Acta Math., 182 (1999), 1–23

Blowup of small data solutions for a class of quasilinear wave equations in two space dimensions, II

by

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Introduction

1. This work is a continuation of our previous work "Blowup of small data solutions for a quasilinear wave equation in two space dimensions" [6]. We consider in both quasilinear wave equations in $\mathbf{R}^{2+1}$,

$$L(u) \equiv \partial_t^2 u - \Delta_x u + \sum_{0 \leqslant i,j,k \leqslant 2} g_{ij}^k \partial_k u \, \partial_{ij}^2 u = 0, \tag{0.1}$$

where

$$x_0 = t, \quad x = (x_1, x_2), \quad g_{ij}^k = g_{ji}^k.$$

We assume that the Cauchy data are C^∞ and small,

$$u(x,0) = \varepsilon u_1^0 + \varepsilon^2 u_2^0 + \ldots, \quad \partial_t u(x,0) = \varepsilon u_1^1 + \varepsilon^2 u_2^1 + \ldots, \tag{0.2}$$

and supported in a fixed ball of radius M.

We could with minor changes handle as well more general equations of the form

$$\partial_t^2 u - \Delta_x u + \sum g_{ij}(\nabla u) \partial_{ij}^2 u = 0, \tag{0.1$'$}$$

with $g_{ij}(0) = 0$, because cubic and higher-order terms play no crucial role in the blowup. We restrict ourselves to (0.1) because previous papers used here have been written in this framework, and also for simplicity.

Following [10], we define

$$g(\omega) = \sum g_{ij}^k \widehat{\omega}_i \widehat{\omega}_j \widehat{\omega}_k, \tag{0.3}$$

where

$$r=\sqrt{x_1^2+x_2^2}, \quad x_1=r\cos\omega, \quad x_2=r\sin\omega$$

are the usual polar coordinates in space, and

$$\widehat{\omega}_0=-1, \quad \widehat{\omega}_1=\cos\omega, \quad \widehat{\omega}_2=\sin\omega.$$

Our aim is to study the existence of smooth solutions to this problem, more precisely the lifespan $\bar{T}_\varepsilon$ of such solutions and the breakdown mechanism when these solutions stop being smooth. In space dimensions two or three, this problem has been introduced and extensively studied by John (see his survey [12] and the references therein), then by Klainerman [13], [14], Hörmander [10], [11] and many authors. Using some crude approximations by solutions of Burgers' equation, Hörmander [10] has obtained in dimensions two and three explicit lower bounds for the lifespan. The result in dimension two is

$$\liminf \varepsilon \bar{T}_\varepsilon^{1/2} \geqslant (\max g(\omega)\partial_\sigma^2 R^{(1)}(\sigma,\omega))^{-1} \equiv \bar{\tau}_0. \tag{0.4}$$

Here, the "first profile" $R^{(1)}$ is defined as

$$R^{(1)}(\sigma,\omega)=\frac{1}{2\sqrt{2\pi}}\int_{s\geqslant\sigma}\frac{1}{\sqrt{s-\sigma}}[R(s,\omega,u_1^1)-\partial_s R(s,\omega,u_1^0)]\,ds, \tag{0.5}$$

where $R(s,\omega,v)$ denotes the Radon transform of the function v,

$$R(s,\omega,v)=\int_{x_1\cos\omega+x_2\sin\omega=s} v(x)\,dx.$$

It was suggested in [11] that these lower bounds should be sharp.

In our previous work [6], we were able to prove actual blowup only for the special example of (0.1),

$$(\partial_t^2-\Delta)u=(\partial_t u)(\partial_t^2 u).$$

It was not clear then whether this result was likely to be true in fact for the general equation (0.1), or if it was a consequence of the special structure of the nonlinear terms.

In the present work, we prove that actual blowup takes place at the suggested time for a general equation (0.1), (0.2) (see Lifespan Theorem 1 of Part I). The only assumption we need is the "generic" condition on the Cauchy data:

(ND) The function $-g(\omega)\partial_\sigma^2 R^{(1)}(\sigma,\omega)$ has a unique strictly negative nondegenerate minimum at a point (σ_0,ω_0).

In fact, Theorem 1 shows that the full formal asymptotic lifespan computed in [3] is the asymptotic expansion of the true lifespan $\bar{T}_\varepsilon$. Moreover, the method of proof yields an accurate description of the behavior of $\nabla^2 u$ close to the unique blowup point M_ε at time $t{=}\bar{T}_\varepsilon$: it is a geometric blowup of cusp type, according to the terminology of [4] (see Geometric Blowup Theorem 2 of Part I).

Hence, geometric blowup of cusp type seems to occur quite often at times equal to the lifespan for quasilinear hyperbolic equations. We hope that further work will confirm this view (see [8] for a discussion of the stability of this pattern and [7] for a short discussion of other possibilities).

2. The method of proof relies on the blowup techniques introduced in [4]: we show there how to construct blowup solutions by solving in smooth functions a nonlinear system called blowup system. In [6], the special structure of the nonlinear terms in (0.1) made it possible to eliminate unknowns and reduce the *nonlinear* blowup system to a scalar third-order equation. The improvement of the present paper over [6] is so to speak of "algebraic" nature: we display in the general case, using the genuine nonlinearity $g(\omega_0){\neq}0$ implied by (ND), decoupling properties of the *linearized* blowup system (and only at the linearized level) which allow one to find solutions and prove tame estimates: these results are explained in Theorem 3 of Part II. Let us emphasize that this blowup theory has nothing to do with perturbation problems or asymptotic analysis; its only connection with problem (0.1), (0.2) is that, blankly applied to this problem after an adequate preparation using some asymptotic analysis, it yields the solution. For clarity, we develop in Part II the blowup theory for a second-order general quasilinear equation. In Part III, we consider the application of this theory to (0.1), (0.2), and review step by step the proof of [6] to indicate how it extends to the general case: surprisingly enough, only minor changes are needed in the estimates; the approach of the determination of the lifespan as a free boundary problem remains unchanged. We hope that this theory will extend to systems, and will be a tool to approach such problems as the stability of blowup and so on (see [8] for results in this direction).

I. Results for the nonlinear wave equations

Consider the problem (0.1), (0.2) (already outlined in the Introduction) of a quasilinear wave equation in two space dimensions with small compactly supported Cauchy data. Recall the normalized variables usually used:

$$\sigma = r - t, \quad \omega, \quad \tau = \varepsilon t^{1/2}.$$

Using the function g and the first profile $R^{(1)}$ (defined in (0.3) and (0.5)), we make the following "generic" assumption on the Cauchy data:

(ND) The function $-g(\omega)\partial_\sigma^2 R^{(1)}(\sigma,\omega)$ has a unique strictly negative nondegenerate minimum at a point (σ_0,ω_0).

We have then the following theorem.

LIFESPAN THEOREM 1. *The lifespan $\bar{T}_\varepsilon$ of the classical solution of* (0.1), (0.2) *satisfies*

$$\bar{\tau}_\varepsilon \equiv \varepsilon(\bar{T}_\varepsilon)^{1/2} = \bar{\tau}_0 + O(\varepsilon). \tag{1.1}$$

Moreover, there is a point $M_\varepsilon = (x_\varepsilon, \bar{T}_\varepsilon)$ such that, for $t \geqslant \tau_0^2\varepsilon^{-2}$ $(0<\tau_0<\bar{\tau}_0)$ and ε small enough,

(i) *the solution u is of class C^1 and $|u|_{C^1} \leqslant C\varepsilon^2$,*

(ii) *the solution u is of class C^2 away from M_ε with $|u|_{C^2} \leqslant C\varepsilon^2$ there, and satisfies*

$$|\nabla^2 u(\,\cdot\,,t)|_{L^\infty} \leqslant \frac{C}{\bar{T}_\varepsilon - t}, \tag{1.2}$$

$$|\partial_t^2 u(\,\cdot\,,t)|_{L^\infty} \geqslant \frac{1}{C}\cdot\frac{1}{\bar{T}_\varepsilon - t}. \tag{1.3}$$

As in [6], let us remark that the full asymptotics of $\bar{T}_\varepsilon$ and of the location of M_ε has been already computed in [3]; the one term asymptotics of (1.1) is only given for simplicity.

Close to the point M_ε, we have a much better description of u, given by the following theorem.

GEOMETRIC BLOWUP THEOREM 2. *There exist a point $\widetilde{M}_\varepsilon = (\widetilde{m}_\varepsilon, \bar{\tau}_\varepsilon)$, a neighbourhood V of $\widetilde{M}_\varepsilon$ in $\{(s,\omega,\tau): s\in\mathbf{R}, \omega\in S^1, \tau\leqslant\bar{\tau}_\varepsilon\}$, and functions $\phi, v, w \in C^3(V)$ with the following properties:*

(i) *The function ϕ satisfies in V the condition*

$$\begin{gathered} \phi_s \geqslant 0,\ \phi_s(s,\omega,\tau)=0 \Leftrightarrow (s,\omega,\tau)=\widetilde{M}_\varepsilon, \\ \phi_{s\tau}(\widetilde{M}_\varepsilon)<0, \quad \nabla_{s,\omega}(\phi_s)(\widetilde{M}_\varepsilon)=0, \quad \nabla_{s,\omega}^2(\phi_s)(\widetilde{M}_\varepsilon)\gg 0. \end{gathered} \tag{H}$$

(ii) *$w_s = \phi_s v$ and $v_s(\widetilde{M}_\varepsilon)\neq 0$. If we define the map*

$$\Phi(s,\omega,\tau) = (\sigma = \phi(s,\omega,\tau), \omega, \tau),$$

we have $\Phi(\widetilde{M}_\varepsilon) = (|x_\varepsilon| - \bar{T}_\varepsilon, x_\varepsilon/|x_\varepsilon|, \bar{\tau}_\varepsilon) \equiv \bar{M}_\varepsilon$. The function u verifies near M_ε

$$u(x,t) = \frac{\varepsilon}{r^{1/2}} G(r-t, \omega, \varepsilon t^{1/2}),$$

where G is defined near $\bar{M}_\varepsilon$ by

$$G(\Phi)=w.$$

Finally, the functions ϕ, v, w are of class C^k if $\varepsilon\leqslant\varepsilon_k$.

It is of course understood, in Theorems 1 and 2, that the dependence of the various objects on ε is "uniform": the points M_ε and $\widetilde{M}_\varepsilon$ depend continuously on ε, the neighbourhood V can be taken as the intersection of a fixed set with $\{\tau\leqslant\bar{\tau}_\varepsilon\}$, the functions ϕ, v, w are uniformly bounded in C^3, the strict inequalities in (H) are uniform, etc.

Exactly as in [6], we see that the blowup of $\nabla^2 u$ only comes from the singularity of the mapping Φ at the point $\widetilde{M}_\varepsilon$; according to (H), this singularity is of *cusp type* (in the usual sense of classification of mappings): this is exactly what is called in [4] a "geometric blowup of cusp type". Taking into account the fact that equation (0.1) has no special structure in its nonlinearity (in the sense that the coefficients g_{ij}^k are arbitrary), this result seems to indicate that geometric blowup of cusp type occurs very often for quasilinear hyperbolic equations. We hope that further work on various other equations or systems will confirm this view (see [7] for a short discussion of more complicated cases).

We can easily deduce from Theorem 3 the following corollary, which we can view as some "blowup criterion" (see [15] or [7]):

COROLLARY. *Assume that the data of a solution u of* (0.1) *satisfy* (ND) *and that ε is small enough. If u is smooth for $t<T\leqslant\bar{T}_\varepsilon$ and, for some C,*

$$|\nabla^2 u(\,\cdot\,,t)|_{L^2}\leqslant C,$$

then $T<\bar{T}_\varepsilon$.

We wonder if it is possible to prove such a statement directly by some "functional analysis" method.

II. Blowup of a quasilinear second-order equation

This section is self-contained. In Part III, we will explain how to use this theory to obtain the results of Part I about quasilinear wave equations.

We have developed in [4] a general theory of "blowup solutions" and "blowup systems". However, we do not know in general how to solve the blowup system of a given equation or system. We were able to solve this blowup system only in the special cases considered in [5] and [6].

If we start with a second-order scalar equation, we can of course write it as a first-order system to which the theory of [4] applies, but this is rather tedious: we develop here along the same lines another approach, in which we keep in sight as far as possible the scalar character of the original equation.

Let us consider, in some domain of $\mathbf{R}^n$ with coordinates $(x_1, ..., x_n)$, the quasilinear equation

$$P(u) \equiv \sum p_{ij}(x, y, u, \nabla u)\partial_{ij}^2 u + q(x, y, u, \nabla u) = 0. \tag{2.1}$$

Here, we set for simplicity $x_1 = x$, $y = (x_2, ..., x_n)$, $\nabla u = (\partial_x u, \partial_y u)$. We will also use the notations

$$\bar{\partial} = (0, \partial_2, ..., \partial_n), \quad \hat{\phi} = (-1, \partial_2\phi, ..., \partial_n\phi).$$

We introduce the change of variables

$$\Phi(s, y) = (x = \phi(s, y), y) \tag{2.2}$$

and the new functions

$$w(s, y) = u(\phi(s, y), y), \quad v(s, y) = (\partial_x u)(\phi(s, y), y). \tag{2.2$'$}$$

Note that necessarily $w_s = \phi_s v$. We set then $\mathcal{A} \equiv w_s - \phi_s v$, and call the equation $\mathcal{A} = 0$ the "*auxiliary equation*".

In this section, as in [6], we have in mind the construction of singular solutions of $P(u) = 0$; thus we are interested in points (s, y) where $\phi_s = 0$ and $v_s \neq 0$, because $(\partial_x^2 u)(\Phi) = v_s \phi_s^{-1}$.

1. The following elementary proposition describes *the blowup system of P*.

PROPOSITION AND DEFINITION II.1. *With the above notations, we have*

$$\begin{aligned}
&(\nabla u)(\Phi) = \bar{\partial} w - \hat{\phi} v, \\
&(\partial_{ij}^2 u)(\Phi) = \bar{\partial}_{ij}^2 w - v\bar{\partial}_{ij}^2 \phi - (\hat{\phi}_i \bar{\partial}_j v + \hat{\phi}_j \bar{\partial}_i v) + \hat{\phi}_i \hat{\phi}_j \left(\frac{v_s}{\phi_s}\right), \\
&P(u)(\Phi) = \mathcal{E}\frac{v_s}{\phi_s} + \mathcal{R},
\end{aligned}$$

with

$$\begin{aligned}
&\mathcal{E} = \sum p_{ij}(\phi, y, w, \bar{\partial} w - \hat{\phi} v)\hat{\phi}_i \hat{\phi}_j, \\
&\mathcal{R} = \sum p_{ij}(\phi, y, w, \bar{\partial} w - \hat{\phi} v)[\bar{\partial}_{ij}^2 w - v\bar{\partial}_{ij}^2 \phi - (\hat{\phi}_i \bar{\partial}_j v + \hat{\phi}_j \bar{\partial}_i v)] + q(\phi, y, w, \bar{\partial} w - \hat{\phi} v).
\end{aligned}$$

We call the system

$$\mathcal{E} = 0, \quad \mathcal{R} = 0, \quad \mathcal{A} = 0 \tag{2.3}$$

the "blowup system", *the first equation of* (2.3) *the* "eikonal equation", *the second equation the* "residual equation". *To any smooth solution of* (2.3) *corresponds through* (2.2′) *one or several singular solutions of* (2.1) (*depending on the branch of inverse of* Φ *we choose*), *and such solutions are called* "(geometric) blowup solutions".

Remark. If ϕ, v, w are smooth functions solutions of the blowup system, only the second-order derivatives of u may blowup at a point where $\phi_s=0$. This is in accordance with what we expect from a quasilinear second-order equation.

The equation $\mathcal{E}=0$ has a simple geometric interpretation: if we set formally

$$\Phi^{-1}=(\psi(x,y),y),$$

we have

$$(\nabla\psi)(\Phi)=-\frac{1}{\phi_s}\hat{\phi}.$$

Hence $\mathcal{E}=0$ is equivalent to

$$\sum p_{ij}(x,y,u,\nabla u)(\partial_i\psi)(\partial_j\psi)=0,$$

that is, the (singular) Lagrangean manifold $\Lambda=(x,y,\nabla\psi)$ is characteristic for the linearized equation of (2.1).

2. *Linearization of the blowup system.* To compute the linearized blowup system, we must introduce some notations. We set, for arbitrary smooth functions ϕ, v, w,

$$\gamma=\Big(\sum\partial_{\nabla u}p_{ij}\hat{\phi}_i\hat{\phi}_j\Big)\hat{\phi}, \tag{2.4}$$

$$Z_1=\sum p_{ij}(\hat{\phi}_i\bar{\partial}_j+\hat{\phi}_j\bar{\partial}_i),\quad Q=\sum p_{ij}\bar{\partial}^2_{ij}, \tag{2.5}$$

$$a_{ij}=\partial_x p_{ij}+\partial_u p_{ij}v+\partial_{\nabla u}p_{ij}\bar{\partial}v,\quad a_0=\sum a_{ij}\hat{\phi}_i\hat{\phi}_j,\quad b_0=\sum\partial_u p_{ij}\hat{\phi}_i\hat{\phi}_j,$$

$$Z_2=\Big(\sum\partial_{\nabla u}p_{ij}\hat{\phi}_i\hat{\phi}_j\Big)\bar{\partial},\quad K_{ij}=\bar{\partial}^2_{ij}w-v\bar{\partial}^2_{ij}\phi-(\hat{\phi}_i\bar{\partial}_j v+\hat{\phi}_j\bar{\partial}_i v),$$

$$c_0=\sum K_{ij}\partial_u p_{ij}+\partial_u q,$$

$$c_1=\sum K_{ij}\partial_{\nabla u}p_{ij}+\partial_{\nabla u}q,$$

$$c_2=\sum K_{ij}a_{ij}+Qv+\partial_x q+\partial_u qv+\partial_{\nabla u}q\bar{\partial}v,$$

$$a_1=\gamma v_s-Z_1\phi_s-a_0\phi_s,\quad a_2=a_0+c_1\hat{\phi}+Q\phi.$$

It is understood here that the summations are taken for all i,j, and that p_{ij} and its various derivatives are taken at $(\phi,y,w,\bar{\partial}w-\hat{\phi}v)$.

With these notations, we list first certain technical identities.

LEMMA II.2. *We have the identities*

$$Z_2\mathcal{A}+b_0\mathcal{A}-\mathcal{E}_s=a_1,$$
$$Q\mathcal{A}+c_1\bar{\partial}\mathcal{A}+c_0\mathcal{A}-\mathcal{R}_s=Z_1v_s+(a_2-a_0)v_s-c_2\phi_s.$$

We denote now by

$$\mathcal{E}'_{\phi,v,w}(\dot{\phi},\dot{v},\dot{w})\equiv\mathcal{E}'(\dot{\phi},\dot{v},\dot{w})\equiv\mathcal{E}'$$

the differential of $\mathcal{E}$ at the point (ϕ,v,w), and similarly for $\mathcal{R}$ and $\mathcal{A}$.

The following proposition describes the linearized system of the blowup system.

PROPOSITION II.2. *We set* $\dot{z}=\dot{w}-v\dot{\phi}$, *and have then*

(i) $\mathcal{E}'(\dot{\phi},\dot{v},\dot{w})=-\gamma\dot{v}+Z_1\dot{\phi}+Z_2\dot{z}+a_0\dot{\phi}+b_0\dot{z}$,

(ii) $\mathcal{R}'(\dot{\phi},\dot{v},\dot{w})=Q\dot{z}-Z_1\dot{v}+c_1\bar{\partial}\dot{z}+c_0\dot{z}+c_2\dot{\phi}+(a_0-a_2)\dot{v}$,

(iii) $\mathcal{A}'(\dot{\phi},\dot{v},\dot{w})=\dot{z}_s+v_s\dot{\phi}-\phi_s\dot{v}$.

The straightforward computation is left to the reader.

Remark. In establishing the blowup system (2.3), we keep in mind that Φ,v,w cannot be separately determined, because we can always replace Φ by $\Phi\Phi_1$ (Φ_1 being a diffeomorpism), and then replace v,w by $v(\Phi_1),w(\Phi_1)$. What we need here is that Φ should be of corank one wherever it is not invertible. The choice (2.2) is then no restriction and has the advantage of being simple and leading to (relatively) easy computations. Of course, the structure of the linearized blowup system also reflects this indeterminacy between Φ, v and w. Generally speaking, if $u(\Phi)=w$, we have $\dot{u}(\Phi)+u'(\Phi)\dot{\Phi}=\dot{w}$, hence $\dot{w}-u'(\Phi)\dot{\Phi}$ is indeed the "good unknown" for the linearized system (this fact has many applications in nonlinear problems involving free boundaries, see for instance [1]). Here, $\dot{z}$ is this good unknown, because $v=(\partial_x u)(\Phi)$.

Finally, let us compute $\Phi' Z_1$: we find

$$\tfrac{1}{2}\Phi' Z_1=\Big(-\sum_{j>1}p_{1j}\phi_j+\sum_{\substack{i>1\\ j>1}}p_{ij}\phi_i\phi_j,\,-p_{1j}+\sum_{i>1}p_{ij}\phi_i\Big).$$

On the other hand, if $p=\sum p_{ij}(x,y,u,\nabla u)\xi_i\xi_j$ is the principal symbol of the linearized equation on a solution u corresponding to a solution (ϕ,v,w) of the blowup system, we have on Λ

$$-\tfrac{1}{2}\phi_s\pi_*H_p=\Big(-p_{11}+\sum p_{1j}\phi_j,\,-p_{1j}+\sum_{i>1}p_{ij}\phi_i\Big).$$

The eikonal equation

$$p_{11}-2\sum_{j>1}p_{1j}\phi_j+\sum_{i,j>1}p_{ij}\phi_i\phi_j=0$$

shows that in fact

$$\Phi' Z_1 = -\phi_s \pi_* H_p.$$

3. *The genuinely nonlinear case.* Let us consider now a smooth solution (ϕ, v, w) of the blowup system in a domain D.

Definition II.3. If the function

$$\gamma = \sum \partial_{\partial_k u} p_{ij}(\phi, y, w, v, \partial_y w - v\partial_y \phi)\,\hat{\phi}_i \hat{\phi}_j \hat{\phi}_k$$

does not vanish in D, we say that we are in the genuinely nonlinear case.

This terminology is justified by the following fact: for a blowup solution such as u, the main contribution to the matrix u'' at the blowup point is given by the matrix of rank one $\hat{\phi}^t \hat{\phi}$ (see also Proposition 2.2.1 of [4]); hence $\gamma \neq 0$ asserts the effective dependence of the symbol $\sum p_{ij}\hat{\phi}_i\hat{\phi}_j$ on ∇u in the relevant direction $-\hat{\phi}$ (it is the same situation as that described by Lax for first-order systems). In this case, we can express $\dot{v}$ in terms of $\dot{\phi}, \dot{z}$ according to Proposition II.2. The remarkable fact is that the resulting system in $\dot{\phi}, \dot{z}$ almost decouples, as indicated in the following theorem.

THEOREM 3. *For the linearized system of* (2.3), *we have, in the genuinely nonlinear case, the identities*

$$\begin{aligned}
\mathcal{F}_1 \equiv{}& Z_1\dot{z}_s - \phi_s Q\dot{z} - \left[c_1\phi_s\bar{\partial} + \frac{1}{\gamma}(a_0\phi_s + Z_1\phi_s)Z_2\right]\dot{z} - \left[c_0\phi_s + \frac{b_0}{\gamma}(a_0\phi_s + Z_1\phi_s)\right]\dot{z} \\
&+ a_2\dot{z}_s + \frac{a_1}{\gamma}Z_1\dot{\phi} + \left[Z_1 v_s + (a_2 - a_0)v_s - c_2\phi_s + \frac{a_0 a_1}{\gamma}\right]\dot{\phi} \\
={}& -\phi_s\mathcal{R}' + (Z_1 + a_2)\mathcal{A}' - \frac{1}{\gamma}(a_0\phi_s + Z_1\phi_s)\mathcal{E}',
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
\mathcal{F}_2 \equiv{}& Z_1^2\dot{\phi} + \left(a_2 - \frac{Z_1\gamma}{\gamma}\right)Z_1\dot{\phi} + \left[Z_1 a_0 - a_0^2 + a_0\left(a_2 - \frac{Z_1\gamma}{\gamma}\right) - \gamma c_2\right]\dot{\phi} - \gamma Q\dot{z} \\
&+ Z_1 Z_2\dot{z} - \gamma c_1\bar{\partial}\dot{z} + \left(a_2 - a_0 - \frac{Z_1\gamma}{\gamma}\right)(Z_2\dot{z} + b_0\dot{z}) + Z_1(b_0\dot{z}) - \gamma c_0\dot{z} \\
={}& \left(Z_1 + a_2 - a_0 - \frac{Z_1\gamma}{\gamma}\right)\mathcal{E}' - \gamma\mathcal{R}'.
\end{aligned} \tag{2.7}$$

The point of this theorem is that, thanks to Lemma II.2, the coefficients α_1 and α_2 of the terms involving $Z_1\dot{\phi}$ and $\dot{\phi}$ in the first equation are small if $\mathcal{E}, \mathcal{R}, \mathcal{A}$ and their derivatives are small. In a Nash–Moser scheme aimed at solving $\mathcal{E}=0, \mathcal{R}=0, \mathcal{A}=0$, we could view these terms as "quadratic errors". However, we cannot just neglect them, because this would correspond to solving the linearized system up to quadratic errors *divided by* ϕ_s, which is not acceptable in the framework of smooth functions.

In applications, we will solve *exactly* the equations

$$\mathcal{F}_1 = -\phi_s\dot{g} + (Z_1 + a_2)\dot{h} - \frac{1}{\gamma}(a_0\phi_s + Z_1\phi_s)\dot{f},$$

$$\mathcal{F}_2 = \left(Z_1 + a_2 - a_0 - \frac{Z_1\gamma}{\gamma}\right)\dot{f} - \gamma\dot{g}$$

in some domain D, and then determine $\dot{v}$ by the equation $\mathcal{E}' = \dot{f}$. For the functions $\dot{\phi}, \dot{v}, \dot{w}$ thus obtained, we have then

$$\mathcal{E}'(\dot{\phi}, \dot{v}, \dot{w}) = \dot{f}, \quad \mathcal{R}' = \dot{g}, \quad (Z_1 + a_2)(\mathcal{A}' - \dot{h}) = 0.$$

If the geometry of D and the boundary conditions are appropriate, this will yield $\mathcal{A}' = \dot{h}$, and the linearized system is exactly solved.

III. Application to quasilinear wave equations with small data

In this part, we apply the theory of Part II to equation (0.1). The surprising fact is that, with the help of this theory, the proofs of Theorems 1 and 2 for the general case require only a little extra work compared to the proof of [6]. Thus this part is divided into three sections:

(i) First, we recall the general strategy of the proof of [6].

(ii) Second, we point out the differences between the general case at hand and the special case of [6].

(iii) Finally, we scan the proof of [6], step by step, to explain what minor modifications have to be done to get a complete proof of the general case.

The idea of the proof is to construct a piece of blowup solution to (0.1) in a strip

$$-C_0 \leqslant r - t \leqslant M, \quad \tau_0^2\varepsilon^{-2} \leqslant t \leqslant \bar{T}_\varepsilon, \quad 0 < \tau_0 < \bar{\tau}_0,$$

close to the boundary of the light cone. This gives an upper bound for the lifespan, which turns out to be the correct one. Of course, this is not surprising, because the first blowup of the solution is believed to take place in such a strip, and not far inside the light cone. The proof of the theorems is thus devoted to this construction, which is done in four steps.

Step 1. *Asymptotic analysis, normalization of variables and reduction to a local problem.* We choose a number $0 < \tau_0 < \bar{\tau}_0$ and use here asymptotic information on the behavior of u for $r - t \geqslant -C_0$ and $\varepsilon t^{1/2}$ close to τ_0. Thus, we are far away from any possible blowup at this stage, because of (0.2). According to [1], the solution in this domain behaves like a smooth function (depending smoothly also on ε and $\varepsilon^2 \ln \varepsilon$) of the variables

$$\sigma = r - t, \quad \omega, \quad \tau = \varepsilon t^{1/2}.$$

This is why we set

$$u(x,t) = \frac{\varepsilon}{r^{1/2}} G(\sigma, \omega, \tau).$$

Writing equation (0.1) for G in these new variables, we are left with solving a local problem for G in a domain

$$-C_0 \leqslant \sigma \leqslant M, \quad \tau_0 \leqslant \tau \leqslant \bar{\tau}_\varepsilon,$$

where $\bar{\tau}_\varepsilon = \varepsilon \overline{T}_\varepsilon^{1/2}$ is still unknown. At this stage, we have a *free boundary problem*, the upper boundary of the domain being determined by the first blowup time.

Step 2. *Blowup of the problem.* To solve the free boundary problem of Step 1, we introduce as in (2.2) of Part II a singular (still unknown) change of variables

$$\Phi: (s, \omega, \tau) \mapsto (\sigma = \phi(s, \omega, \tau), \omega, \tau), \quad \phi(s, \omega, \tau_0) = s.$$

The idea is to obtain G in the form

$$G(\Phi) = w$$

for *smooth* functions ϕ and w, and arrange at the same time to have ϕ_s vanish at one point $\widetilde{M}_\varepsilon = (\widetilde{m}_\varepsilon, \bar{\tau}_\varepsilon)$ of the upper boundary of the domain. Thus, we will have

$$w_s = G_\sigma \phi_s,$$

and the technical condition (ii) of Theorem 2 gives $G_\sigma(\Phi) = v$, hence

$$G_{\sigma\sigma}(\Phi) = v_s / \phi_s.$$

We see that $u, \nabla u$ will remain continuous and that $\nabla^2 u$ will blowup at some point, in accordance with the expected behavior of u.

Note that instead of looking for a *singular* solution G of the normalized original equation as in Step 1, we are now looking for a *smooth* solution (ϕ, v, w) of the blowup system. However, we cannot just solve for τ close to τ_0: we have to reach out to attain a point where $\phi_s = 0$.

Finally, introducing an unknown real parameter (corresponding to the height of the domain), we can reduce the free boundary problem at hand to a problem on a fixed domain.

Step 3. *Existence and tame estimates for the linearized problem.* The genuinely nonlinear character of the problem (in the sense of Definition II.3) is implied by the condition (ND). The theory of Part II (Theorem 3) tells us that in this case, the linearized blowup system decouples approximately. This allows us to obtain existence of a solution and tame estimates by doing so for a scalar third-order equation, as in the special case of [6]. The (unknown) point where ϕ_s vanishes is a degeneracy point for this equation. Energy estimates can then be obtained using an appropriate multiplier.

Step 4. *Back to the solution* u. Having w and ϕ, we deduce G and thus obtain a piece of solution $\tilde{u}$ of (0.1) with the desired properties. It remains to prove that $\tilde{u}=u$ where $\tilde{u}$ is defined, and that u does not blowup anywhere else.

We indicate now the two main differences between the present work and [6]:

(i) In the special case of [6], the full blowup system could be reduced to a single scalar third-order equation on ϕ. Here, this is no longer possible, but the theory of Part II (Theorem 3) shows that the *linearized* blowup system almost decouples into an ordinary differential equation and a scalar third-order equation, very close to that of [6].

(ii) We do not assume $g(\omega)\neq 0$ as in [6]. The condition (ND) only tells us that $g(\omega)$ will be nonzero for ω close to ω_0. Thus we have to *localize* the (global in ω) estimates of [6] to prove estimates for the linearized blowup system in a local domain of appropriate geometry in s, τ and ω.

Everything else is essentially the same, in particular, the analysis of the nondegeneracy condition (H) and the "fundamental lemma" are unchanged.

We proceed finally with the step-by-step analysis of the modifications of the proof of [6].

Step 1. *Asymptotic analysis, normalization of variables and reduction to a local problem.* The asymptotic analysis "close to the boundary of the light cone" is exactly the same as in [6]. It leads us to set

$$u(x,t)=\frac{\varepsilon}{r^{1/2}}G(\sigma,\omega,\tau)$$

with

$$r=|x|,\quad x=r(\cos\omega,\sin\omega),\quad \sigma=r-t,\quad \tau=\varepsilon t^{1/2}.$$

We fix $0<\tau_0<\bar{\tau}_0$ (see (0.4)). Results from [2] indicate that close to τ_0, G behaves essentially as a smooth function of its arguments (uniformly in ε). We start from time τ_0 to reach the actual (unknown) blowup time $\bar{\tau}_\varepsilon$, which we expect to be close to $\bar{\tau}_0$. In fact, τ_0 will have to be chosen very close to $\bar{\tau}_0$, as will be explained in §3.1 of Step 2.

For completeness, though it is not really necessary, we indicate the expression of $L(u)$ in the normalized variables σ,ω,τ.

PROPOSITION III.1. *Set*

$$\widetilde{\omega}=(0,-\sin\omega,\cos\omega),\quad \bar{\omega}=(0,\cos\omega,\sin\omega),\quad \widehat{\omega}=(-1,\cos\omega,\sin\omega),\quad R=\tau^2+\varepsilon^2\sigma.$$

Then, for a smooth function q of its arguments,

$$\begin{aligned}\frac{r}{\varepsilon^2}L(u)=&-\frac{R^{1/2}}{\tau}\,\partial^2_{\sigma\tau}G+\varepsilon^2\frac{R^{1/2}}{4\tau^2}\,\partial^2_{\tau}G-\frac{\varepsilon^2}{R^{3/2}}\,\partial^2_{\omega}G+\varepsilon^2 q(\sigma,\omega,\tau,G,\nabla G)\\ &+\sum g^k_{ij}\left[\widehat{\omega}_k\partial_\sigma G+\varepsilon^2\left(-\frac{\bar{\omega}_k}{2R}\,G+\frac{\widetilde{\omega}_k}{R}\,\partial_\omega G+\frac{\delta^k_0}{2\tau}\,\partial_\tau G\right)\right]\\ &\times\left[\widehat{\omega}_i\widehat{\omega}_j\partial^2_\sigma G+\varepsilon^2\left(\frac{\widehat{\omega}_i\widetilde{\omega}_j+\widehat{\omega}_j\widetilde{\omega}_i}{R}\,\partial^2_{\sigma\omega}G+\frac{\widehat{\omega}_i\delta^j_0+\widehat{\omega}_j\delta^i_0}{2\tau}\,\partial^2_{\sigma\tau}G\right)\right.\\ &\left.+\varepsilon^4\left(\frac{\delta^i_0\delta^j_0}{4\tau^2}\,\partial^2_\tau G+\frac{\widetilde{\omega}_i\delta^j_0+\widetilde{\omega}_j\delta^i_0}{2\tau R}\,\partial^2_{\tau\omega}G+\frac{\widetilde{\omega}_i\widetilde{\omega}_j}{R^2}\,\partial^2_\omega G\right)\right]\equiv P(G).\end{aligned}\tag{1.1}$$

We want to solve the equation $P(G)=0$ in an appropriate subdomain of

$$\sigma\leqslant M,\quad \tau_0\leqslant\tau\leqslant\bar{\tau}_\varepsilon,$$

with two trace conditions on $\{\tau=\tau_0\}$ corresponding to that for u and G supported in $\{\sigma\leqslant M\}$.

Step 2. *Blowup of the problem and reduction to a Goursat problem on a fixed domain.*

1. *Formal blowup.* The equation $P(G)=0$ computed in (1.1) is of the form (2.1) studied in Part II, with

$$x=\sigma,\quad y=(\omega,\tau),\quad u=G,\quad w(s,\omega,\tau)=G(\phi,\omega,\tau),\quad v(s,\omega,\tau)=G_\sigma(\phi,\omega,\tau).$$

It is of course very tedious to compute the blowup system explicitely, and *we need not do that.* It is enough to see what happens for $\varepsilon=0$. Equation (1.1) reduces then to Burgers' equation

$$-\partial^2_{\sigma\tau}G+g(\omega)(\partial_\sigma G)(\partial^2_{\sigma\sigma}G),$$

and we have

$$\mathcal{E}=\phi_\tau+g(\omega)v,\quad \mathcal{R}=-v_\tau,\quad \mathcal{A}=w_s-\phi_s v.$$

Hence

$$Z_1=\partial_\tau,\quad Z_2=0,\quad Q=0,\quad \gamma=-g(\omega).$$

The linearized blowup system is (still for $\varepsilon=0$)

$$\mathcal{E}'=\dot{\phi}_\tau+g\dot{v},\quad \mathcal{R}'=-\partial_\tau\dot{v},\quad \mathcal{A}'=\dot{z}_s+v_s\dot{\phi}-\phi_s\dot{v}.$$

Since $g(\omega_0)\neq 0$, we are in the genuinely nonlinear case in a domain where ω is close enough to ω_0, which we will always assume in the rest of this work. The identities of Theorem 3 read, for $\varepsilon=0$,

$$\mathcal{F}_1=\partial_\tau \dot{z}_s+\left(v_s+\frac{\phi_{s\tau}}{g}\right)\dot{\phi}_\tau+v_{s\tau}\dot{\phi}=-\phi_s\mathcal{R}'+\partial_\tau\mathcal{A}'+\frac{\phi_{s\tau}}{g}\mathcal{E}', \tag{2.1.1}$$

$$\mathcal{F}_2=\dot{\phi}_{\tau\tau}=\partial_\tau\mathcal{E}'+g\mathcal{R}'. \tag{2.1.2}$$

Finally, we need to know the main terms (that is, the ε^2-terms) of Q: they are

$$Q=\varepsilon^2\left(\frac{1}{4\tau}\partial_\tau^2-\frac{1}{\tau^3}\partial_\omega^2\right)+O(\varepsilon^4). \tag{2.1.3}$$

2. *Reduction to a free boundary Goursat problem.*

2.1. *A local solution of the blowup system.* By the implicit-function theorem, we can solve in ϕ_τ the equation

$$\sum p_{ij}(\phi,\omega,\tau,G(\phi,\omega,\tau),\nabla G(\phi,\omega,\tau))\hat{\phi}_i\hat{\phi}_j=0$$

in the form

$$\phi_\tau=E(\omega,\tau,\phi,\phi_\omega). \tag{2.2.1}$$

We can solve locally in s and τ close to τ_0 the Cauchy problem (2.2.1) with initial value $\phi(s,\omega,\tau_0)=s$. Calling $\bar{\phi}$ the obtained solution, we set then

$$\bar{w}=G(\bar{\phi},\omega,\tau),\quad \bar{v}=G_\sigma(\bar{\phi},\omega,\tau).$$

It follows that

$$\bar{w}_s=\bar{\phi}_s\bar{v},\quad \bar{\partial}\bar{w}-\hat{\phi}\bar{v}=\nabla G(\bar{\phi},\omega,\tau).$$

Hence the eikonal equation and the auxiliary equation are satisfied, and so is the residual equation.

2.2. *Straightening out a characteristic surface.* We will see in Step 3 that solving the linearized system reduces essentially to solving the main term in $\mathcal{F}_1$ (Theorem 3) whose principal part is

$$Z_1\partial_s-\phi_s Q.$$

As in [6], in order to obtain a characteristic Goursat problem, we consider the "nearly horizontal" surface $\Sigma=\{\tau=\psi(s,\omega)+\tau_0\}$ through $\{\tau=\tau_0, s=M\}$ which is characteristic for this operator taken on $\bar\phi, \bar v, \bar w$. We perform then in the nonlinear blowup system the (known) change of variables

$$X = s, \quad Y = \omega, \quad T = \Big(1 - \chi\Big(\frac{\tau - \tau_0}{\eta}\Big)\Big)(\tau - \tau_0) + (\tau - \tau_0 - \psi)\chi\Big(\frac{\tau - \tau_0}{\eta}\Big), \tag{2.2.2}$$

where χ is zero near one and one near zero, and $\eta > 0$ is small enough.

We now work in a subdomain of

$$X \leqslant M, \quad 0 \leqslant T \leqslant \bar T = \bar\tau_\varepsilon - \tau_0.$$

§2.3 of [6] has no equivalent here, so we jump to

2.4. *Construction of an approximate solution in the large.* For $\varepsilon = 0$, the exact solution $\bar\phi_0, \bar v_0, \bar w_0$ of the blowup system is

$$\bar\phi_0 = X - gT\partial_\sigma R^{(1)}(X, Y, \tau_0), \quad \bar v_0 = \partial_\sigma R^{(1)}(X, Y, \tau_0), \quad \bar w_0 = R^{(1)} - \tfrac{1}{2} gT(\partial_\sigma R^{(1)})^2.$$

Gluing together $\bar\phi_0, \bar v_0, \bar w_0$ with the true local solution $\bar\phi, \bar v, \bar w$ yields as in [6] an approximate solution $\bar\phi^{(0)}, \bar v^{(0)}, \bar w^{(0)}$ for which $\mathcal{E} = \bar f^{(0)}, \mathcal{R} = \bar g^{(0)}, \mathcal{A} = \bar h^{(0)}$. These right-hand sides are smooth, flat on $\{X = M\}$, zero near $\{T = 0\}$, and vanish for $\varepsilon = 0$.

2.5. *The condition* (H). For the sake of completeness, and because it is an essential point, we repeat here what has been said in §2.5 of [6].

We say that ϕ satisfies the condition (H) in a domain D bounded below and above by $\{T = 0\}$ and $\{T = \bar T\}$ if ϕ_X vanishes (appropriately) only at some point $M = (m, \bar T)$ of D. More precisely, in D,

$$\begin{gathered} \phi_X \geqslant 0, \ \phi_X(X, Y, T) = 0 \ \Leftrightarrow \ (X, Y, T) = M, \\ \phi_{XT}(M) < 0, \quad \nabla_{X,Y}(\phi_X)(M) = 0, \quad \nabla^2_{X,Y}(\phi_X)(M) \gg 0. \end{gathered} \tag{H}$$

The approximate solution $\bar\phi^{(0)}$ from §2.4 satisfies, thanks to (ND), this condition (H) at time

$$\bar T = T_0 \equiv (-\inf -g\partial_X^2 R^{(1)}(X, Y, \tau_0))^{-1} = \bar\tau_0 - \tau_0. \tag{2.2.3}$$

3. *Reduction to a Goursat problem on a fixed domain and condition* (H).

3.1. *Reduction to a fixed domain.* Exactly as in [6], to be free to adjust the height of the domain, we perform a change of variables depending on a parameter λ close to zero:

$$X = x, \quad Y = y, \quad T = T(t, \lambda) \equiv t + \lambda t(1 - \chi_1(t)), \tag{2.3.1}$$

where χ_1 is one near zero and zero near T_0, and T_0 is defined in (2.2.3). We hope that the reader will not confuse these coordinates with the original coordinates! To describe the fixed domain D_0 in which we will work, we first consider the following picture in the plane $\{x=M\}$:

(i) Fix ω_1 small enough to have $g\neq 0$ for $\omega_0-\omega_1\leqslant\omega\leqslant\omega_0+\omega_1$.

(ii) Fix $0<\nu<\frac{1}{4}\bar{\tau}_0^2$, and consider the points I_j with coordinates

$$I_1=(y=\omega_0-\omega_1,t=0),\quad I_2=(\omega_0+\omega_1,0),\quad I_3=(\omega_0-\omega_1+T_0/\nu,T_0),$$
$$I_4=(\omega_0+\omega_1-T_0/\nu,T_0),\quad I_5=(\omega_0-\omega_1+2T_0/\nu,0),\quad I_6=(\omega_0+\omega_1-2T_0/\nu,0).$$

We choose $\bar{\tau}_0-\tau_0$ so small that $\nu<\frac{1}{4}\tau_0^2$ and $0<T_0=\bar{\tau}_0-\tau_0<\frac{1}{2}\nu\omega_1$, so that $(\omega_0,0)$ lies in the interior of the segment I_5I_6. For some large constant A_0, we denote by $\widetilde{D}_0$ the cylindrical domain

$$\widetilde{D}_0=\{-A_0\leqslant x\leqslant M,\ (y,t)\in(I_1I_3I_4I_2)\}.$$

For technical reasons, our actual domain D_0 will be a slight modification of $\widetilde{D}_0$: for some $\eta_1>0$, consider the lines

$$\delta_1=\Big\{t=0,\ y-(\omega_0-\omega_1)=-\frac{\eta_1}{\nu}(x-M)\Big\},\quad \delta_2=\Big\{t=0,\ y-(\omega_0+\omega_1)=\frac{\eta_1}{\nu}(x-M)\Big\}.$$

The domain D_0 is the domain bounded by the planes $x=-A_0, x=M$, $t=0, t=T_0$, the plane containing (δ_1, I_1I_3), and the plane containing (δ_2, I_2I_4); these planes have normal $n_\pm=(-\eta_1,\pm\nu,1)$. It is understood that A_0 and small η_1 are chosen such that $\bar{\phi}^{(0)}$ satisfies (H) for a point M interior to the upper boundary of D_0.

We denote now by $\widetilde{\mathcal{E}}(\lambda,\phi,v,w)$ the nonlinear equation $\mathcal{E}$ of the blowup system transformed by the two successive changes of variables (2.2.2) and (2.3.1); we use similar notations for the other equations. Our aim is to solve the new nonlinear system in (λ,ϕ,v,w) in the domain D_0, starting with

$$\lambda^{(0)}=0,\quad \phi^{(0)}=\bar{\phi}^{(0)},\quad v^{(0)}=\bar{v}^{(0)},\quad w^{(0)}=\bar{w}^{(0)}.$$

Since the solution we start from has already all the good traces on $\{x=M\}$ and $\{t=0\}$, we need only solve the linearized system in flat functions.

3.2. *Structure of the linearized system.* By the same lemma as in [6] (Lemma 3.1 of Part III), we have, with $q=\partial_\lambda T/\partial_t T$, the identity

$$\partial_\lambda\widetilde{\mathcal{E}}+\partial_\phi\widetilde{\mathcal{E}}(\phi_t q)+\partial_v\widetilde{\mathcal{E}}(v_t q)+\partial_w\widetilde{\mathcal{E}}(w_t q)=q\widetilde{\mathcal{E}}_t,$$

and similarly for the other equations. Thus the linearized system

$$(\widetilde{\mathcal{E}})'(\dot\lambda,\dot\phi,\dot v,\dot w)=\dot f,\quad (\widetilde{\mathcal{R}})'=\dot g,\quad (\tilde{\mathcal{A}})'=\dot h$$

can be written

$$\widetilde{(\mathcal{E}')}(\dot{\Phi},\dot{V},\dot{W})=\dot{f}-q\dot{\lambda}\widetilde{\mathcal{E}}_t,$$

and similarly for the other equations. Here,

$$\dot{\Phi}=\dot{\phi}-\dot{\lambda}q\phi_t,\quad \dot{V}=\dot{v}-\dot{\lambda}qv_t,\quad \dot{W}=\dot{w}-\dot{\lambda}qw_t,\quad \dot{Z}=\dot{W}-v\dot{\Phi},$$

and $\widetilde{(\mathcal{E}')}$ denotes the linear system obtained from the linearized blowup system (Proposition II.2 of Part II) in the original variables s,ω,τ by the two successive changes of variables (2.2.2) and (2.3.1). As in [6], we neglect the "quadratic errors" $q\dot{\lambda}\widetilde{\mathcal{E}}_t$ and so on.

The idea for adjusting $\dot{\lambda}$ is the following: once $\dot{\Phi}$ is known, we want to have $\phi+\dot{\phi}$ satisfy again condition (H) for some point on the upper boundary of our fixed domain D_0. This can be achieved by picking up $\dot{\lambda}$ appropriately; this is what we call the "fundamental lemma" (see §§3.3 and 3.4 of [6]). Note that, at this stage, it is the nondegeneracy condition (ND) which ensures the stability of the vanishing pattern of ϕ_X under perturbations. The iteration scheme is identical to that of [6, §4], which we do not repeat here. Hence, it is enough to solve the transformed linear system

$$\widetilde{(\mathcal{E}')}=\dot{f},\quad \widetilde{(\mathcal{R}')}=\dot{g},\quad \widetilde{(\mathcal{A}')}=\dot{h} \tag{2.3.2}$$

in D_0.

Step 3. Existence and tame estimates for the linearized problem.

1. *Structure of the linearized system.* In order to write down the transformed linearized system, let us denote by $\widetilde{Z}_1,\tilde{\partial}_s\equiv S,\widetilde{Q}$, and so on, the transformed operators of Z_1,∂_s,Q, and so on. We normalize $\widetilde{Z}_1$ to have it be ∂_t for $\varepsilon=0$, so that (as in [6])

$$Z=\partial_t+\varepsilon^2 z_0\partial_y,\quad S=\partial_x+\varepsilon^2 s_0\partial_t,$$

and the transformed linearized system has the form

$$ZS\dot{Z}+\varepsilon^2(S\phi)N\dot{Z}+\varepsilon^2 l_1(\dot{Z})+\alpha_1 Z\dot{\Phi}+\alpha_2\dot{\Phi}=\dot{f}_1, \tag{3.1.1$_\mathrm{a}$}$$

$$Z^2\dot{\Phi}+\beta_1 Z\dot{\Phi}+\beta_2\dot{\Phi}+\varepsilon^2 ZH\dot{Z}+\varepsilon^2\beta_3\partial_y^2\dot{Z}+\varepsilon^2 l_1'(\dot{Z})=\dot{f}_2. \tag{3.1.1$_\mathrm{b}$}$$

Here,

(i) $l_1(\dot{Z})$, $l_1'(\dot{Z})$ are linear combinations of $\nabla\dot{Z}$ and $\dot{Z}$, while H is a linear combination of Z and ∂_y,

(ii) $N=N_1Z^2+2\varepsilon^2N_2Z\partial_y+N_3\partial_y^2$, with

$$N_1=-\frac{1}{4(\partial_t T)(\tau_0+T(t,\lambda))}+O(\varepsilon^2),\quad N_3=\frac{\partial_t T}{(\tau_0+T(t,\lambda))^3}+O(\varepsilon^2),$$

(iii) all the coefficients of Z, S, N, l_1, l_1', H, and also the α's and the β's, involve at most third-order derivatives of ϕ, w and second-order derivatives of v.

Finally, remember that in the process of solving the blowup system, we linearize only on functions for which

(3.1.2) $\{t=0\}$ is characteristic for $ZS+\varepsilon^2(S\phi)N$,

(3.1.3) the coefficients α_1 and α_2 can be made arbitrarily small.

Note also that D_0 is an influence domain for Z, so that solving (3.1.1) yields exact solutions of the linearized system (2.3.2), in accordance with §3 of Part II.

2. *Energy inequality for the linearized system.* We replace ε^2 by ε and set

$$\widetilde{P} \equiv ZSZ+\varepsilon(S\phi)NZ.$$

We set $\dot{Z}=Z\dot{k}$ in the linearized system (3.1.1), so we have now to solve the system

$$\widetilde{P}\dot{k}+\varepsilon l_1(Z\dot{k})+\alpha_1 Z\dot{\Phi}+\alpha_2\dot{\Phi}=\dot{f}_1, \tag{3.2.1$_\mathrm{a}$}$$

$$Z^2\dot{\Phi}+\beta_1 Z\dot{\Phi}+\beta_2\dot{\Phi}+\varepsilon ZHZ\dot{k}+\varepsilon\beta_3\partial_y^2 Z\dot{k}+\varepsilon l_1'(Z\dot{k})=\dot{f}_2. \tag{3.2.1$_\mathrm{b}$}$$

There are two main differences with the treatment of [6]: first, we do not have to solve only for $\widetilde{P}$, but for a coupled system; second, we want to prove estimates in D_0, and hence we have to check that the geometry of D_0 is correct.

With the notations

$$A=S\phi, \quad \delta=T_0-t, \quad g=\exp h(x-t), \quad p^2=\delta^\mu g, \quad |\cdot|_0=|\cdot|_{L^2(D_0)},$$

we have the following energy inequality.

PROPOSITION 3.2. *Fix $\mu>1$. Then there exist $\eta_0>0$, $\alpha_0>0$, $\varepsilon_0>0$, h_0 and $C>0$ such that, for all smooth ϕ, v, w satisfying* (H), (3.1.2) *and*

$$|\phi-\phi^{(0)}|_{C^4}+|v-v^{(0)}|_{C^4}+|w-w^{(0)}|_{C^4}\leqslant\eta_0,$$

for all $0\leqslant\varepsilon\leqslant\varepsilon_0$, $h\geqslant h_0$, if $|\alpha_1|+|\alpha_2|\leqslant\alpha_0$, we have the inequality

$$\begin{aligned} h|pSZ\dot{k}|_0^2+h|pZ^2\dot{k}|_0^2+\varepsilon h|p\partial_y Z\dot{k}|_0^2+\varepsilon^2\int\delta^{\mu-1}g(S\phi)(1+\delta h)|\partial_y^2\dot{k}|^2 \\ +|pZ\dot{\Phi}|_0^2+h|p\dot{\Phi}|_0^2\leqslant C|p\dot{f}_1|_0^2+Ch^{-1}|p\dot{f}_2|_0^2. \end{aligned} \tag{3.2.2}$$

Here, the functions $\dot{k}, \dot{\Phi}$ are supposed to be smooth and to satisfy (3.2.1) *and*

$$\dot{k}(x,y,0)=\dot{k}_t(x,y,0)=\dot{k}_{tt}(x,y,0)=0, \quad \dot{k}(M,y,t)=0, \quad \dot{\Phi}(x,y,0)=\dot{\Phi}_t(x,y,0)=0.$$

Proof. (a) We first extend the proof of (4.2.2) of [6] to our domain D_0. All we have to do is to check the sign of the boundary terms on the part of the boundary of D_0 which we have not already checked, that is, on the "lateral" planes with normals $n_\pm=(-\eta_1,\pm\nu,1)$. We assume that ε is small enough to have $\eta_1-\varepsilon s_0\geqslant 0$. Using just u here instead of $\dot{k}$ to stick to the notations of [6], we see that twice these terms on the plane with normal n_+ is the integral of

$$
\begin{aligned}
&(1+\varepsilon z_0\nu)a(SZu)^2+(Z^2u)^2[(-\eta_1+\varepsilon s_0)(d-\varepsilon a(S\phi)N_1)+\varepsilon d(S\phi)(2\varepsilon\nu N_2+N_1(1+\varepsilon z_0\nu))]\\
&\quad+(\partial_y Zu)^2[\varepsilon(S\phi)(1+\varepsilon z_0\nu)(-dN_3+\varepsilon cN_1)-2\varepsilon^3c(S\phi)\nu N_2+\varepsilon(\eta_1-\varepsilon s_0)(a(S\phi)N_3-c)]\\
&\quad+(\partial_y^2u)^2\varepsilon^2N_3c(S\phi)(1+\varepsilon z_0\nu)+2(Z^2u)(SZu)\varepsilon a(S\phi)[(1+\varepsilon z_0\nu)N_1+\varepsilon N_2\nu]\\
&\quad+2\varepsilon(Z^2u)(\partial_y Zu)[(\eta_1-\varepsilon s_0)\varepsilon a(S\phi)N_2+\nu(S\phi)(dN_3-\varepsilon cN_1)]\\
&\quad+2\varepsilon(SZu)(\partial_y Zu)[(1+\varepsilon z_0\nu)\varepsilon N_2a(S\phi)+\nu(a(S\phi)N_3-c)]\\
&\quad+2\varepsilon(SZu)(\partial_y^2u)c(1+\varepsilon z_0\nu)+2\varepsilon^2(Z^2u)(\partial_y^2u)N_1c(S\phi)(1+\varepsilon z_0\nu)\\
&\quad+4\varepsilon^3(\partial_y Zu)(\partial_y^2u)N_2c(S\phi)(1+\varepsilon z_0\nu).
\end{aligned}
$$

With the same choices $a=A^{-1}\delta^\mu g$, $c=c'\delta^\mu g$, $d=-d'\delta^\mu g$ as in [6], we can write these terms as a sum of two squares and two quadratic forms as follows:

$$
\begin{aligned}
&=(1+\varepsilon z_0\nu)a\bigg[SZu+\varepsilon(S\phi)\left(N_1+\frac{\varepsilon N_2\nu}{1+\varepsilon z_0\nu}\right)Z^2u\\
&\qquad+\varepsilon(S\phi)\left(\frac{\nu(N_3-c')}{1+\varepsilon z_0\nu}+\varepsilon N_2\right)\partial_y Zu+\varepsilon c'(S\phi)\partial_y^2u\bigg]^2\\
&\quad+\varepsilon^2(S\phi)(1+\varepsilon z_0\nu)\delta^\mu gc'(N_3-c')\bigg[\partial_y^2u-\frac{\varepsilon N_2\nu}{(1+\varepsilon z_0\nu)(N_3-c')}Z^2u\\
&\qquad+\left(\frac{\varepsilon N_2}{N_3-c'}-\frac{\nu}{1+\varepsilon z_0\nu}\right)\partial_y Zu\bigg]^2\\
&\quad+\delta^\mu g(\eta_1-\varepsilon s_0)[(d'+\varepsilon N_1)(Z^2u)^2+2\varepsilon^2N_2(Z^2u)(\partial_y Zu)+\varepsilon(N_3-c')(\partial_y Zu)^2]\\
&\quad+\varepsilon\delta^\mu g(S\phi)[(-d'N_1+O(\varepsilon))(Z^2u)^2-(2\nu d'N_3+O(\varepsilon))(Z^2u)(\partial_y Zu)\\
&\qquad+(d'N_3+O(\varepsilon))(\partial_y Zu)^2].
\end{aligned}
$$

The first quadratic form in the factor of $\eta_1-\varepsilon s_0$ is clearly positive for small ε. The second is positive for small ε only if $\nu^2<-N_1/N_3$: taking into account the explicit form of N_1 and N_3, and the choice of ν, this condition is satisfied for small ε. The boundary terms on the other plane for which ν is changed into $-\nu$ are handled similarly.

We have thus proved that the energy estimate (4.2.2) of [6] about the operator $\widetilde{P}$ is valid in a local domain such as D_0.

(b) We combine now this estimate with the standard estimate for Z:

$$h|pv|_0^2 \leqslant C|pZv|_0^2, \quad v(x,y,0)=0.$$

We write $(3.2.1_{\mathrm{b}})$ in the form

$$Z(Z\dot{\Phi}+\varepsilon HZ\dot{k}+\varepsilon\beta_3\partial_y^2\dot{k}) = -\beta_1 Z\dot{\Phi}-\beta_2\dot{\Phi}-\varepsilon[\beta_3\partial_y^2, Z]\dot{k}-\varepsilon l_1'(Z\dot{k})+\dot{f}_2,$$

and obtain then

$$|p\dot{E}|_0^2 \leqslant Ch^{-1}E_0, \tag{3.2.3}$$

with

$$\dot{E} = Z\dot{\Phi}+\varepsilon HZ\dot{k}+\varepsilon\beta_3\partial_y^2\dot{k}$$

and

$$E_0 = |p\dot{f}_2|_0^2+|pZ\dot{\Phi}|_0^2+|p\dot{\Phi}|_0^2+\varepsilon^2(|p\nabla Z\dot{k}|_0^2+|p\partial_y^2\dot{k}|_0^2+|p\nabla\dot{k}|_0^2).$$

For some small $\eta_2>0$ to be chosen, we also obtain

$$\eta_2|pZ\dot{\Phi}|_0^2 \leqslant \eta_2 Ch^{-1}E_0+C\varepsilon^2|p\nabla Z\dot{k}|_0^2+C\eta_2\varepsilon^2|p\partial_y^2\dot{k}|_0^2. \tag{3.2.4}$$

Using the inequality on $\widetilde{P}$, we have now

$$\begin{aligned} &h|pSZ\dot{k}|_0^2+h|pZ^2\dot{k}|_0^2+\varepsilon h|p\partial_y Z\dot{k}|_0^2+\varepsilon^2|p\partial_y^2\dot{k}|_0^2 \\ &\qquad \leqslant C|\dot{f}_1|_0^2+C\varepsilon^2|p\nabla Z\dot{k}|_0^2+C|p\dot{\Phi}|_0^2+C\alpha_0^2(|p\dot{E}|_0^2+\varepsilon^2|p\partial_y^2\dot{k}|_0^2). \end{aligned} \tag{3.2.5}$$

Adding (3.2.3), (3.2.4) and (3.2.5), we first choose α_0 and η_2 such that

$$C\alpha_0^2+C\eta_2<1,$$

then choose h_0 big enough to absorb all the remaining terms such as

$$|pZ\dot{\Phi}|_0^2, \quad |p\dot{\Phi}|_0^2, \quad \varepsilon^2|p\nabla Z\dot{k}|_0^2, \quad \varepsilon^2|p\nabla\dot{k}|_0^2$$

on the left-hand side of the inequality. □

3. *Higher-order inequalities.* This section is entirely identical to the corresponding section of [6]: Lemma IV.3.1, which is a lemma on $\widetilde{P}$, remains valid, and Lemma IV.3.2 is simplified because there is no Z to the left of $[K,\widetilde{P}]$. On the other hand, the commutation of factors such as K with $(3.2.1)_{\mathrm{b}}$ only produces harmless terms. As in [6], we see that we can obtain a control in the $\widetilde{H}^s$ without decreasing ε with s, but only by increasing h.

The statement corresponding to Proposition IV.3.2 of [6] is here

PROPOSITION 3.3. *There exist $\eta_0>0$, $\alpha_0>0$, an integer n_0 and $\varepsilon_0>0$ such that for smooth functions ϕ, v, w satisfying* (H), (3.1.2) *and*

$$|\phi-\phi^{(0)}|_{C^6}+|v-v^{(0)}|_{C^6}+|w-w^{(0)}|_{C^6}\leqslant\eta_0,$$

and all integer s, if $|\alpha_1|+|\alpha_2|\leqslant\alpha_0$ and $0\leqslant\varepsilon\leqslant\varepsilon_0$, there exists $C_s>0$ for which we have the inequality

$$|\dot\Phi|_s+|\dot V|_s+|\dot Z|_s\leqslant C[|(\dot f,\dot g,\dot h)|_{s+n_0}+(1+|(\phi,v,w)|_{s+n_0})|(\dot f,\dot g,\dot h)|_{n_0}]. \tag{3.3.1}$$

Here, the functions $\dot\Phi, \dot V, \dot Z$ are supposed to be smooth and flat on $\{t=0\}$ and $\{x=M\}$, and satisfy the transformed linearized system (2.3.2).

4. *Existence of flat solutions.* It is a consequence of the solvability of $\widetilde{P}$ in flat functions.

PROPOSITION 3.4. *Let ϕ, v, w, α_1, α_2 and ε satisfy the assumptions of Proposition* 3.3. *Then for all smooth $\dot f_1, \dot f_2$, flat on $\{t=0\}$ and $\{x=M\}$, there exists a unique smooth solution of* (3.2.1), *flat on $\{t=0\}$ and $\{x=M\}$. The corresponding smooth and flat $\dot\Phi, \dot V, \dot Z$ satisfy the tame estimate* (3.3.1).

Proof. First, we extend the fields Z, S, the operator $\widetilde{P}$ and the various coefficients of (3.2.1) to transform (3.2.1) into a system global in ω, with the same properties as (3.2.1). In particular, we assume that we can solve the extended $\widetilde{P}$ in smooth flat functions as in Proposition IV.4 of [6], and that we have the same estimates.

Next, we use the following fixed point scheme:

$$\widetilde{P}\dot k^{(n+1)}+\varepsilon l_1(Z\dot k^{(n)})+(\alpha_1Z+\alpha_2)\dot\Phi^{(n)}=\dot f_1,$$
$$Z\dot E^{(n+1)}+\beta_1Z\dot\Phi^{(n)}+\beta_2\dot\Phi^{(n)}+\varepsilon[\beta_3\partial_y^2,Z]\dot k^{(n)}+\varepsilon l_1'(Z\dot k^{(n)})=\dot f_2,$$

with, as before,

$$\dot E^{(n)}=Z\dot\Phi^{(n)}+\varepsilon HZ\dot k^{(n)}+\varepsilon\beta_3\partial_y^2\dot k^{(n)}.$$

We start from $\dot k^{(0)}=0$, $\dot\Phi^{(0)}=0$. We denote by $\|\cdot\|$ the norm whose square is the left-hand side of (3.2.2), where we put an additionnal coefficient η_2 in front of the terms involving $\dot\Phi$, and by $|||\cdot|||$ the norm

$$|||(\dot k,\dot\Phi)|||^2=\|(\dot k,\dot\Phi)\|^2+|p\dot E|_0^2,\quad \dot E=Z\dot\Phi+\varepsilon HZ\dot k+\varepsilon\beta_3\partial_y^2\dot k.$$

We prove, exactly as in the proof of (3.2.2), that for an appropriate choice of η_2 and h_0, we have the contraction ($K_0<1$)

$$Q_{n+1}\leqslant K_0Q_n,$$

with

$$Q_{n+1} = |||(\dot{k}^{(n+1)} - \dot{k}^{(n)}, \dot{\Phi}^{(n+1)} - \dot{\Phi}^{(n)})|||.$$

Moreover, by commuting factors $K = T^l$, $l \leqslant s$, as in [6], we obtain the same type of inequalities with a control of the sum of the norms of the terms

$$(K(\dot{k}^{(n+1)} - \dot{k}^{(n)}), K(\dot{\Phi}^{(n+1)} - \dot{\Phi}^{(n)}))$$

for the various K of order at most s. Thus we see that the sequence of smooth and flat functions $\dot{k}^{(n)}, \dot{\Phi}^{(n)}$ converges in all H^s to a unique smooth and flat solution of (3.2.1). □

Step 4. *Back to the solution* u.

1. *The constructed piece of solution belongs to* u. In the previous sections, we have obtained a solution λ, ϕ, v, w of the blowup system in a domain D_0, with ϕ satisfying the nondegeneracy condition (H) and ϕ_x vanishing at a point $\widehat{M}_\varepsilon$. The two changes of variables (2.2.2) and (2.3.1) being close to the identity when ε is small, the transformed domain D_1 of D_0 back to the original blowup variables s, ω, τ is as closed as we want to D_0. We extend D_1 down to the plane $\{\tau = \tau_0\}$ in a domain D_2 which is now bounded by vertical and horizontal planes,

$$-A_0 \leqslant s \leqslant M, \quad \tau_0 \leqslant \tau \leqslant \bar{\tau}_\varepsilon,$$

and by two lateral surfaces as close as we want to planes with normal $n_\pm = (-\eta_1, \pm\nu, 1)$ (see §3.1 of Step 2). Recall that D_2 contains on its upper boundary the image $\widetilde{M}_\varepsilon$ of $\widehat{M}_\varepsilon$ where ϕ_s vanishes. We can now "recut" this domain, that is, find a subdomain D_3 of D_2 with a simpler geometry, but still containing the crucial point $\widetilde{M}_\varepsilon$: we replace the lateral surfaces by planes with normals $(0, \pm\nu, 1)$. The image D_4 of D_3 by Φ is again a cylindrical domain (with a trapezoidal basis in (ω, τ)) that we can recut as in [6] into a domain

$$-A_0 + C(\tau^2 - \tau_0^2) \leqslant \sigma \leqslant M, \quad \tau_0 \leqslant \tau \leqslant \bar{\tau}_\varepsilon,$$

which is also laterally bounded by planes with normals $(0, \pm\nu, 1)$. We denote by $\overline{M}_\varepsilon$ the image of $\widetilde{M}_\varepsilon$ by Φ. The image D_5 of this domain in the original variables (x, t) is an influence domain for the linearized equation of (0.1) on $\bar{u} = (\varepsilon / r^{1/2}) G$, thanks to the choice of ν and C big enough. Thus, by uniqueness, the constructed piece of solution $\bar{u}$ in D_5 coincides with the true solution u of the Cauchy problem, whose second-order derivatives blowup at M_ε.

2. *The function* u *does not blowup anywhere else.* The proof is completely analogous to that of [6]: we extend first, in a strip close to the light cone, the obtained function

G globally in ω to an approximate G which blows up only at $\bar{M}_\varepsilon$. Then we extend this approximate G into the interior of the light cone, and complete the proof by the standard energy inequality argument.

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Received October 1, 1997