A THEORY OF RADON MEASURES ON LOCALLY COMPACT SPACES.

By

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1. Introduction.

In functional analysis one is often presented with the following situation: a locally compact space X is given, and along with it a certain topological vector space $\mathcal{E}$ of real functions defined on X; it is of importance to know the form of the most general continuous linear functional on $\mathcal{E}$. In many important cases, $\mathcal{E}$ is a superspace of the vector space $\mathfrak{C}$ of all real, continuous functions on X which vanish outside compact subsets of X, and the topology of $\mathcal{E}$ is such that if a sequence (f_n) tends uniformly to zero and the f_n collectively vanish outside a fixed compact subset of X, then (f_n) is convergent to zero in the sense of $\mathcal{E}$. In this case the restriction to $\mathfrak{C}$ of any continuous linear functional μ on $\mathcal{E}$ has the property that $\mu(f_n) \to 0$ whenever the sequence (f_n) converges to zero in the manner just described. It is therefore an important advance to determine all the linear functionals on $\mathfrak{C}$ which are continuous in this sense.

It is customary in some circles (the Bourbaki group, for example) to term such a functional μ on $\mathfrak{C}$ a "Radon measure on X". Any such functional can be written in many ways as the difference of two similar functionals, each having the additional property of being positive in the sense that they assign a number ≥ 0 to any function f satisfying $f(x) \geq 0$ for all $x \in X$. These latter functionals are termed "positive Radon measures on X", and it is to these that we may confine our attention.

It is a well known theorem of F. Riesz (Banach [1], pp. 59—61) that if X is the compact interval $[0,1]$ of the real axis, then any positive linear functional μ on $\mathfrak{C}$ has a representation in the form

$$\mu(f) = \int_0^1 f(x)\, d\, V(x).$$

$V(x)$ being a certain bounded, non-decreasing point-function on $[0,1]$. When X is a general locally compact space, the problem has been treated (albeit in a rather incidental fashion) in Halmos' book "Measure theory", which will be denoted by [H] in the sequel. It is shown ([H], Chapter X, Theorem D, p. 247) that every positive linear functional μ on $\mathfrak{C}$ of the type described has a representation

$$\mu(f) = \int_X f(x)\, d m(x) \tag{1.1}$$

where m is a certain Borel measure on X ([H], pp. 223–4). The result for a general compact space X is also proved in the recent publication [2].

At the moment when I was ready to submit the present account for publication, I received from E. Hewitt a reprint of his recent paper written jointly with H. S. Zuckerman [3]. This joint paper forms the sequel to an article bearing the same title which is written by Hewitt alone and which is not yet published. The overlap between the present theory and that developed in [3] is not large and the two accounts are in many respects complementary. I wish to express here my thanks to Professor Hewitt for reading and commenting upon the present paper in its MS form; several of his suggestions have been incorporated with advantage. Apart from fragmentary indications devoid of all detail which have appeared in the writings of Cartan and Godement, I believe that the account in [3] and that given below are the only ones which develop the theory of Radon measures in the particular manner adopted (in which the novel feature is the appearance of the measures as functionals).

The main aim of the paper is to show that a positive Radon measure on X defines a countably additive measure function on X by integration with respect to which we can recover the original functional. The method we adopt has several advantages over that of Halmos. For example, if X is not countable at infinity, the class of Borel sets in X as defined by Halmos does not include all open sets. These sets are so simple that it is desirable that they be measurable for the measure m in (1.1). Again, the class of Borel sets is not necessarily closed under complementation: in our treatment, the class of measurable sets does enjoy this property, includes all the wide Borel sets defined in § 2, and is closed under the operation (A) of Lusin and therefore includes all the analytic subsets of X. Finally, Halmos does not show that the measure m in (1.1) is regular for all Borel sets: we show that it is possible to arrange that it has this property on a significantly wider class of sets.

None of these remarks has any significance when X is countable at infinity, and it cannot be denied that in the contrary case the theory of the measure-function is not as complete as one might wish. The defects appear to be irrevocably connected with the pathology of the infinite in measure theory. However, these difficulties appear only when one insists on introducing the measure-function, of which there is strict need only for the purposes of comparison with Halmos' treatment. The integration theory appears to be satisfactory in most important respects.

The method we adopt is similar to that exploited by McShane in his book "Integration" in order to discuss the Lebesgue and Lebesgue-Stieltjes integrals on the real axis and Euclidean spaces. However, some of the proofs given there need important modifications in the general case. In addition, McShane's treatment corresponds to studying the method over each compact subset of X separately. There is no need to do this, and the process is often inconvenient. Nevertheless, it will often be convenient to make references to McShane's proofs whenever they are applicable, and in doing this we shall refer to his book by the symbol [M].

The Bourbaki volume on integration, hereafter referred to as [B], appeared whilst the present paper was in the hands of the referee. Bourbaki's treatise uses a method different from that discussed here, presumably in order to have a uniform treatment of real-valued and vector valued functions. This is made possible by completing $\mathfrak{C}$ as a uniform structure, rather than as a partially ordered set. The method used here is linked with that of Bourbaki by means of Exercices 5 and 6 on pp. 172—173 of [B]. Other comparisons with [B] will be made at the appropriate places. Several proofs which formed part of the original version of this paper have been omitted and references to [B] substituted. My thanks are due to the referee for suggestions in this and many other directions.

2. Preliminaries Concerning Measures.

We shall adopt the notation and terminology of [H] as far as is convenient.

Let us consider first certain non-void sets $\mathbf{F}$ of subsets of X, a locally compact (Hausdorff) space with points x. $\mathbf{F}$ is said to be a σ-ring if it is closed under countable unions and differences ([H], p. 24); $\mathbf{F}$ is a σ-algebra if it is a σ-ring and contains X as a member, in which case it is closed under complementation. $\mathbf{B}$ denotes the set of all Borel subsets of X ([H], p. 219), the minimal σ-ring containing all the compact subsets of X. $\mathbf{B}^*$ is the set of all "wide" Borel sets, the minimal σ-ring containing all closed subsets of X. Always, $\mathbf{B}<\mathbf{B}^*$, and $\mathbf{B}=\mathbf{B}^*$ if and only if X is countable at infinity.

A subset of X is called bounded if it is relatively compact (that is, if it has a compact closure); it is called σ-bounded if it is the union of countably many bounded sets. The set of all σ-bounded sets in X is a σ-ring containing $\mathbf{B}$. Any σ-bounded open set is a member of $\mathbf{B}$ ([H], Theorem A, p. 219).

If $\mathbf{F}$ is a σ-ring of subsets of X, a measure on $\mathbf{F}$ will mean a set-function m, defined, positive (that is, ≥ 0), possibly $+\infty$, and countably additive on $\mathbf{F}$; m is locally bounded on $\mathbf{F}$ in case $m(S) < +\infty$ for every bounded set $S \in \mathbf{F}$.

By a measure-space (over X) we shall understand a pair $\{\mathbf{F}, m\}$ formed of a σ-ring $\mathbf{F}$ of subsets of X, assumed to be at least as large as $\mathbf{B}$, together with a measure m on $\mathbf{F}$; $\{\mathbf{F}, m\}$ is said to be locally bounded if m is locally bounded on $\mathbf{F}$, that is if $m(K) < +\infty$ for all compact subsets K of X. A subset $S \in \mathbf{F}$ is said to be

a) inner-$\{\mathbf{F}, m\}$-regular if $m(S) = \sup\ m(K)$ for all compact sets $K < S$;

b) outer-$\{\mathbf{F}, m\}$-regular if $m(S) = \inf\ m(G)$ for all open sets $G \in \mathbf{F}$ with $G > S$;

S is $\{\mathbf{F}, m\}$-regular if it satisfies both a) and b). The measure-space $\{\mathbf{F}, m\}$ is itself said to be regular if every set $S \in \mathbf{F}$ is $\{\mathbf{F}, m\}$-regular.

By a Borel measure space (over X) we mean a measure-space of the form $\{\mathbf{B}, m\}$: such a measure-space is completely defined by its measure-function m, and it is simpler to speak of a Borel measure, a locally bounded Borel measure, et cetera. These notions agree with those of Halmos.

Any measure-space $\{\mathbf{F}, m\}$ defines a Borel measure when we restrict m to $\mathbf{B}$: the resulting Borel measure (-space) is termed the Borel restriction of $\{\mathbf{F}, m\}$.

We can now summarise some of the results we shall prove:

1) With every positive Radon measure μ on X is associated a locally bounded measure-space $\{\mathbf{F}, \mu\}$ in which $\mathbf{F}$, the set of sets termed "measurable for μ", is a σ-algebra at least as large as $\mathbf{B}^*$ and containing all the analytic subsets of X (Theorems 5 and 9).

2) The set $\mathbf{F}_{\mathrm{reg}}$ of sets which are $\{\mathbf{F}, \mu\}$-regular invariably contains $\mathbf{B}$ and all open sets; $\{\mathbf{F}, \mu\}$ is regular at least whenever X is the union of countably many open sets of finite measure, hence surely whenever either X is countable at infinity or μ is bounded (Theorem 7 and corollaries).

3) The Borel restriction of $\{\mathbf{F}, \mu\}$ is a locally bounded and regular Borel measure (Theorem 8).

4) Every locally bounded and regular Borel measure is the Borel restriction of the measure-space $\{\mathbf{F}, \mu\}$ associated with a unique positive Radon measure μ (Theorem 10).

3. Preliminaries Concerning Functions.

By a function we shall always mean a real-valued function on X; continuity is taken to imply finiteness, but discontinuous functions may take the values $\pm\infty$. If f is a continuous function, K_f, the support of f, is the closure of the set $\{x \in X: f(x) \neq 0\}$: K_f is thus the complement of the maximal open set on which f vanishes (cf. § 9).

$\mathfrak{C}(X)$ (denoted by $\mathcal{L}$ in [H], p. 240) is the real vector space of continuous functions f on X for each of which K_f is compact. This class of functions is denoted by $\underset{\infty\,\infty}{\mathfrak{C}}(X, R)$ in [3].

Given two functions f and g, $f \leq g$ means that $f(x) \leq g(x)$ for each $x \in X$; we then say that f minorises g, and that g majorises f. We say that f is positive if $f \geq 0$. $\mathfrak{C}^+$ is the set of positive functions in $\mathfrak{C}$.

On occasions it will be useful to employ the notion of directed sets and systems of functions or of subsets of X. In either case, the distinction between increasing and decreasing directed sets and systems will be quite obvious. The only point requiring explicit mention in this connection is this: if I and J are directed sets, $I \times J$ will stand for the directed set having as elements ordered pairs (i, j) with $i \in I$ and $j \in J$ and having as partial order the relation $(i, j) < (i', j')$ signifying that $i < i'$ and $j < j'$.

The following four lemmas will be required.

Lemma 1: *If $K \subset X$ is compact, and if N is a neighbourhood of K, there is a continuous function f such that*

$$0 \leq f \leq 1, \quad f = 1 \text{ on } K, \quad f = 0 \text{ on } X - N;$$

in particular, $f \in \mathfrak{C}^+$ if N is bounded.

A proof of this is given in [H], Theorem B, p. 216.

Lemma 2: *Any open set $G \subset X$ is the union of an increasing directed set of bounded open sets, each having its closure contained in G.*

Proof: For each $p \in G$, select an open neighbourhood $N(p)$ of p such that $\overline{N(p)}$ is compact and contained in G: this is possible because X is locally compact and hence regular. The required directed set of open sets results on taking the finite unions of the selected neighbourhoods $N(p)$.

We now introduce two further sets of functions which play a fundamental role in the integration theory, namely: $\mathfrak{L} = \mathfrak{L}(X)$, the set of lower semicontinuous functions each majorising some function in $\mathfrak{C}$; and $\mathfrak{U} = \mathfrak{U}(X)$, the set of upper semicontinuous functions each minorising some function in $\mathfrak{C}$. In an obvious notation, $\mathfrak{U} = -\mathfrak{L}$ and $\mathfrak{L} = -\mathfrak{U}$, so that every proposition concerning $\mathfrak{L}$ has an analogue for $\mathfrak{U}$ which is obtained by change of sign. We remark that by convention a function in $\mathfrak{L}$ may be $+\infty$ at some or all points, but is nowhere $-\infty$; an analogous convention applies to $\mathfrak{U}$. If φ is a function which admits the value $+\infty$ at the point x_0, lower semicontinuity of φ at the point x_0 means that for each *finite* number α, there is a neighbourhood of x_0 (in general depending upon α) throughout which $\varphi(x) > \alpha$. A similar remark applies to upper semicontinuity at a point where the value $-\infty$ is attained.

Lemma 3: *Any function in $\mathfrak{L}$ is the upper envelope of those functions in $\mathfrak{C}$ which it majorises.*

This assertion is trivial; cf. [B], p. 103, Lemme 1. For future reference we observe that the lemma may be stated thus: if $\varphi \in \mathfrak{L}$, the set D_φ of functions $f \in \mathfrak{C}$ majorised by φ is an increasing directed set, and we have at each point of X the relation $\varphi = \lim\limits_{f \in D_\varphi} f$.

The final lemma of this group has interest for those spaces X which satisfy certain countability restrictions, but it is not essential to the general argument. I justify its inclusion on the grounds that, wherever it is applicable, it simplifies a number of subsequent proofs.

Lemma 4: *In order that every function $\varphi \in \mathfrak{L}$ be the limit of an increasing (countable) sequence of functions in $\mathfrak{C}$, it is necessary and sufficient that X satisfies the two countability restrictions:*

(c) *X is countable at infinity;*

(c′) *every open set in X is an F_σ-set.*

Proof: The necessity is trivial. A sketch of the proof of sufficiency follows. To begin with, there is no loss of generality involved in considering only positive functions. Obviously, the subset $\mathfrak{L}_0$ of $\mathfrak{L}$ formed of limits of increasing sequences of functions in $\mathfrak{C}$ is itself closed under the operations of taking limits of increasing sequences and of forming finite linear combinations with positive coefficients. Now every positive φ in $\mathfrak{L}$ is easily seen to be the limit of an increasing sequence of finite linear combinations with positive coefficients of characteristic functions of open sets in X. If (c) is true, the characteristic function of any open set in X is the limit of an increasing sequence of characteristic functions of bounded open sets. If (c′) is true, Lemma 1 shows immediately that any such function is in $\mathfrak{L}_0$.

4. The Definition of Radon Measures on X.

It is obvious that the definition of a Radon measure on X given in § 1 above is identical with that described in Définition 1, p. 50 of [B]. We therefore take its formulation for granted and, for the purposes of future reference, denote it hereafter as Definition 1. Observe that the type of continuity required of a linear functional μ on $\mathfrak{C}$ in order that it be a Radon measure is equivalent to the existence for each compact subset K of X of a finite, positive number M_K such that

$$|\mu(f)| \leq M_K \cdot \sup_{x \in X} |f(x)| \tag{4.1}$$

is valid for all $f \in \mathfrak{C}$ with $K_f < K$; cf. equation (2), p. 50 of [B], where our $\mathfrak{C}$ is denoted by $\mathcal{K}$.

As has been said, it is not difficult to show that any linear functional μ on $\mathfrak{C}$ which satisfies the inequalities (4.1) is decomposable in many ways into the form

$$\mu = \mu' - \mu'' \tag{4.2}$$

where μ' and μ'' are positive Radon measures on X. Further, there exists always a unique so-called "minimal decomposition"

$$\mu = \mu^+ - \mu^- \tag{4.3}$$

of this form having the property that for any decomposition (4.2), $\mu' - \mu^+$ and $\mu'' - \mu^-$ are both positive Radon measures. See [B], Théorème 2, p. 54.

For a positive linear functional on $\mathfrak{C}$, the continuity is automatically arranged. Thus one might define a Radon measure as a linear functional on $\mathfrak{C}$ having at least one decomposition (4.2) into the difference of two positive linear functionals, thereby avoiding all reference to continuity.

We shall show in § 12 how $\mathfrak{C}$ may be provided with a topology making it into a locally convex topological vector space in such a manner that the Radon measures are precisely those linear functionals on $\mathfrak{C}$ which are continuous for this topology. Meanwhile, the notion of continuity already defined serves equally well.

From this point onwards until explicit mention to the contrary, we shall consider a positive Radon measure on X, fixed once for all.

5. The Integral defined by μ.

We aim to show that the domain of the functional μ can be extended to a wide class of functions having many of the properties of the Lebesgue-summable functions on the real axis, the functional μ playing the role of an operation of integration. In anticipation of this we shall, for $f \in \mathfrak{C}$, write $\mu(f)$ in the form $\int_X f(x)\,d\mu(x)$ or simply $\int f\,d\mu$ when no confusion can arise: this is pure symbolism at present.

The first stage in the extension of the domain of μ is to define the integral for functions in $\mathfrak{L}$ or in $\mathfrak{U}$.

Definition 2: *If* $\varphi \in \mathfrak{L}$ *we define*

$$\int \varphi\,d\mu = \sup \int f\,d\mu \quad \text{for } f \in \mathfrak{C},\ f \leq \varphi;$$

and if $\psi \in \mathfrak{U}$ *we define*

$$\int \psi\,d\mu = \inf \int g\,d\mu \quad \text{for } g \in \mathfrak{C},\ g \geq \psi.$$

Remark: This definition of $\int \varphi \, d\mu$ for $\varphi \in \mathfrak{L}$ agrees with that of $\mu^*(\varphi)$ for positive φ given in Définition 1, p. 104 of [B].

Since the intersection of $\mathfrak{L}$ and $\mathfrak{U}$ is exactly $\mathfrak{C}$, these definitions will be shown to be compatible provided we make certain that for $f_0 \in \mathfrak{C}$, $\mu(f_0) = \sup \mu(f) = \inf \mu(g)$, where f and g separately range over $\mathfrak{C}$ and are further subject to $f \leq f_0$ and $g \geq f_0$ respectively. But this is obvious.

If $\varphi \in \mathfrak{L}, -\infty < \int \varphi \, d\mu \leq +\infty$; if $\psi \in \mathfrak{U}, -\infty \leq \int \psi \, d\mu < +\infty$. Finally, if Θ belongs to $\mathfrak{L}$ or to $\mathfrak{U}, -\Theta$ belongs to $\mathfrak{U}$ or to $\mathfrak{L}$ respectively, and in either case $\int (-\Theta) \, d\mu = -\int \Theta \, d\mu$ · For this reason it is always enough to prove properties of integrals of functions in $\mathfrak{L}$.

Proposition 1: *If φ (automatically in $\mathfrak{L}$) is the limit of an increasing directed system $(\varphi_i)_{i \in I}$ of functions in $\mathfrak{L}$, then*

$$\int \varphi \, d\mu = \lim_{i \in I} \int \varphi i \, d\mu.$$

For positive functions, this is just Théorème 1, p. 105 of [B]; removal of the restriction that the functions be positive involves no difficulty.

Remarks: (i) If we take in place of $(\varphi_i)_{i \in I}$ the directed set D_φ, we recover Definition 1, which may be written in the form

$$\int \varphi \, d\mu = \lim_{f \in D_\varphi} \int f \, d\mu.$$

(ii) If X satisfies (c) and (c′), every φ in $\mathfrak{L}$ is the limit of a monotone ascending countable sequence $(f_n) < \mathfrak{C}$ so that we could then define

$$\int \varphi \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu,$$

the right member being independent of the particular sequence (f_n) used.

Proposition 2: (a) *The integral is additive and homogeneous on $\mathfrak{L}$ (and therefore on $\mathfrak{U}$ as well).*

(b) *If $\varphi \in \mathfrak{L}$ and $\psi \in \mathfrak{U}$, and if $\psi \leq \varphi$, then $\int \psi \, d\mu \leq \int \varphi \, d\mu$.*

Proof: (a) follows easily from Proposition 1. It is also equivalent to the conjunction of Proposition 1 and Théorème 2, pp. 104—106, of [B].

(b) There is no loss of generality in assuming both integrals finite. Write φ and ψ in the forms

$$\varphi = \lim_{i \in I} f_i,$$

$$\psi = \lim_{j \in J} g_j,$$

where $(f_i)_{i\in I}$ is an increasing, and $(g_j)_{j\in J}$ a decreasing, directed system of functions in $\mathfrak{C}$. Then $(f_i-g_j)_{(i,j)\in I\times J}$ is an increasing directed system of functions in $\mathfrak{C}$ with limit the positive function $\varphi-\psi$ in $\mathfrak{L}$. So by Proposition 1,

$$0\leq\int(\varphi-\psi)\,d\mu=\lim_{(i,j)}\int(f_i-g_j)\,d\mu$$
$$=\lim_{i\in I}\int f_i\,d\mu-\lim_{j\in J}\int g_j\,d\mu,$$

since both limits on the right, namely $\int\varphi\,d\mu$ and $\int\psi\,d\mu$, are finite. This yields (b).

Remark: The essential application of (b) is the proof of the second half of Remark (iii) following the next definition, which is essentially Theorem 4.8 of [3]. But in [3] the assertion (b) is replaced by a rather delicate argument, given loc. cit. Theorem 3.4, on the possibility of injecting a function of $\mathfrak{C}$ between φ and ψ. The use of directed systems avoids this and makes the proof more transparent.

Now we can enter upon the second and final stage of the extension process, for which we formulate.

Definition 3: *For an arbitrary function f we define the upper μ-integral*

$$\overline{\int} f\,d\mu=\inf\int\varphi\,d\mu\quad\text{for}\quad\varphi\in\mathfrak{L},\ \varphi\geq f;$$

and the lower μ-integral

$$\underline{\int} f\,d\mu=\sup\int\psi\,d\mu\quad\text{for}\quad\psi\in\mathfrak{U},\ \psi\leq f.$$

We say that f is widely summable (for μ), or that $\int f\,d\mu$ exists, in case $\underline{\int} f\,d\mu=\overline{\int} f\,d\mu$ (possibly infinite), the value of $\int f\,d\mu$ being then the common value of the upper and lower integrals of f. Finally, f is summable (for μ) in case its integral exists and is finite.

Remarks: (i) Our definition of upper μ-integral coincides with that given for positive functions in Définition 3, p. 109 of [B]; see also Exercices 5 and 6, p. 172–3 of [B].

(ii) It is easy to see that any f in $\mathfrak{L}$ or in $\mathfrak{U}$ is widely summable, and that its integral according to Definition 3 agrees with that according to Definition 2. In particular, any $f\in\mathfrak{C}$ is summable.

(iii) Always $\overline{\int} f\,d\mu=-\underline{\int}(-f)\,d\mu$ and $\underline{\int} f\,d\mu\leq\overline{\int} f\,d\mu$.

(iv) There appears to be no term standard in integration theory which corresponds to our "widely summable": "integrable" would be misleading since a function can be widely summable without being in any sense measurable for μ.

(v) *If $f\leq g$, $\underline{\int} f\,d\mu\leq\underline{\int} g\,d\mu$, $\overline{\int} f\,d\mu\leq\overline{\int} g\,d\mu$.*

It is very easy to see that the upper integral is a convex and positive-homogeneous functional of the integrand, the only proviso being that the sum of two functions occurring in the statement of convexity be well defined. However, later results show that a summable function can be infinite only at the points of a set which is negligible in integration, and so it is legitimate to state without any proviso the basic

Theorem 1: *The set of functions summable for μ is a real vector space, addition and subtraction being taken modulo sets which are null for μ* (see § 6), *and the integral is a linear functional of the integrand on this set.*

The next theorem is perhaps the most fundamental of all.

Theorem 2: *Let (f_n) be an increasing sequence of widely summable functions such that $\int f_1 d\mu > -\infty$. Then $f = \lim_{n\to\infty} f_n$ is widely summable and*

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu.$$

The proof demands only a slight modification of that given in [M], Section 15.7s, p. 81. We remark that, even in the classical theories, Theorem 2 is not valid for increasing directed systems of widely summable (or even of summable) functions. See also [B], Théorème 5, p. 138, and Exercice 5, p. 172.

The next theorem may also be proved by standard means given for example in [M], Section 15.6s, p. 77; see also Proposition 12 and Corollaire, p. 136 of [B].

Theorem 3: *If f and g are summable, so also are the functions* $\sup(f, g)$ *and* $\inf(f, g)$. *Further, if f and g are bounded and summable, so is $f.g$.*

As a consequence, f is summable if and only if all of the functions $f^+ = \sup(f, 0)$, $f^- = \sup(-f, 0)$, $|f| = f^+ + f^-$ enjoy this property. The second assertion of Theorem 3 is of course only provisional but is necessary for the discussion of the measure-function generated by μ.

We end this § with the following characterisation of summable functions, the proof of which is an easy consequence of our definitions and of Theorem 2.

Theorem 4 (Vitali-Carathéodory): *For f to be summable, it is necessary and sufficient that there exist two functions Φ and Ψ with the properties;*
(a) *Φ is summable and is the limit of a decreasing sequence (φ_n) of summable functions in $\mathfrak{L}$, each majorising f; Ψ is summable and is the limit of an increasing sequence (Ψ_n) of summable functions in $\mathfrak{U}$, each minorising f;*
(b) $\int \Phi d\mu = \int \Psi d\mu$.
Whenever f is summable, $\int f d\mu$ is the common value of the integrals in (b).

Remarks: (i) One can say that Φ, Ψ and f are equal on a set which is null for μ in the sense of Definition 5 to follow.

(ii) Compare Théorème 3 and Corollaire, p. 151 of [B].

6. The Measure-Function generated by μ.

For a general locally compact space X, there is some difficulty in defining a notion of measurability which is desirable in all respects. Several alternative notions, which are known to be entirely equivalent in the classical cases, lead to widely divergent theories in the absence of suitable countability restrictions on X. Of any notion of measurability one will desire that on the one hand it be local (or analysable into the conjunction of local restrictions) and, on the other hand, that these local restrictions may be pieced together so as to form some sort of global restrictions on the set. It seems very difficult to discover how to piece together more than countably many such local restrictions in any one step. When X is countable at infinity, this is all that is required. But in the contrary case difficulties arise which I am unable to resolve satisfactorily.

We adopt a strictly local definition of measurability which admits as measurable many sets which are not included in the definitions of Halmos, but it appears that one cannot include these additional sets without sacrificing the regularity of the associated measure-space. However, nothing is lost by doing this since one can recover the results of Halmos by suitable restriction of the sets. The theory laid out in [3] is also included.

Our notion of measurability agrees with that prescribed in [B], Définition 2, p. 181; this results from Proposition 2, p. 182 of [B]. Agreement is also attained over the notions of sets which are globally null and those which are locally null, termed respectively "négligeable" and "localement négligeable" in [B], Définition 2, p. 118 and Définition 3, p. 183. Our interior and exterior measures μ_i and μ_e agree respectively with Bourbaki's μ_* and μ^*, these latter being introduced in [B], Exercice 7, p. 173, and Définition 4, p. 113: agreement is ensured by our Theorem 6 or by [B], Proposition 19, p. 114, and Exercice 7, p. 173.

Definition 4: *Let E be any subset of X, χ_E its characteristic function. We say that E is measurable (for μ) if and only if for every compact set $K \subset X$, the function $\chi_{E \cap K}$ is summable (for μ). When this is the case, the ($\mu-$) measure of E is by definition the number*

$$\mu(E) = \sup_K \int \chi_{E \cap K} \, d\mu.$$

The results concerning the integral already at our disposal lead to

Theorem 5: *The set* $\mathbf{F}=\mathbf{F}(\mu)$ *of subsets of* X *which are measurable for* μ *is* a *σ-algebra containing* $\mathbf{B}^*$, *and* $\{\mathbf{F},\mu\}$ *is a locally bounded measure-space over* X.

The interior and exterior $(\mu-)$ measures of an arbitrary set $E\subset X$ are introduced via the equations

$$\mu_i(E)=\sup\ \mu(K)\ \text{for}\ K\ \text{compact},\ K\subset E,$$

$$\mu_e(E)=\inf\ \mu(G)\ \text{for}\ G\ \text{open},\ G\supset E.$$

It is plain that both μ_i and μ_e are positive set-functions, that $\mu(E)$ lies between $\mu_i(E)$ and $\mu_e(E)$ whenever E is measurable, that $\mu_i(E)\leq\mu_e(E)$ whatever the set E, and finally that μ_e is countably sub-additive.

Definition 5: *A set* $E\subset X$ *is said to be null (for* μ*) in case* $\mu_e(E)=0$.

Any subset of a null set, and any countable union of null sets, is again null.

The next theorem relates the interior and exterior measures of a set with the lower and upper integrals of the characteristic function of that set. The theorem is proved in [M], Section 20.5s, pp. 111–4 for bounded sets, but it is rather essential for our purposes that this restriction be removed. Since the necessary modifications are not difficult to formulate, we shall merely state the result.

Theorem 6: *For an arbitrary set* $E\subset X$,

$$\mu_i(E)=\underline{\int}\chi_E\,d\mu,\ \ \mu_e(E)=\overline{\int}\chi_E\,d\mu.$$

The $\{\mathbf{F},\mu\}$-regularity of a set $E\in\mathbf{F}$ is equivalent to the equality $\mu_i(E)=\mu_e(E)$; Theorem 6 tells us that this in turn is equivalent to the wide summability of χ_E. This observation is one of the most important consequences of Theorem 6. Other consequences which are worth mentioning here are contained in the following corollaries.

Corollary 1: *If* E *is null,* χ_E *is summable, and* $\mu_i(E)=\mu_e(E)=\mu(E)=\int\chi_E\,d\mu=0$.

Corollary 2: *If* E *is null, if* f *is widely summable, and if* $g=f$ *save perhaps on* E, *then* g *is widely summable and* $\int g\,d\mu=\int f\,d\mu$.

Corollary 3: *If* f *is positive, and if* $\overline{\int}f\,d\mu=0$, *then* $f=0$ *save perhaps on a null set.*

Corollary 4: *If* f *is summable, then:* a) *it is zero outside the union of a σ-bounded set and a null set, and* b) *it is finite save perhaps at the points of a null set.*

Of these, Corollary 1 is immediate from Theorem 6. As for Corollary 2, it suffices to show that

$$\overline{\int}g\,d\mu\leq\int f\,d\mu; \tag{6.1}$$

for, granted this, we can replace in it g by $-g$ and f by $-f$ to conclude that $\underline{\int} g\,d\mu \geq \int f\,d\mu$, whence it follows that g is widely summable and that $\int g\,d\mu = \int f\,d\mu$. To prove (6.1), let us first assume that $\int f\,d\mu > -\infty$. Define for each integer n the function $g_n = f + n\,.\,\chi_E$. Since E is null, each g_n is widely summable and $\int g_n\,d\mu = \int f\,d\mu$. The sequence (g_n) is increasing, and $g \leq \lim_{n\to\infty} g_n$. By Theorem 2, $\lim_{n\to\infty} g_n$ is widely summable and

$$\bar{\int} g\,d\mu \leq \int (\lim_{n\to\infty} g_n)\,d\mu = \lim_{n\to\infty} \int g_n\,d\mu = \int f\,d\mu,$$

which is (6.1). If now $\int f\,d\mu = -\infty$, we wish to show that $\bar{\int} g\,d\mu = -\infty$ also. But, for any finite number α, there is $\varphi \in \mathfrak{L}$ majorising f such that

$$\int \varphi\,d\mu < \alpha - 1.$$

For each $k = 1, 2, \ldots$, enclose E is an open set U_k of measure less than 2^{-k}. The function $\varphi_1 = \varphi + \sum_{k=1}^{\infty} \chi_{U_k}$ is in $\mathfrak{L}$ and majorises g, since it is equal to $+\infty$ on E. Hence

$$\bar{\int} g\,d\mu \leq \int \varphi_1\,d\mu = \int \varphi\,d\mu + \sum_{k=1}^{\infty} \mu(U_k) < \alpha - 1 + \sum_{k=1}^{\infty} 2^{-k} = \alpha.$$

Since this is true for any finite α, (6.1) is again verified and the proof thus completed.

The proof of Corollary 3 runs along standard lines. Finally, consider a) of Corollary 4: the proof of b) is similar. We may assume that f is positive. Being summable, it is majorised by a summable function in $\mathfrak{L}$, and hence we may assume that f itself is positive, summable, and in $\mathfrak{L}$. Then there is an increasing sequence $(f_n) < \mathfrak{C}$ such that $f_n \leq f$ and $\lim_{n\to\infty} \int f_n\,d\mu = \int f\,d\mu$. Let $f_0 = \lim_{n\to\infty} f_n$, so that f_0 is summable (Theorem 2), minorises f, and $\int f_0\,d\mu = \int f\,d\mu$. Thus we have both $f - f_0 \geq 0$ and $\int (f - f_0)\,d\mu = 0$. By Corollary 3, $f = f_0$ save on a null set. This completes the proof since f_0 is zero outside the union of the compact supports of the f_n.

Remark: Assertion a) of Corollary 4 is a simple but significant result when one is dealing with spaces X which are not countable at infinity.

7. Concerning Regularity.

We recall from § 1 that $\mathbf{F}_{\text{reg}} = \mathbf{F}_{\text{reg}}(\mu)$ denotes the set of sets $E \in \mathbf{F}(\mu)$ which are $\{\mathbf{F}, \mu\}$-regular, that is which are such that χ_E is widely summable (according to the remarks following Theorem 6). We can prove

Theorem 7: (1) *Every set $E \in \mathbf{F}$ is inner-$\{\mathbf{F}, \mu\}$-regular.*

(2) *Every set in $\mathbf{F}_{\text{reg}}$ having finite measure is almost σ-bounded (that is, is the union of a σ-bounded set and a null set).*

(3) *Every almost σ-bounded measurable set is in* $\mathbf{F}_{\text{reg}}$.

(4) *Every measurable set E is the disjoint union $P \cup Q$, where P is σ-bounded and measurable (hence in $\mathbf{F}_{\text{reg}}$) and where Q is "locally null" in the sense that $\mu_e(Q \cap K) = 0$ for every compact set K.*

(5) *Every open set is in* $\mathbf{F}_{\text{reg}}$.

Proof: (1) Let $E \in \mathbf{F}$ and let $\alpha < \mu(E)$. There is then a compact set K_0 such that $\int \chi_{E \cap K_0} d\mu > \alpha$. Since $\chi_{E \cap K_0}$ is summable, Theorem 6 shows that $\int \chi_{E \cap K_0} d\mu = \mu_i(E \cap K_0)$. Hence there is a compact set $K \subset E \cap K_0$, a fortiori $K \subset E$, such that $\mu(K) > \alpha$. Therefore $\mu_i(E) \geq \mu(K) > \alpha$. So $\mu_i(E) \geq \mu(E)$, and the assertion is proved.

(2) Let $E \in \mathbf{F}_{\text{reg}}$ have finite measure. Then χ_E is summable by Theorem 6; and by Corollary 4, χ_E is zero outside the union of a σ-bounded set and a null set, which yields our assertion.

(3) Let $E \in \mathbf{F}$ be almost σ-bounded. We can then write $E = \lim_{n\to\infty} (E_n \cup N)$, where the increasing sequence (E_n) is composed of bounded measurable sets, and where N is null. Since E_n is bounded and measurable, χ_{E_n} is summable, hence so also is $\chi_{E_n \cup N}$. By Theorem 2, χ_E is at any rate widely summable, and so $E \in \mathbf{F}_{\text{reg}}$.

(4) If $E \in \mathbf{F}$, we can find an increasing sequence (K_n) of compact sets such that $\chi_{E \cap K_n}$ is summable and

$$\lim_{n\to\infty} \int \chi_{E \cap K_n} d\mu = \mu(E).$$

Let $P = \lim_{n\to\infty} (E \cap K_n)$: P is measurable, contained in E, and is σ-bounded. Let $Q = E - P$: Q is measurable and, if C is any compact set, Theorem 2 yields

$$\mu(E \cap C) = \int \chi_{E \cap C} d\mu = \lim_{n\to\infty} \int \chi_{E \cap K_n \cap C} d\mu = \lim_{n\to\infty} \mu(E \cap K_n \cap C) = \mu(P \cap C).$$

Thus $\mu(Q \cap C) = 0$. Since $Q \cap C$ is bounded and measurable, this last is equivalent by 3) to $\mu_e(Q \cap C) = 0$, so that Q is locally null.

(5) This is immediate since the characteristic function of any open set, being in $\mathfrak{L}$, is widely summable.

Corollary 1: *If X itself is almost σ-bounded relative to μ (or equivalently, if X is the union of countably many open sets of finite measure), then $\mathbf{F}_{\text{reg}}$ exhausts $\mathbf{F}$. This relation therefore holds in the classical cases in which X is countable at infinity, or if the measure μ is bounded.*

Corollary 2: *Always,* $\mathbf{F}_{\text{reg}} \subset \mathbf{B}$.

We observe here that for a set Q to be locally null, it is necessary and sufficient that it be measurable and have measure zero. We prefer the term "locally null" since, although in the classical cases $\mu(Q)=0$ is equivalent to $\mu_e(Q)=0$, this is no longer the case in general: this is shown by example in § 13 in connection with the Haar measures on topological groups. The example there constructed is a set Z which is locally null and has infinite exterior measure.

According to Theorem 7, (4), in order that $\mathbf{F}_{\text{reg}}$ shall exhaust $\mathbf{F}$, it is necessary and sufficient that every locally null set be null.

Theorem 8: *The Borel restriction of* $\{\mathbf{F}, \mu\}$ *is a locally bounded and regular Borel measure over* X.

Proof: In view of Theorem 7, Corollary 2, it remains only to show that if $E \in \mathbf{B}$, then $\mu(E)=\inf \mu(H)$ for $H \in \mathbf{B}$, H open, and $H \supset E$. For this it is enough to show that any open set $G \supset E$ contains some open Borel set $H \supset E$. However, since E is σ-bounded, we have $E \subset \bigcup_{n=1}^{\infty} O_n$, where each O_n is bounded and open. Putting $H = \bigcup_{n=1}^{\infty} (G \cap O_n)$, we see that $G \supset H \supset E$, that H is open, and that H is σ-bounded and so belongs to $\mathbf{B}$ (see § 2). This completes the proof.

We turn next to a brief consideration of the notion of measurability in the sense of Carathéodory. The exterior measure μ_e is defined for all subsets of X and satisfies the conditions:

(C 1) $\mu_e(E) \leq \mu_e(F)$ whenever $E \subset F$;

(C 2) $\mu_e(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mu_e(E_n)$ for any sequence (E_n) of sets. These are two of the three conditions characterising a Carathéodory outer measure on X (see [4], § 4, p. 43). The third condition is formulated loc. cit. only for metric spaces X and is to the effect that $\mu_e(E \cup F) = \mu_e(E) + \mu_e(F)$ shall be true whenever the sets E and F are at positive distance. We shall replace it here by the condition

(C 3) $\mu_e(E \cup F) = \mu_e(E) + \mu_e(F)$ whenever the sets E and F have disjoint neighbourhoods.

The properties (C 1) and (C 2) have already been noted in § 6. As for (C 3), if E and F can be enclosed in disjoint open sets G and H respectively, and if U is any open set containing $E \cup F$, then $U \cap G$ and $U \cap H$ are disjoint open sets containing E and F respectively. Therefore

$$\mu(U) \geq \mu(U \cap G) + \mu(U \cap H) \geq \mu_e(E) + \mu_e(F).$$

Taking the infimum on the left, we derive

$$\mu_e(E \cup F) \geq \mu_e(E) + \mu_e(F),$$

which, combined with (C 2), yields (C 3).

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Let us consider the set $\mathbf{F}_c = \mathbf{F}_c\ (\mu)$ of sets $E \subset X$ which are measurable in the sense of Carathéodory relative to the outer measure μ_e: $E \in \mathbf{F}_c$ if and only if

$$\mu_e\,(P \cup Q) = \mu_e\,(P) + \mu_e(Q) \tag{7.1}$$

is true for arbitrary sets $P \subset E$ and $Q \subset X - E$, or, equivalenty, if and only if

$$\mu_e\,(A) = \mu_e\,(A \cap E) + \mu_e\,(A - A \cap E) \tag{7.2}$$

is true for all sets $A \subset X$. By (C 2), we do not alter the content of either (7.1) or (7.2) on replacing therein "$=$" by "$\geq$".

The arguments of [4], pp. 44–50 are independent of (C 3) and show that $\mathbf{F}_c$ is a σ-algebra which is closed under the operation (A) of Lusin.

The proof given on p. 51 of [4] that $\mathbf{F}_c$ contains all the wide Borel sets uses the property (C 3) (in its original form for metric spaces X). However, the proof of the essential Lemma (7.1) loc. cit. can be put through in the present case. In the terminology of Bourbaki, Topologie générale, Chapitre IX, 1, Nos. 4 and 5, the space X is uniformisable; take any "écart" φ on X which is uniformly continuous relative to any uniform structure compatible with the topology of X; this φ can replace the metric in Saks' argument. The proof of Theorem (7.4), p. 52 of [4] is then as before.

Finally, it is easy to prove by familiar arguments (see for example the ultimate paragraph on p. 117 of [M]) that $\mathbf{F}_c \subset \mathbf{F}$. On the other hand, it does not seem possible to show (along the lines of Section 20.10s, p. 116 of [M]) that conversely $\mathbf{F} \subset \mathbf{F}_c$, though this is surely the case if X is almost σ-bounded relative to μ.

To summarise:

Theorem 9: $\mathbf{F}_c$ *is a σ-algebra, closed under the operation (A) of Lusin and containing all the wide Borel sets and hence all the analytic sets. Always* $\mathbf{F}_c \subset \mathbf{F}$ *and* $\mathbf{F}_c = \mathbf{F}$ *at least whenever X is almost σ-bounded relative to μ.*

8. Further Theorems on the Integral.

There is now no difficulty in defining the notion of measurability of functions and in proving most of the standard results: see for example [H], p. 76 et seq. In fact, the only familiar result which is not proved to remain valid in the general case is the summability of the characteristic function of a measurable set of finite measure: this is certainly true if X is almost σ-bounded, but may be false in the contrary case (see § 13). However, it remains true that a measurable function which is dominated by a summable function is itself summable.

The path is then open to prove in the standard fashion both Fatou's lemma and Lebesgue's theorem on the termwise integration of dominated sequences of summable functions. Neither the statements nor the proofs of these familiar results need delay us here.

9. The Support of a Radon Measure.

This useful notion is defined first for a positive Radon measure μ. We know that

$$\mu(G) = \sup \int g\, d\mu \quad \text{for } g \in \mathfrak{C},\ g \leq \chi_G,$$

G being any open set in X. It is important for our purpose that this be refined to the extent of asserting that

$$\mu(G) = \sup \int f\, d\mu \quad \text{for } f \in \mathfrak{C},\ f \leq 1,\ K_f \subset G. \tag{9.1}$$

The truth of (9.1) follows from Lemma 2, from Proposition 1, and from the equality $\mu(G) = \int \chi_G\, d\mu$, this last being an almost immediate consequence of the definition of $\mu(G)$.

From (9.1) it follows easily that the union of any set of open sets each of measure zero again has measure zero. Hence there exists a maximal open set of measure zero. The complement of this set is by definition the support K_μ of the positive Radon measure μ.

In case μ is not positive, we employ the minimal decomposition $\mu = \mu^+ - \mu^-$ and define the support K_μ of μ to be the union of K_{μ^+} and K_{μ^-}.

This is a consistent generalisation of the notion of support of a continuous function φ in the following sense, the situation described being the most usual in which both notions are in use simultaneously. If X is a group on which a Haar measure is denoted symbolically by dx (see § 13), to any continuous function φ on X one may assign the Radon measure μ defined by

$$\int f\, d\mu = \int f \cdot \varphi\, dx \quad \text{for all } f \in \mathfrak{C}.$$

It then turns out that the support of the measure μ is none other than the support of φ.

The substance of this § may be compared with that of pp. 67–73 of [B]. Our definition of support is exactly that of Bourbaki for positive measures, and agreement for other measures is ensured by his Proposition 2, p. 70.

10. Borel Measures as Positive Radon Measures.

Let us assume given a locally bounded and regular Borel measure m over X. Then we may construct the corresponding integral $\int f\, dm$ exactly as in [H], Chapter V.

The integral is defined and finite at any rate for all $f \in \mathfrak{C}$. Hence there exists one and just one positive Radon measure μ on X determined by the equation

$$\mu(f) \equiv \int f \, d\mu = \int f \, dm \text{ for } f \in \mathfrak{C}. \tag{10.1}$$

Our immediate aim is to show that m is none other than the Borel restriction of the measure-space $\{\mathbf{F}, \mu\}$ associated with μ. Since, according to Theorem 8, the Borel restriction of $\{\mathbf{F}, \mu\}$ is a regular Borel measure, it is necessary merely to verify that $\mu(K) = m(K)$ for every compact set $K \subset X$. Now the appropriate definitions give, on the one hand, that

$$m(K) = \int \chi_K \, dm,$$

and on the other hand that

$$\mu(K) = \int \chi_K \, d\mu = \inf \int f \, d\mu = \inf \int f \, dm$$

for all $f \in \mathfrak{C}$ satisfying $f \geq \chi_K$. Thus we have only to show that

$$\int \chi_K \, dm = \inf \int f \, dm. \tag{10.2}$$

But it is plain that the left member minorises the right, the measure m being positive. On the other hand, since m is regular, for every $\varepsilon > 0$ there is an open set $G \in \mathbf{B}$ containing K and such that $m(G) < m(K) + \varepsilon$. We may plainly assume that G is bounded. Then by Lemma 1 we can choose $f \in \mathfrak{C}$ satisfying $f = 1$ on K, $0 \leq f \leq 1$, $f = 0$ on $X - G$. Hence $\chi_K \leq f \leq \chi_G$ and so

$$\int f \, dm \leq \int \chi_G \, dm = m(G) < m(K) + \varepsilon.$$

Since ε is arbitrary, this proves (10.2). We are thus free to state

Theorem 10: *Any locally bounded and regular Borel measure over X is the Borel restriction of a unique $\{\mathbf{F}, \mu\}$ linked with a positive Radon measure μ on X.*

The substance of this § may be compared with that of pp. 164—169 of [B], though Theorem 10 is not an immediate consequence of Bourbaki's Théorème 5, p. 165, since the given Borel measure m is not necessarily finite-valued.

11. Radon Measures on Product Spaces.

It is desirable that the product of two Radon measures be defined in terms of their appearance as functionals on the spaces $\mathfrak{C}$, rather than by the standard method used for measure-functions (see for example [H], Chapter VII). The way in which this may be done is described fully in the proof of Théorème 1, p. 89 of [B], and we content ourselves with recalling the main result. As usual, it is enough to consider only positive measures.

Let X and Y be two locally compact spaces with points x and y respectively; let $Z=X\times Y$ be their topological product with points $z=(x, y)$. Z is again a locally compact space. Suppose given two positive Radon measures, μ on X and ν on Y. There is then a unique positive Radon measure λ on Z, termed the product of μ and ν and denoted by $\lambda=\mu\otimes\nu$, such that

$$\int h\,d\lambda=\int f\,d\mu\cdot\int g\,d\nu \tag{11.1}$$

holds for every function $h(z)=h(x, y)$ on Z having the form $f(x)\cdot g(y)$ with $f\in\mathfrak{C}(X)$ and $g\in\mathfrak{C}(Y)$. In view of the fact (proved in Lemme 1, p. 89 of [B]) that the functions on Z of the form

$$\sum_{i=1}^{n} f_i(x)\cdot g_i(y), \tag{11.2}$$

with $f_i\in\mathfrak{C}(X)$ and $g_i\in\mathfrak{C}(Y)$, are dense in $\mathfrak{C}(Z)$, it follows by continuity from (11.1) that

$$\int h\,d\lambda=\int d\mu(x)\int h(x, y)\,d\nu(y)=\int d\nu(y)\int h(x, y)\,d\mu(x) \tag{11.3}$$

is valid for every $h\in\mathfrak{C}(Z)$; see Théorème 2, p. 91 of [B]. This is to say that the Fubini theorem is automatically ensured for functions $h\in\mathfrak{C}(Z)$.

However, the fact that (11.3) is valid for all $h\in\mathfrak{C}(Z)$ does not imply trivially that the same relation is significant and valid for, say, all h summable for λ. That this is true is the assertion of the Fubini theorem, which we shall study next. It is in fact interesting that the Fubini theorem *does* hold since the measure-spaces involved are not generally subject to the condition of being σ-finite in the sense of Halmos ([H], p. 146), and it is known that the theorem is not unrestrictedly true in the absence of this condition.

Theorem 11 (Fubini): *If $h(z)=h(x, y)$ is summable for $\lambda=\mu\otimes\nu$, then:*

(1) *for almost all $[\mu]$ $x\in X$, $h(x, y)$ is summable in y for ν;*

(2) *the function of x defined almost everywhere $[\mu]$ as $\int h(x, y)\,d\nu(y)$ is summable for μ;*

(3) $\int d\mu(x)\int h(x, y)\,d\nu(y)=\int h(z)\,d\lambda(z)$.

By symmetry, the same is true with X and Y and μ and ν interchanged. Thus, in particular, the order of integrations in the iterated integral is irrelevant.[1]

Proof: The basic idea of the following proof has its origin in some unpublished lecture notes of Professor J. L. B. Cooper, though the generality of the spaces X and Y involved demands some modifications.

[1] Our proof shows that (3) is significant and valid for all h in $\mathfrak{L}(Z)$ or in $\mathfrak{U}(Z)$, summable for λ or not.

Let us denote by $\mathfrak{X}$ the set of functions summable for λ for which the theorem is true; $\mathfrak{X}$ is known to include $\mathfrak{C}(Z)$. We prove first that $\mathfrak{X}$ has two properties, namely:

(i) $\mathfrak{X}$ contains all functions in $\mathfrak{L}(Z)$ or $\mathfrak{U}(Z)$ which are summable for λ;

(ii) $\mathfrak{X}$ contains the limit of any monotone sequence of its members, provided only that this limit is summable for λ. If X and Y both satisfy conditions (c) and (c') of Lemma 4, § 3, (i) is a consequence of (ii): this is the case envisaged by Professor Cooper. In general, (i) must be proved independently in the following manner.

(i) Consider any $\varphi \in \mathfrak{L}(Z)$ which is summable for λ. Write $\varphi = \lim_{i \in I} h_i$, where the h_i form an increasing directed system of functions in $\mathfrak{C}(Z)$. For a fixed $x \in X$, $\varphi(x, y) \in \mathfrak{L}(Y)$, being in fact equal to $\lim_{i \in I} h_i(x, y)$. By Proposition 2, $\varphi^*(x) \equiv \int \varphi(x, y)\, d\nu(y) = \lim_{i \in I} \int h_i(x, y)\, d\nu(y) \equiv \lim_{i \in I} h_i^*(x)$, say. Here $h_i^*(x) \in \mathfrak{C}(X)$, so that $\varphi^* \in \mathfrak{L}(X)$. Further, since $h_i \in \mathfrak{C}(Z)$, it appears that $\varphi^*(x)$ is finite for each x. Thus (1) is true for φ and all x. Since $\varphi^*(x)$ is the limit of the increasing directed system $(h_i^*)_{i \in I}$ of functions in $\mathfrak{C}(X)$, we can apply Proposition 2 once again to deduce that $\varphi^*(x)$ is summable for μ and that

$$\begin{aligned}\int \varphi(z)\, d\lambda(z) &= \lim_{i \in I} \int h_i(z)\, d\lambda(z) = \lim_{i \in I} \int d\mu(x) \int h_i(x, y)\, d\nu(y)\\ &= \lim_{i \in I} \int h_i^*(x)\, d\mu(x) = \int \varphi^*(x)\, d\mu(x)\\ &= \int d\mu(x) \int \varphi(x, y)\, d\nu(y).\end{aligned}$$

Thus (2) and (3) are valid for φ. The case of a function $\psi \in \mathfrak{U}(Z)$ summable for λ is deducible by a change of sign.

(ii) Suppose that h, summable for λ, is the limit of an increasing sequence (h_n) of functions in $\mathfrak{X}$. Let $E_n \subset X$ with $\mu_e(E_n) = 0$ be such that $h_n(x, y)$ is summable in y for ν whenever $x \in X - E_n$. Putting $E = \bigcup_{n=1}^{\infty} E_n$, we see that $\mu_e(E) = 0$ and that, if $x \in X - E$, the functions $h_n(x, y)$ are collectively summable in y for ν for all n. Hence, by Theorem 2, if $x \in X - E$, $h(x, y)$ is widely summable for ν and

$$\int h(x, y)\, d\nu(y) = \lim_{n \to \infty} \int h_n(x, y)\, d\nu(y).$$

Thus (1) is true of h. By the same theorem, and by Theorem 6, Corollary 2, $\int h(x, y)\, d\nu(y)$ is widely summable for μ, and

$$\begin{aligned}\int d\mu(x) \int h(x, y)\, d\nu(y) &= \lim_{n \to \infty} \int d\mu(x) \int h_n(x, y)\, d\nu(y)\\ &= \lim_{n \to \infty} \int h_n(z)\, d\lambda(z)\end{aligned}$$

since $h_n \in \mathfrak{X}$, which

$$= \int h(z)\,d\lambda(z) \neq \infty$$

by Theorem 2 once more. (1) and (2) now follow automatically (Theorem 6, Corollary 4, § 6) and the proof of (ii) is thereby completed.

Granted (i) and (ii), if h is summable for λ, and if Φ and Ψ are the two functions linked with h by the Vitali-Carathéodory theorem, then Φ and Ψ are both in $\mathfrak{X}$. So we have

$$\left.\begin{aligned}\int h(z)\,d\lambda(z) &= \int \Phi(z)\,d\lambda(z) = \int d\mu(x) \int \Phi(x,\,y)\,d\nu(y),\\ \int h(z)\,d\lambda(z) &= \int \Psi(z)\,d\lambda(z) = \int d\mu(x) \int \Psi(x,\,y)\,d\nu(y).\end{aligned}\right\} \tag{11.4}$$

But $\Psi \leq h \leq \Phi$ everywhere, so that

$$\int \Phi(x,\,y)\,d\nu(y) \geq \overline{\int} h(x,\,y)\,d\nu(y),$$

and hence also

$$\int d\mu(x) \int \Phi(x,\,y)\,d\nu(y) \geq \overline{\int} d\mu(x)\, \overline{\int} h(x,\,y)\,d\nu(y). \tag{11.5}$$

Similarly,

$$\int d\mu(x) \int \Psi(x,\,y)\,d\nu(y) \leq \underline{\int} d\mu(x)\, \underline{\int} h(x,\,y)\,d\nu(y). \tag{11.6}$$

Since Φ and Ψ belong to $\mathfrak{X}$, and since $\int \Phi\,d\lambda = \int \Psi\,d\lambda = \int h\,d\lambda$, this yields

$$\overline{\int} d\mu(x)\, \overline{\int} h(x,\,y)\,d\nu(y) \leq \underline{\int} d\mu(x)\, \underline{\int} h(x,\,y)\,d\nu(y). \tag{11.7}$$

Let us write $\underline{H}(x) = \underline{\int} h(x,\,y)\,d\nu(y)$ and $\overline{H}(x) = \overline{\int} h(x,\,y)\,d\nu(y)$, so that $\underline{H} \leq \overline{H}$ everywhere. By (11.7) we have a fortiori

$$\underline{\int} \underline{H}(x)\,d\mu(x) \geq \overline{\int} \underline{H}(x)\,d\mu(x),$$

so that $\underline{H}$ is surely widely summable for μ. Likewise, (11.7) implies a fortiori that

$$\underline{\int} \overline{H}(x)\,d\mu(x) \geq \overline{\int} \overline{H}(x)\,d\mu(x),$$

so that $\overline{H}$ is surely widely summable for μ. But we have $\overline{H}(x) \leq \int \Phi(x, y)\,d\nu(y)$, so that $\overline{H}(x)$ is $< +\infty$ almost everywhere $[\mu]$, and further

$$\int \overline{H}(x)\,d\mu(x) \leq \int d\mu(x) \int \Phi(x,\,y)\,d\nu(y) = \int \Phi(z)\,d\lambda(z) \;< +\infty.$$

In a similar manner we see that $\overline{H}(x)$ is $> -\infty$ almost everywhere $[\mu]$, and $\int \underline{H}(x)\,d\mu(X) \geq \int d\mu(x) \int \Psi(x,\,y)\,d\nu(y) = \int \Psi(z)\,d\lambda(z) > -\infty$. Since $\underline{H} \leq \overline{H}$, it results that both are finite almost everywhere $[\mu]$ and that both are summable for μ. And since

$$\int (\overline{H} - \underline{H})\,d\mu \leq \int \Phi\,d\lambda - \int \Psi\,d\lambda = 0,$$

we see that $\overline{H}=\underline{H}$ almost everywhere $[\mu]$. This contains the substance of (1) and (2). Also, (3) is now immediate from (11.7) and (11.4).

This completes the proof of the Fubini Theorem.

The partial converse of Fubini's theorem usually known as the theorem of Tonelli may be proved exactly as in [M], Section 25. 6, p. 145, with the difference that it seems necessary in general to assume a priori that the function on the product space is zero outside a set which is almost σ-bounded relative to the product measure λ.

12. The Topologisation of $\mathfrak{C}(X)$.

It is by no means obvious that the notion of the convergence of sequences in $\mathfrak{C}=\mathfrak{C}(X)$ defined in § 4 is precisely that induced by some true topology on $\mathfrak{C}$. It is a notion which serves to define the idea of convergent sequences (or more generally of filters or directed systems) and with it one can define the notion of a closed set (containing the limit of every convergent filter on that set) and the closure of a set (the set of all limits of convergent filters on that set), but it is no longer certain that the closure of a set is always a closed set, nor that the closure of a set is the smallest closed set containing that set.

From the point of view of functional analysis, it is highly desirable that one discusses the possibility of defining on $\mathfrak{C}$ a true topology which, as far as the continuity of linear functionals in concerned, is equivalent to the pseudo-topology defined in § 4. Further, it is natural to demand of these topologies that they be compatible with the vector space structure of $\mathfrak{C}$ and be locally convex.

Thus we are led to seek those topologies of a locally convex vector space on $\mathfrak{C}$ with the property that, relative to them, the topological dual of $\mathfrak{C}$ is exactly the set $\mathfrak{M}=\mathfrak{M}(X)$ of all Radon measures on X. Such topologies are surely existent. In fact, amongst these topologies there is a least fine and a finest. The least fine is none other than the usual weak topology, $\sigma(\mathfrak{C}, \mathfrak{M})$, on $\mathfrak{C}$ generated by $\mathfrak{M}$: the notation is that of Dieudonné [5]. The finest such topology is that denoted by $\tau(\mathfrak{C}, \mathfrak{M})$ on pp. 64–5 of [6]. Further, the topologies we seek are precisely those which are locally convex and which are at once finer than $\sigma(\mathfrak{C}, \mathfrak{M})$ and less fine than $\tau(\mathfrak{C}, \mathfrak{M})$. Consequently the main interest lies in studying and characterising in a manner not involving $\mathfrak{M}$ explicitly this topology $\tau(\mathfrak{C}, \mathfrak{M})$: in this connection we discount as not sufficiently explicit the characterisation of $\tau(\mathfrak{C}, \mathfrak{M})$ according to the general theorem of Mackey-Arens ("convergence uniform on the convex and weakly compact subsets of $\mathfrak{M}$").[1]

[1] This characterisation becomes explicit as soon as one provides independently the characterisation of the weakly compact subsets of $\mathfrak{M}$ afforded by Theorem 13 below.

When X is countable at infinity, the desired topology may be introduced in an a priori fashion as one of the so-called $(\mathfrak{L}\mathfrak{F})$-topologies studied in [6]: see especially Exemple 1^0, p. 67 of this reference. Denoting by $\mathcal{T}_\omega$ the topology on $\mathfrak{C}$ defined there, from Proposition 4, p. 70 of [6] it follows that a sequence $(f_n) \subset \mathfrak{C}$ is convergent to zero in the sense adopted in § 4 if and only if it converges to zero in the sense of the topology $\mathcal{T}_\omega$ (though this is no longer true if (f_n) is replaced by a general directed system or filter on $\mathfrak{C}$). That $\mathcal{T}_\omega = \tau(\mathfrak{C}, \mathfrak{M})$ whenever X is countable at infinity is shown in Théorème 3, Corollary, p. 76 of [6].

However, if X is not countable at infinity, it is no longer a question of the $(\mathfrak{L}\mathfrak{F})$-spaces of Schwartz and Dieudonné: indeed, one has precisely the situation described in § 14, pp. 99–100 of [6]. In the notation there employed (save that we use $\mathfrak{C}_\alpha$ in place of E_α and $\mathfrak{C}$ in place of E), we write $X = \bigcup_\alpha \Omega_\alpha$, where (Ω_α) is an increasing directed set of bounded open sets, put $\mathfrak{C}_\alpha$ for the vector subspace of $\mathfrak{C}$ formed of functions $f \in \mathfrak{C}$ with $K_f \subset \bar{\Omega}_\alpha$, and take for $\mathcal{T}_\alpha$ the topology of convergence uniform on $\bar{\Omega}_\alpha$; $\mathcal{T}_\omega$ may then be defined as the finest of all locally convex topologies on $\mathfrak{C}$ having the property that, for each α, $\mathcal{T}_\omega$ induces on $\mathfrak{C}_\alpha$ a topology less fine than $\mathcal{T}_\alpha$.[1] As is shown by example loc. cit., a sequence (f_n) may converge to zero for the topology $\mathcal{T}_\omega$ and yet be not convergent to zero in the sense adopted in § 4: in the example constructed, X is one of the familiar pathological spaces of ordinals and $\mathcal{T}_\omega$ turns out to be the topology of convergence uniform on X.[2] In addition, the proof given for the case in which X is countable at infinity of the fact that $\mathcal{T}_\omega = \tau(\mathfrak{C}, \mathfrak{M})$ is no longer available. Nevertheless we will show that this latter result still stands.

Theorem 12: *Whatever the locally compact space X, $\mathcal{T}_\omega = \tau(\mathfrak{C}, \mathfrak{M})$.*

Proof: We have to show, first that $\mathfrak{M}$ is the topological dual of $\mathfrak{C}$ relative to $\mathcal{T}_\omega$, and, second, that no strictly finer locally convex topology on $\mathfrak{C}$ enjoys this property.

Now Proposition 5 of [6] still stands and serves to show that the first assertion is true. For completeness we will give the proof, which is in any case quite brief. Let μ be any linear functional on $\mathfrak{C}$ which is continuous for the topology $\mathcal{T}_\omega$. There is then a neighbourhood V of zero in $\mathfrak{C}$ (relative to the topology $\mathcal{T}_\omega$) such that $|\mu(f)| \leq 1$ if $f \in V$. Now for each α, $V \cap \mathfrak{C}_\alpha$ is a neighbourhood of zero in $\mathfrak{C}_\alpha$ for the topology $\mathcal{T}_\alpha$: that is, there is a positive number R_α such that $f \in \mathfrak{C}_\alpha$ and $\sup_{x \in X} |f(x)| \leq R_\alpha$ together imply $|\mu(f)| \leq 1$. But then it is clear that to every compact set $K \subset X$ corresponds at least one α such that $K \subset \Omega_\alpha$, in which case we deduce easily that

[1] That $\mathcal{T}_\omega$ is independent of the system (Ω_α) used is easily seen by reasoning as in the paragraph beginning at the foot of p. 67 of [6].

[2] See also Exercice 3, p. 65 of [B].

$$|\mu(f)| \leq R_\alpha^{-1} \cdot \sup_{x \in X} |f(x)|$$

holds for all $f \in \mathfrak{C}$ with $K_f \subset K$. Thus μ is a Radon measure on X.

Conversely, if μ is a Radon measure on X, to every α corresponds, according to (4.1), a positive finite number playing the role R_α^{-1} in (12.1). Consequently, if $\varepsilon > 0$, and if I is the interval $(-\varepsilon, \varepsilon)$ of real numbers, $\mu^{-1}(I) \cap \mathfrak{C}_\alpha$ is a neighbourhood of zero in $\mathfrak{C}_\alpha$ for the topology $\mathcal{T}_\alpha$. But then (cf. pp. 66–7 of [6]), $\mu^{-1}(I)$ is a neighbourhood of zero in $\mathfrak{C}$ for the topology $\mathcal{T}_\omega$. Thus μ is continuous for $\mathcal{T}_\omega$, and the first assertion is now completely proved.

Finally, suppose that $\mathcal{T}$ is a locally convex topology on $\mathfrak{C}$ strictly finer than $\mathcal{T}_\omega$. Then, for some α, $\mathcal{T}$ induces on $\mathfrak{C}_\alpha$ a topology strictly finer than $\mathcal{T}_\alpha$. But $\mathfrak{C}_\alpha$, with the topology $\mathcal{T}_\alpha$, is a Banach space and hence is relatively strong in the sense of Mackey ([8], Definition 5, p. 523, and Theorem 10, p. 527). Therefore there is a linear functional λ on $\mathfrak{C}_\alpha$ which is continuous for the topology induced on $\mathfrak{C}_\alpha$ by $\mathcal{T}$ but not continuous for the topology $\mathcal{T}_\alpha$. Since $\mathcal{T}$ is locally convex, the Hahn-Banach theorem shows that λ has an extension $\bar{\lambda}$ to all $\mathfrak{C}$ which is linear and continuous for $\mathcal{T}$. This $\bar{\lambda}$ cannot be a Radon measure on X since if it were, its restriction to $\mathfrak{C}_\alpha$, namely λ, would be continuous for the topology $\mathcal{T}_\alpha$. This completes the proof.

Remarks: (i) The example of a locally compact space X which is not countable at infinity constructed in § 14 of [6] to which we have already referred shows that peculiar things can happen for such spaces. Combined with Theorem 12 it shows for example that, although the space concerned is non-compact, every Radon measure on X is bounded. Of course, this situation can never arise if X is non-compact and countable at infinity: in this case, if (x_n) is any sequence of points of X converging to infinity, and if (μ_n) is any sequence of positive numbers, the equation

$$\mu(f) = \sum_n \mu_n \cdot f(x_n)$$

defines a positive Radon measure on X having total mass $\sum_n \mu_n$. This argument breaks down in general since it may be (and in fact *is*, in the said example) impossible to find a countable sequence (x_n) which converges to infinity.

(ii) The above proof that $\mathcal{T}_\omega = \tau(\mathfrak{C}, \mathfrak{M})$ is direct, but once it is shown that $\mathfrak{M}$ is the topological dual of $\mathfrak{C}$ relative to $\mathcal{T}_\omega$, the remainder follows from the general theory of locally convex spaces. By definition, $\mathfrak{C}$ (with $\mathcal{T}_\omega$) is the inductive limit of the Banach spaces $\mathfrak{C}_\alpha$ and hence has the property of being "tonnelé" in the sense of [7]. Hence, by Proposition 2 of [7], $\mathcal{T}_\omega$ is identical with $\tau(\mathfrak{C}, \mathfrak{M})$.

An interesting consequence of Theorem 12 is the characterisation of the weakly compact subsets of $\mathfrak{M}$ afforded by the next theorem. As already hinted, it is possible to derive this characterisation directly and thus (using the theorem of Mackey-Arens) to give an independent proof of Theorem 12 itself.

Theorem 13: *A subset M of $\mathfrak{M}$ is relatively weakly compact if and only if the members of M are of uniformly bounded variation over each compact subset of X.*

Proof: The condition is sufficient. It implies in fact that for each α there is a finite positive number R_α^{-1} such that

$$\left|\int f\,d\mu\right| \leq R_\alpha^{-1} \tag{12.2}$$

holds for all $\mu \in M$ and all $f \in \mathfrak{C}_\alpha$ with $\sup_{x \in X} |f(x)| \leq 1$. It follows therefore that M is contained in the polar set of the set $V \subset \mathfrak{C}$ defined by

$$V = \bigcup_\alpha \{f \in \mathfrak{C}_\alpha : \sup_{x \in X} |f(x)| \leq R_\alpha\}.$$

Now this set V is a neighbourhood of zero in $\mathfrak{C}$ relative to the topology $\mathcal{T}_\omega$, and it will suffice to show that consequently the polar set V^0 is weakly compact in $\mathfrak{M}$.

This last assertion is well known and is a generalisation of the assertion that, in the dual of a normed vector space, the norm-bounded sets are weakly compact. We indicate the proof for completeness. By definition, V^0 is the set of all $\mu \in \mathfrak{M}$ such that $\left|\int f\,d\mu\right| \leq 1$ for all $f \in V$. For each $f \in \mathfrak{C}$, let $r_f = \sup_{\mu \in V^0} \left|\int f\,d\mu\right| < +\infty$, let J_f denote the compact interval $(-r_f, r_f)$ of the real axis, and consider the topological product

$$P = \prod_{f \in \mathfrak{C}} J_f.$$

By Tychonoff's theorem, P is compact. On the other hand, each $\mu \in V^0$ can be looked upon as a point of P whose co-ordinate with suffix f is the number $\mu(f) \in J_f$. What is more, it is easy to see that if we inject V^0 into P in this manner, the topology of P induces on V^0 the weak topology. Hence we are reduced to showing that V^0, considered as a subset of P, is closed in P. However, supposing that the point $p = (p_f)_{f \in \mathfrak{C}}$ of P is in the closure of V^0, it is very simple to show that the mapping $f \to p_f$ is linear on $\mathfrak{C}$ and also that $|p_f| \leq 1$ if $f \in V$. This shows immediately that $f \to p_f$ is a continuous linear functional on $\mathfrak{C}$, hence is a Radon measure μ on X (Theorem 12), and that this μ is a member of V^0. This completes the proof of the sufficiency of the condition.

The condition is necessary. It is enough to show that if M is weakly compact in $\mathfrak{M}$, then for each α the number

$$\sup \left|\int f\,d\mu\right| \tag{12.3}$$

is finite, μ ranging over M and f over all functions of $\mathfrak{C}_a$ such that $\sup_{x \in X} |f(x)| \leq 1$. In the argument that follows, M is assumed taken with the induced weak topology. For a fixed f, $\int f \, d\mu$ represents a continuous function of μ, so that

$$F(\mu) = \sup \left| \int f \, d\mu \right| (< +\infty),$$

with f ranging over the aforesaid set, is lower semicontinuous in μ. Since M is compact, hence a Baire space, there is a measure $\mu_0 \in M$ and a symmetric weak neighbourhood W of zero in $\mathfrak{M}$ such that

$$F(\mu) \leq c < +\infty$$

whenever $\mu \in M$ and $\mu - \mu_0 \in W$. But then, if $\mu \in W \cap \mathfrak{M}$, we have $F(\mu) \leq F(\mu + \mu_0) + F(\mu_0) \leq 2c$. By compactness of M, there is a finite number μ_i $(1 \leq i \leq n)$ of points of M such that any $\mu \in M$ satisfies $\mu - \mu_i \in \mathfrak{M} \cap W$ for some i (depending upon μ of course). Consequently, if $\mu \in M$,

$$F(\mu) \leq \sup_{1 \leq i \leq n} F(\mu_i) + 2c$$

so that the number (12.3) is indeed finite, and the proof is complete.

Remarks: (i) There are numerous interesting applications of Theorem 13. In particular, if X is a Euclidean space, it includes the famous "choice principle" of de la Vallée Poussin, used frequently in potential theory.

(ii) The direct proof of Theorem 13 can again be replaced by more general arguments. Since $\mathfrak{C}$ is "tonnelé", we have the following interesting fact: in $\mathfrak{M}$, there is identity between the subsets which are (a) relatively weakly compact, (b) weakly bounded, (c) strongly bounded (i. e. bounded with respect to the topology on $\mathfrak{M}$ of convergence uniform on the bounded subsets of $\mathfrak{C}$), and (d) equicontinuous. The hypothesis of Theorem 13 is obviously sufficient to ensure that (b), and hence (a), is true of M. Conversely, (a) is equivalent to (d), which is itself equivalent to the relation $M \subset V^0$, where V is a suitable chosen neighbourhood of 0 in $\mathfrak{C}$; and from this the necessity follows immediately.

(iii) Some of the results of this § are contained in Exercice 1, p. 64 of [B].

13. Haar Measures on Topological Groups.

Let us suppose now that X is a locally compact group: we shall write X multiplicatively, as in the custom when commutativity is not assumed. The neutral element of X is denoted by e.

If f is any function on X, we denote by $L_s f$ the left translate of f corresponding to the point $s \in X$, that is the function $L_s f(x) = f(sx)$. Likewise, $R_s f$ will denote the right translate of f, defined by $R_s f(x) = f(xs)$.

It is natural to single out from all the positive Radon measures on X those which are either left-invariant or right-invariant under translations. A non-zero positive Radon measure on X which is left-invariant in the sense that

$$\mu(f) = \mu(L_s f) \quad (f \in \mathfrak{C},\ s \in X) \tag{14.1}$$

is termed a left Haar measure on X. The notion of right Haar measure on X is defined in an analogous manner. The complete symmetry between left and right, which finds formal expression in the consideration along with X of the opposing group X' having the same elements as X but provided with the new law of composition defined by $(xy)' = yx$, makes it possible to consider only, say, the left Haar measures on X.

Since the notion of positive Radon measure used in this paper is exactly that utilised by André Weil in [9], his proof of the existence and essential uniqueness (up to positive constant factors) of the left Haar measure on X constitutes the natural reference at this point. The present paper serves to fill in all the details implied in Weil's consideration of the Lebesgue spaces $L^p(X)$ constructed relative to the left Haar measure and his appeal to the basic properties of these spaces. For the development of the theory of the Haar measure, we refer the reader to [9], observing here merely that:

(1) X is discrete if and only if each one-point set has strictly positive measure (for any one left or right Haar measure), the same measure (for a given Haar measure) for all points because of translation invariance.

(2) X is compact if and only if it has finite measure for any one (and hence all) left or right Haar measure. When this is the case it is natural to normalise both the left and right Haar measures so that for each of them X has total measure unity. These two normalised measures then coincide and we may thus speak here, as in the case of abelian groups, of *the* Haar measure on X. The corresponding integral coincides for all continuous functions on X with the von Neumann mean value of such functions (each of which is automatically uniformly almost periodic due to the compactness of X).

The theory of product measures has an application in the case of groups which is at once peculiar to this case and of the utmost importance. This is the question of the convolution or product by composition of two measures on the group X. Let X be as before, and suppose given two Radon measures μ and ν on X. We may as usual assume that μ and ν are positive. Let λ be the product measure $\mu \otimes \nu$ on $X \times X$. The convolution $\mu\star\nu$ of μ and ν (in that order) is said to exist if, whatever the function $f\in\mathfrak{C}(X)$, the function F on $X\times X$ defined by $F(x, y)=f(xy)$ is summable for λ, in which case $\mu\star\nu$ is by definition the positive Radon measure on X specified by the equation

$$\begin{aligned}\int f\,d(\mu\star\nu)=\int F\,d\lambda=\int d\mu(x)\int f(xy)\,d\nu(y)\\ =\int d\nu(y)\int f(xy)\,d\mu(x)\end{aligned}$$

for all $f\in\mathfrak{C}(X)$, the last three equalities being consequences of the Fubini theorem. Since the function F is continuous on $X\times X$, part (i) of the proof of Theorem 11 shows that in any case the last three of the above equalities hold good at least whenever $f\in\mathfrak{C}^{+}(X)$. This fact provides the simplest method of determining the existence of $\mu\star\nu$ in those cases having the greatest importance. For $\mu\star\nu$ exists if and only if F is summable for λ whatever the function $f\in\mathfrak{C}^{+}(X)$; and for such an f, the summability for λ of F is equivalent to the finiteness of either (and hence both) of the iterated integrals $\int d\mu(x)\int f(xy)\,d\nu(y)$ or $\int d\nu(y)\int f(xy)\,d\mu(x)$. For example, the two most important cases may be dismissed in this way: $\mu\star\nu$ surely exists if either (1) at least one of μ or ν has a compact support, or (2) both μ and ν are bounded.

The question of the convolution of functions may be treated either as a special case of that of Radon measures, or it may be treated with equal success independently and in the standard fashion.

We terminate this § by discussing an example of a set which is locally null but not (globally) null. For the construction we take a locally compact, non-discrete group X having a non-countable, discrete subgroup Z. For example, let X be the direct product of the real axis with the discrete topology by the torus group T, and let Z be the subgroup of X defined as the direct product of the aforesaid real axis by the trivial subgroup of T: this simple example was suggested to me by Dr. C. H. Dowker. We shall show that Z, although measurable and having zero measure, has a non-summable characteristic function; it will appear in fact that $\mu_i(Z)=0$ and $\mu_e(Z)=+\infty$, μ denoting any Haar measure on X.

To begin with, Z is measurable and has measure zero because, whatever the compact set K of X, $K\cap Z$ is finite and hence has measure zero (X being non-discrete, every one-point set has zero measure). This shows also that $\mu_i(Z)=0$.

To show that χ_Z is non-summable, it is enough to show that Z is not almost σ-bounded in X. The last paragraph shows that the intersection of Z with any σ-bounded set is countable. Hence we have only to show that $Z\cap G$ is countable whenever the open set G has finite measure. But, since Z is discrete, it is easy to see that we can find an open neighbourhood N of e such that $zN \cap z'N = \emptyset$ for any two distinct points z and z' of Z. It follows that to each point $z \in Z \cap G$ corresponds an open neighbourhood $U(z)$ of z such that $U(z) \subset G$ and $U(z) \cap U(z') = \emptyset$ if z and z' are distinct points of $Z \cap G$. For any finite set $\{z_1, \ldots, z_n\}$ of distinct points of $Z \cap G$ we have accordingly

$$\sum_{r=1}^{n} \mu(U(z_r)) \leq \mu(G) < +\infty,$$

whence it follows at once that the set of $z \in Z \cap G$ such that $\mu(U(z)) > 0$ is countable. But this set must be none other than $Z \cap G$ itself since any Haar measure assigns a strictly positive measure to any non-void open set. This argument proves at the same time that $\mu_e(Z) = +\infty$.

14. The Duals of Some Topological Vector Spaces of Functions.

We shall now complete the circle by discussing a few useful examples of the situation described in the opening paragraph of this paper. Recall that this situation is as follows: Given a topological vector space $\mathcal{E}$ of real continuous functions on the locally compact space X, required to identify the topological dual $\mathcal{E}'$ of $\mathcal{E}$.

The number of possible examples in plainly unlimited. We shall merely discuss a few of the most frequently occurring instances, the method being adequately indicated in this fashion.

1°. When X is compact and $\mathcal{E}$ is the Banach space of all real continuous functions on X, taken with the uniform norm:

$$\|f\| = \sup_{x \in X} |f(x)|, \tag{14.1}$$

the solution of the problem is immediate by very definition of a Radon measure; see [B], pp. 41–48. A slight extension of this result arises when X is locally compact and $\mathcal{E}$ the space of all real continuous functions on X which tend to zero at infinity: here too the solution is immediate ([B], Exercice 9, p. 67). In either case, *the dual $\mathcal{E}'$ may be identified with the set $\mathfrak{M}^1(X)$ of all bounded Radon measures on X (the condition of boundedness being void if X is compact) in such a manner that to any $u \in \mathcal{E}'$ corresponds a unique $\mu \in \mathfrak{M}^1(X)$ such that*

$$u(f) = \int_X f(x)\, d\mu(x) \tag{14.2}$$

for all $f \in \mathcal{E}$. See also [2], p. 39; and [10].

Two consequences are perhaps worth noting. To begin with, the useful criterion given by Banach ([1], Théorème 8, p. 224) for the weak convergence of sequences of functions in $\mathcal{E}$ is now just a combination of the Banach-Steinhaus theorem and the Lebesgue theorem on the termwise integration of sequences of summable functions. Secondly, regarding $\mathcal{E}$ is an algebra under pointwise mutiplication, use of the notion of support of a Radon measure leads to a very direct proof of a well known theorem on the structure of closed ideals in $\mathcal{E}$, namely: *Let $\mathfrak{J}$ be any closed ideal in $\mathcal{E}$, and let E be the set of points x of X at which at least one of the members of $\mathfrak{J}$ is non-zero; then $\mathfrak{J}$ contains those and only those functions of $\mathcal{E}$ which vanish on $\complement E$ (at least).* The proof of this requires only the observation that, if μ is a Radon measure and f a continuous function, and if the measure defined symbolically by $d\nu = f\,d\mu$ is zero, then any point x at which $f(x) \neq 0$ belongs to the complement of the support of μ.

2°. When X is non-compact, a number of important examples are covered by the following set-up. Let $\mathfrak{S}$ be a set of subsets A of X; without loss of generality one may assume that the sets A are closed and that $\mathfrak{S}$ is an increasing directed set. In all examples met in practice, the union of the sets $A \in \mathfrak{S}$ is X itself. Let then $\mathcal{E}$ be the space of all real continuous functions f on X which satisfy the condition that, for each $A \in \mathfrak{S}$ and each $\varepsilon > 0$, there is a compact set $K \subset X$ such that

$$|f(x)| \leq \varepsilon \text{ for } x \in A - K. \tag{14.3}$$

On $\mathcal{E}$ one takes the topology of the $\mathfrak{S}$-convergence, that is the topology defined by the seminorms

$$N_A(f) = \sup_{x \in A} |f(x)|$$

with $A \in \mathfrak{S}$. $\mathcal{E}$ is then a separated locally convex space. It is not difficult to show that: *the dual $\mathcal{E}'$ may be identified with the set of bounded Radon measures, each of which has its support contained in some set $A \in \mathfrak{S}$, the identification between an element $u \in \mathcal{E}'$ and the corresponding measure μ being obtained by* (14.2). Too see this, observe first that the restriction to $\mathfrak{C}$ of any given $u \in \mathcal{E}'$ is continuous on $\mathfrak{C}$, so that there is a Radon measure μ such that (14.2) is valid for all $f \in \mathfrak{C}$. On the other hand, the continuity of u (on $\mathcal{E}$) implies the existence of a set $A \in \mathfrak{S}$ and a finite, positive number M such that

$$|u(f)| \leq M . N_A(f)$$

for all $f \in \mathcal{E}$. If in this relation we restrict f to $\mathfrak{C}$, it is easily concluded that μ has its support contained in A and is in addition bounded (having a norm in $\mathfrak{M}^1(X)$ at most M). This being so, (14.2) extends to all $f \in \mathcal{E}$ by virtue of the condition (14.3).

The two most important examples of the present situation arise when $\mathfrak{S}$ comprises all the finite subsets of X or all the compact subsets of X; in either case, $\mathcal{E}$ embraces all continuous functions on X, the condition (14.3) being void. The second instance here has been discussed by Hewitt ([11], Theorem 21).

It is perhaps interesting to observe that if $\mathfrak{S}$ has a countable base (i. e., if there exists a sequence (A_n) of sets $\in \mathfrak{S}$ such that every $A \in \mathfrak{S}$ is contained in some A_n), the space $\mathcal{E}$, which is in any case complete, is metrisable. As a consequence, the subsets of $\mathcal{E}'$ which are weakly bounded have the property that their members are supported by a *fixed* set $A \in \mathfrak{S}$ and are further of uniformly bounded total variation.

The method is limited in two main directions, each of which may be instanced by an example.

3°. Suppose that X is non-compact and that $\mathcal{E}$ is the space of all bounded, real continuous functions on X taken with the norm (14.1). The equation (14.2) defines a member u of $\mathcal{E}'$ if and only if the Radon measure μ is bounded. On the other hand, as is well known, not every element u of the dual can be represented in the form (14.2). It is shown in [12] that, in order to obtain the general member of $\mathcal{E}'$, the right member of (14.2) must be supplemented by a linear combination of two generalised limits at infinity on X. Alternatively, a representation (14.2) may be effected for all $u \in \mathcal{E}'$ by replacing X by its Čech compactification. It is of course true that a respresentation of the form (14.2) can be accorded to every $u \in \mathcal{E}'$, but only on condition that μ be allowed to be a general bounded, finitely-additive set-function.

4°. If X is compact and if $\mathcal{E}$ comprises discontinuous functions, it is again true in general that not every u can be represented in the form (14.2) with μ a Radon measure on X. An example is provided by taking for $\mathcal{E}$ the vector space generated by all bounded, real, semicontinuous functions on X, the norm being (14.1) once more. It is quite easy to show that a general $u \in \mathcal{E}'$ has a decomposition $u = u' + u''$ in which u' has a representation (14.2) whilst u'' is orthogonal to all continuous functions (without necessarily being zero on $\mathcal{E}$ itself).

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