$. Hence,

$$(4) \qquad \lim_{\eta\to\infty} p\,\mathfrak{a}_{\varkappa_\eta}\, x = q, \text{ and,}$$

since $\{p\,\mathfrak{a}_{\varkappa_\eta}\, x\}$ is a subsequence of $\{p\,\mathfrak{a}_\eta\, x\}$, $p\,\mathfrak{a}\, x = q$. This shows that $p\,\mathfrak{a}\, x = q$ has a solution for each $p, q \in M$. The argument which follows indicates how one may show the uniqueness of such a solution and the continuity of the x in this equation. The equation $y\,\mathfrak{a}\, p = q$ may be treated, of course, in analogous fashion and we will conclude that $\mathfrak{a} \in \mathfrak{Q}_\tau$.

Define x_η by $p\,\mathfrak{a}_\eta\, x_\eta = q$. Then,

$$(5) \qquad \delta(p\,\mathfrak{a}_\eta\, x_\eta,\; p\,\mathfrak{a}\, x) = 0; \quad \forall\, \eta.$$

$$(6) \qquad \varepsilon > 0 :) : \exists\, \alpha > 0_\xi\, \eta > \alpha \cdot) \cdot \delta(p\,\mathfrak{a}_\eta\, x,\; p\,\mathfrak{a}\, x) < \varepsilon,$$

since $\lim_{\eta\to\infty} \mathfrak{a}_\eta = \mathfrak{a}$.

From (5), (6), and the triangle inequality,

$$(7) \qquad \varepsilon > 0 :) : \exists\, \alpha > 0_\xi\, \eta > \alpha \cdot) \cdot \delta(p\,\mathfrak{a}_\eta\, x_\eta,\; p\,\mathfrak{a}_\eta\, x) < \varepsilon.$$

Also, since $\mathfrak{a}_\eta \in \mathfrak{Q}_\tau$, we have

(8) $$\varepsilon > 0 :) : \exists \lambda > 0 \vdots q, q' \in M \wedge \delta(q, q') < \lambda \wedge p\,\mathfrak{a}_\eta x = q \wedge p\,\mathfrak{a}_\eta x' = q' \cdot) \cdot \delta(x, x') < \varepsilon.$$

Combining (7) and (8) one obtains:

(9) $$\varepsilon > 0 \vdots) \vdots \exists \mu > 0 \vdots) \vdots \exists \alpha > 0 \vdots \eta > \alpha :) : \delta(p\,\mathfrak{a}_\eta x_\eta, p\,\mathfrak{a}_\eta x) < \mu \cdot) \cdot \delta(x, x_\eta) < \varepsilon.$$

Hence, $\lim_{\eta \to \infty} x_\eta = x$.

The replacement of the constant p and q by $\{p_\eta\}$ and $\{q_\eta\}$ in the equations $p\,\mathfrak{a}_\eta x = q$ to obtain the equations $p_\eta \mathfrak{a}_\eta x_\eta = q_\eta$ raises no real additional difficulties when $\lim_{\eta \to \infty} q_\eta = q$ and $\lim_{\eta \to \infty} p_\eta = p$.

Theorem 5.4. *$\mathfrak{L}_\tau$ and $\mathfrak{G}^*_\tau$ are closed.*

6. Unsolved problems.

The writers intend further study along the lines of the present paper at a later date. One of the major problems which we have studied without success is:

Given a metric space M, under what conditions does there exist a compactum K so that the space of groupoids over K is congruent to M? This is, of course, the characterization (metric) problem for spaces of groupoids (in the sense of this paper) among metric spaces.

The case where M is finite appears of little interest. The problem appears very difficult when M is infinite. Some necessary conditions were given in the early part of this paper but we have found no non-trivially sufficient conditions. The only obvious approach lies in trying to construct K from M which means finding the metric structure of M from metric relations among two-variable operators upon it and this we have been unable so far to do.

One may also note that mappings of a compactum into itself may be considered as elements of the space $\mathfrak{G}$ which are independent of one of the arguments and may thus be included in the study of $\mathfrak{G}$.

Another interesting question is: *What algebraic conditions (if any) characterize the closures of $\mathfrak{S}$, $\mathfrak{Q}$, $\mathfrak{L}$, and $\mathfrak{G}^*$? Stated differently: What types of groupoids (on a compactum) may be approximated by semigroups, by loops or quasigroups, by groups?*

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